Lecture of 2012-06-07 **scribe: B. Kolesnik**

1 TASEP animation

This lecture began by investigating a TASEP animation available on Patrik Ferrari's webpage[1]. The animation depicts the TASEP growth process. Some aspects of the simulation are discussed: the deterministic limit shape, the fluctuations about the limit shape and their growth rate, and the behaviour of the system started with flat initial conditions.

We began by setting the initial conditions so as to observe the corner growth process in Russian coordinates. With every trial the evolution of a random growth process and the limit shape (the parabola tangent to the axis) are drawn in time. We remarked that after a certain amount of time the process stays very close to the limit shape. As mentioned in the previous lecture, at time t we have fluctuations around the limit shape of $O(t^{1/3}) \ll O(\sqrt{t})$ and non-vanshing correlations at scales $O(t^{2/3})$; and at larger distances we have approximate independence. It was pointed out that the growth process tends to be above the limit shape. This is because the Airy process, which is used to model the growth, is asymmetric.

We also observed the growth process with alternating holes and particles. This gives a flat height function initially. As seen in the animation, and as one would expect, the growth process remains relatively flat over time. Still, we observe fluctuations of $O(t^{1/3})$ and correlations of $O(t^{2/3})$.

We noted that our observations of the TASEP growth process are typical of the KPZ universality class. There is a general heuristic which says that many 'fairly simple' growth process should have these $t^{1/3}, t^{2/3}$ scaling parameters and a growth process which can be described by the Airy process.

2 The KPZ universality class

Ballistic deposition is a very simple growth process, believed to be in the Kardar-Parisi-Zhang (KPZ) universality class.

Consider first the following process. With rate 1 (i.e. at times determined by an $Exp(1)$ random variable) boxes of unit dimensions are added to growing stacks at each integer. This results in a stack of boxxes at every vertex z, of height $h_t(z)$. See Figure 1.

By the CLT, the height of each stack of boxes — independent of all others — at time t is $t + O(\sqrt{t})$. Since stacks are independent, there is no correlation between h_t at nearby points. Since there is no interaction between columns of boxes, the dynamics of this process is not so interesting.

One way to modify the pure Poisson deposition model to get non-trivial behaviour, known as **ballisic deposition**, is to make the boxes 'sticky.' When

[1] http://www-wt.iam.uni-bonn.de/∼ferrari/animations/ContinuousTASEP.html

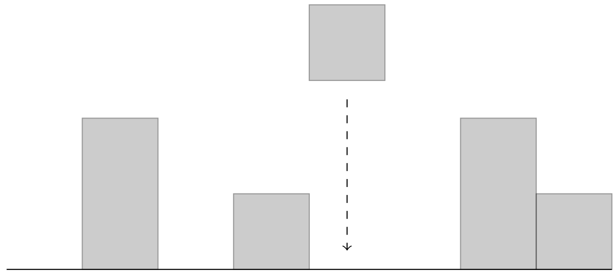

Figure 1: Poisson deposition.

a box is dropped into a column, it comes to a stop once it is fully adjacent to any other box, either in the same column, or in a neighbouring column, and not necessarily on top of a box in its own column. See Figure 2. In this way, holes may be created.

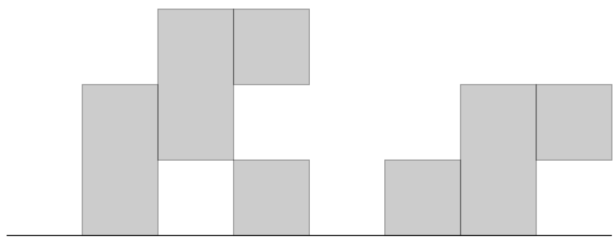

Figure 2: ‘Sticky box’ process.

This gives the process a *smoothing mechanism*: the inter-column interaction ensures that neighbouring columns will typically be of a similar height, since a large height difference takes many boxes to form but a single one in the lower column would reset the difference. Moreover, we expect any valleys in the configuration of growth to be quickly bridged over.

This process has a second important property: The rate at which the height of a columns grow depends on the slope of the height function. Indeed, if the height function in a certain area of the configuration has a very large slope, then there are many places for a box to stick on to and leave much empty space below. Since such empty space remains unoccupied for the duration of the process (i.e. boxes dropped subsequently cannot reach this space), in an interval where the height function has a large slope, the height tends to increase much faster than in flatter areas. See Figure 3.

The third noteworthy property of the process is that the randomness of the growth process is independent across space and time. That is, there is independent growth at different locations and at different times.

These three characteristics of the ‘sticky box’ process roughly characterize the KPZ universality class.

General belief: If a growth process has (i) a smoothing mechanism, (ii)

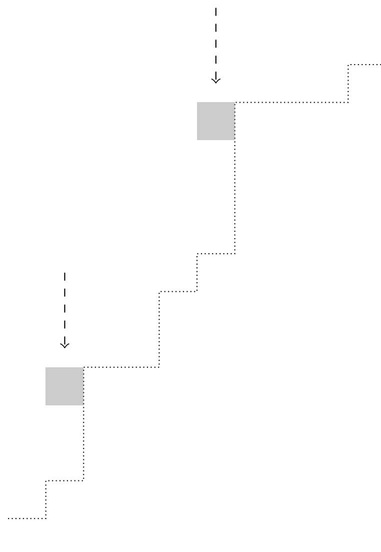

Figure 3: 'Sticky' boxes added to an interval with height function of large slope.

growth rate/height function slope dependence and (*iii*) space-time independent noise, then it has $t^{1/3}, t^{2/3}$ scaling parameters and a height function which can be described by the Airy process in the scaling limit (as does TASEP).

Another example of a process with these three properties is the 'random Tetris' process. In this process, random Tetris pieces are dropped at random locations. In fact, the conditions are all satisfied even if we use only horizontal domino pieces. See Figure 4.

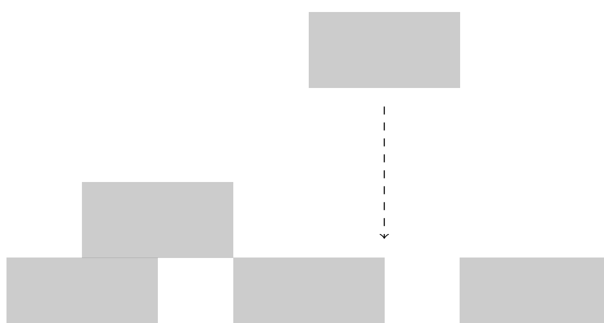

Figure 4: 'Tetris' process with horizontal dominos.

3 Limit shape of the corner growth process

Our next objective is to formally derive the parabolic limit shape of the corner growth process. The first step is to understand the stationary distributions of the process.

3.1 Stationary measures for (TAS)EP

Much of the following holds for exclusion processes in general. In this section, however, we restrict our attention to TASEP.

For $\rho \in [0, 1]$, let μ_ρ denote the product measure with density ρ. That is, every site is occupied w.p. ρ, and unoccupied w.p. $\bar{\rho} = 1 - \rho$. As we shall see, these are stationary measures for the exclusion process. Furthermore, they are the only 'interesting ones.' We state this as theorem.

Theorem 1. *μ_ρ, for $\rho \in [0, 1]$, are all the ergodic, stationary measures for the TASEP.*

We note that Theorem 1 holds for a fairly general class of exclusion processes; for instance on $\mathbb{Z}$ and in higher dimensions. In particular, the result holds for the following process: a collection of particles jump, at rate 1 and independently, to a new position according to a sample from a distribution μ on $\mathbb{Z}$, so long as this position is unoccupied. For instance, if $\text{supp}(\mu) = \{1\}$ we have TASEP; and if $\text{supp}(\mu) = \{-1, 1\}$ we have SEP if $\mu(\pm 1) = 1/2$ and ASEP otherwise. However, observe that in this more general process, the particles are not necessarily restricted to nearest neighbour transitions.

In regards to the statement of Theorem 1, since every stationary distribution has an ergodic decomposition (i.e. has a representation as the linear combination of ergodic, stationary measures), we are mainly concerned with the stationary measures which are ergodic.

Proof. *(μ_ρ is stationary)* We first consider the dynamics of a finite system $\mathbb{Z}_M$, the circle of length M. We define the T(ASEP) dynamics here in the obvious way; and obtain a Markov chain with state-space of order 2^M. Note that, since the number of particle is preserved with every step of the chain, the process is reducible. Specifically, the Markov chain consists of a collection of irreducible classes with a fixed number of particles.

We now show that for any k, the uniform measure on the $\binom{M}{k}$ configurations is stationary.

For a configuration of particles on $\mathbb{Z}_M$, define a bundle of particles to be string of occupied sites with unoccupied sites on either side. Suppose that $\mathcal{C}$ is a configuration on $\mathbb{Z}_M$ with k particles and b bundles. It is clear that after one move $\mathcal{C}$ is now one of b configurations; and conversely, there are exactly b configurations of k particles on $\mathbb{Z}_M$ from which after one move the configuration moves into $\mathcal{C}$. Hence, the uniform measure is stationary, since the rates at which the process moves out of or into any particular configuration are equal.

We note that by considering arithmetic progressions with comment increment (as determined by μ) instead of bundles, we can show that the above claim also holds for the more general EP described after the statement of Theorem 1.

We see now that the product measures are stationary. This is because drawing from the product measure is equivalent to sampling a binomial number of the M particles. Given this procedure selects k particles, they are uniformly likely to be in any of the possible configurations with k particles. Hence, the product measures, for any parameter ρ, are stationary.

To obtain the full result on $\mathbb{Z}$, we take a limit in such a way so that up to time t, we consider the process on a circle $\mathbb{Z}_n$ of sufficient length so that there has been no activity on $\mathbb{Z}$ that has not taken place on $\mathbb{Z}_n$. $\square$

3.2 The standard coupling

Before we go on to show that the μ_ρ are the only stationary distributions, we introduce a very useful coupling.

Consider two corner growth process with initial conditions g_0 and h_0, so that h dominates g, i.e. $g_0 \leq h_0$. For example, see Figure 5.

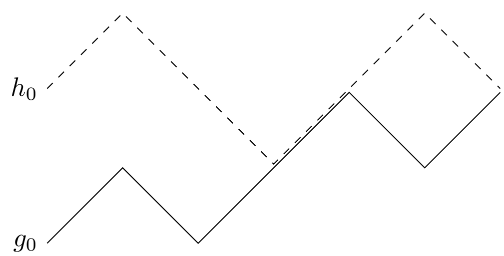

Figure 5: Two initial conditions. g is dominated by h: $g_0 \leq h_0$.

There exists a coupling such that for all times t of the processes, $g_t \leq h_t$. This coupling is often referred to as the *standard (or basic) coupling.* The coupling works as follows: we add squares at the same locations for both g and h so longs as it is possible to do so. By induction we see that for all t, $g_t \leq h_t$ since at any given time, s say, if it is possible to add a square to a location in g_s but not h_s, then $g_s < h_s$ in this location. Similarly, if it is possible to add a square to a location in h_s but not g_s, then adding a square here will clearly maintain its dominance.

We note that there is slight ambiguity in the definition of this procedure: if a square is added to a location where both g and h can receive a square [where g but not h can receive a square], and where there is a strict positive difference between g and h at this location, we can either add a square to both g and h or just to h [to g but not h or neither g nor h]. Under either interpretation the dominance of h is preserved. For our purposes, in either of the two situations described above, we will always make the former choice (i.e. we will add a square wherever possible); but note that the latter interpretation is useful for other situations.

3.3 Lower bound on the limit shape of the corner growth process

The standard coupling can help us find information about the limit shape of the corner growth process.

Consider two initial conditions g_0 and h_0, where $g_0 \leq h_0$, g_0 is determined by a i.i.d. random walk passing through the origin, and h_0 is the corner at the origin. See Figure 6. Under the square adding procedure described in the previous subsection, we have $g_t \leq h_t$ for all t. As we shall see, the growth of the stationary growth process, g_t, provides us with a lower bound on the limit shape of the corner growth process.

Writing the above formally, fix $\rho \in [0, 1]$ and let g_0 be determined by a doubly infinite RW with steps $+1$ w.p. ρ and -1 w.p. $\bar{\rho}$. We note that g_0 corresponds to a sample from μ_ρ. Also, we have $h_0(x) = |x|$, since h is initially the corner. See Figure 6.

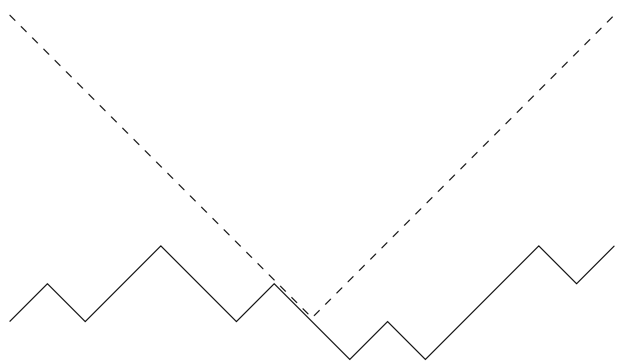

Figure 6: SRW and corner initial conditions.

We now analyse the evolution of g_t. Since the particle system is stationary, the configuration at time t will be another sample from the product measure; so we will again see a random walk, however it will be shifted up and no longer pass through the origin. See Figure 7. Also, since the measure is ergodic and we add squares at rate 1, we have by the LLN that

$$g_0(x) \approx (1 - 2\rho)x.$$

and

$$g_t(x) \approx (1 - 2\rho)x + \rho(1 - \rho)t + (\text{smaller order terms}). \tag{1}$$

Note that to increase the height function we need an 'up-step' followed by a 'down-step.' This occurs w.p. $\rho(1 - \rho)$. Once the new square is added, the height function increases by 2.

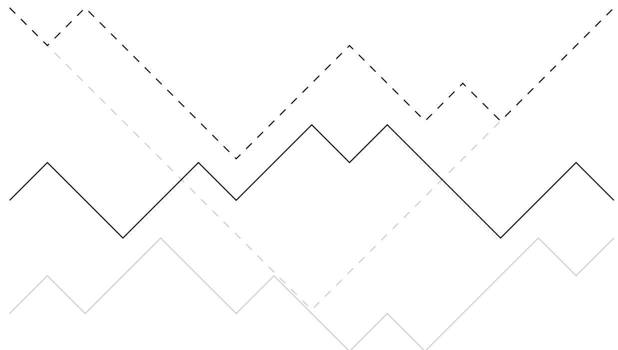

Figure 7: TASEP at time t started from SRW and corner initial conditions.

We are set to derive a lower bound on the limit shape. We consider x of the order t in what follows, and so put $x = ty$. By (1), we have

$$h_t(ty) \geq (1-2\rho)ty + 2\rho(1-\rho)t + \text{(smaller order terms)}. \tag{2}$$

We note that as $t \to \infty$, $h_t(ty)/t$ converges to the parabolic limit shape $\varphi(y)$. Hence

$$\varphi(y) \approx h_t(ty)/t \geq (1-2\rho)y + 2\rho(1-\rho) + \text{(terms vanishing in the limit)} \tag{3}$$

The arguments above are valid for any ρ. Taking derivatives, we find the optimal choice

$$\rho^*(y) = \begin{cases} (1-y)/2 & y \in [-1,1] \\ 1 & y < -1 \\ 0 & y > 1. \end{cases}$$

Then, substituting into (3), we find

$$\varphi(y) = \begin{cases} (1+y^2)/2 & y \in [-1,1] \\ -y & y < -1 \\ y & y > 1. \end{cases}$$

This (truncated) parabolic shape is the envelope of all lower bounds corresponding to choices of $\rho(y)$.

We note here that the above argument does not negate the possibility that the limit shape is higher than the parabola we have found. However, as we shall see in the next lecture, it turns out nothing is lost in our argument.

To summarise this subsection, we have seen that stationary distributions can be useful for analysing of the limit shape of a growth process.

3.4 Second-class particles

Towards showing that the product measures are the only 'interesting' stationary measures, we introduce the concept of second-class particles.

We begin by defining the basic coupling for particle systems. Imagine two rows of particles, drawn one row above the other. In other words, at any given time, we depict the two systems as an array with two rows and infinitely many columns. Some entries of the array contain particles. The empty entries are available for a particle to move into.

The idea of the basic coupling in this setup is that, with rate 1, any particles in a column attempt to move into the column to their right. The important difference here is that the random times are associated to the locations of the array, and not the particles themselves. So, for example, if there are no particles in a column when an attempted jump time occurs, then the configuration does not change. (This is highly reminiscent of the graphical representation for the TASEP.)

Also note that this coupling can be generalised, in the natural way, to any number of systems. In many cases it is very useful to use this generalisation of the basic coupling. For instance, in the previous section, to derive the parabolic lower bound for the limit shape of the corner growth process, we coupled uncountably many processes (one for each slope) to get the envelope.

We now consider the possible local situations in the coupling of two particle processes. We begin by assuming the top row has strictly more particles than the bottom row. Then any column of the array is one of

$$\hat{\infty} = \begin{array}{c} \cdot \\ \cdot \end{array}$$

$$\hat{2} = \begin{array}{c} \circ \\ \cdot \end{array}$$

$$\hat{1} = \begin{array}{c} \circ \\ \circ \end{array}$$

where a '$\circ$' denotes a particle, and a '$\cdot$' the absence thereof. By the TASEP rules, we see that looking along any two adjacent columns in the configuration:

$$\hat{1}/\hat{\infty} \longrightarrow \hat{\infty}/\hat{1}, \quad \text{with rate 1}$$

$$\hat{2}/\hat{\infty} \longrightarrow \hat{\infty}/\hat{2}, \quad \text{with rate 1}$$

$$\hat{1}/\hat{2} \longrightarrow \hat{2}/\hat{1}, \quad \text{with rate 1.}$$

In summary, if we denote the initial condition of the two systems by $\eta_0 \in \{\hat{1}, \hat{2}, \hat{\infty}\}^{\mathbb{Z}}$ and the configuration at time t by $\eta_t \in \{\hat{1}, \hat{2}, \hat{\infty}\}^{\mathbb{Z}}$, then we have that for any i, if $\eta(i) < \eta(i+1)$ (in the natural ordering $\hat{1} < \hat{2} < \hat{\infty}$) then we swap them with rate 1.

In the above dynamics, we say that the element $\hat{2}$ is a second-class particle. This is because it is someplace between a particle ($\hat{1}$) and a hole ($\hat{\infty}$). It acts as a particle with respect to the holes, and as a hole with respect to the particles.

More generally, for two systems of particles, depicted in array form, we make additional definitions for the state of column:

$$\uparrow = \begin{array}{c} \circ \\ \cdot \end{array}$$

$$\downarrow = \begin{matrix} \cdot \\ \circ \end{matrix}$$

The dynamics of this process is similar to that on $\{\hat{1}, \hat{2}, \hat{\infty}\}^{\mathbb{Z}}$, in that $\uparrow$ and $\downarrow$ interact with $\hat{1}$ and $\hat{\infty}$ in the same way as did $\hat{2}$. Additionally, we have the interaction between $\uparrow$ and $\downarrow$:

$$\uparrow / \downarrow \longrightarrow \hat{\infty}/\hat{1}, \quad \text{with rate } 1$$

$$\downarrow / \uparrow \longrightarrow \hat{\infty}/\hat{1}, \quad \text{with rate } 1.$$

So this process has two second-class particles. Observe that when these second-class particles interact they annihilate each other, in the sense that when they are adjacent they attempt, at rate 1, to change into a configuration without a $\uparrow$ or a $\downarrow$.

We are prepared to prove the remainder of Theorem 1.

Proof. (The product measures are the only ergodic, stationary measures.) Suppose ν is an ergodic, stationary measure for TASEP (but as discussed above, what follows holds in greater generality).

Put $\rho = \nu(\text{particle at } 0)$. We apply the basic coupling to the pair of initial conditions $X_0 =_d \nu$ and $Y_0 =_d \mu_\rho$. Looking along either row we just see the regular TASEP dynamics. So, by stationary, we have $X_t =_d \nu$ and $Y_t =_d \mu_\rho$. Initially X and Y are independent, but of course, for any $t > 0$, X_t and Y_t are no longer independent. Observe that, by the choice of ρ, the density of particles in the two rows is equal. Moreover, the density of $\uparrow$'s and $\downarrow$'s in the rows is equal. By ergodicity, this density converges to 0 as $t \to \infty$. This is because with positive probability $\uparrow$'s and $\downarrow$'s with eventually interact (whereupon they are annihilated), and since it is not possible that for some large t, there are very few $\uparrow$'s and many $\downarrow$'s (or vice versa) at time t. Thus

$$\mathbf{P}(X_t(n) = Y_t(n)) \to 1, \quad t \to \infty$$

in any finite window $n \in [-N, N]$. It follows that $\nu = \mu_\rho$. $\square$