PERMUTATIONS GENERATED BY A DEPTH 2 AND INFINITE STACK IN SERIES ARE ALGEBRAIC

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Abstract. We prove that the class of permutations generated by passing an ordered sequence $12\dots n$ through a stack of depth 2 and an infinite stack in series is in bijection with an unambiguous context-free language, where a permutation of length n is encoded by a string of length $3n$. It follows that the sequence counting the number of permutations of each length has an algebraic generating function. We use the explicit context-free language to compute the generating function:

$$\sum_{n\geq 0} c_n t^n = \frac{(1+q)\left(1+5q-q^2-q^3-(1-q)\sqrt{(1-q^2)(1-4q-q^2)}\right)}{8q}$$

where c_n is the number of permutations of length n that can be generated, and $q \equiv q(t) = \frac{1-2t-\sqrt{1-4t}}{2t}$ is a simple variant of the Catalan generating function. This in turn implies that $c_n^{1/n} \to 2+2\sqrt{5}$.

1. Introduction

Let $p = p_1p_2\dots p_n$ and $q = q_1q_2\dots q_k$ be permutations of length $n \geq k$. We say p *avoids* q if there are no k indices $i_1 < \cdots < i_k$ so that for all s, t,

$$p_{i_s} < p_{i_t} \qquad \text{if and only if} \qquad q_s < q_t.$$

For example, 25413 avoids 123 since it has no increasing subsequence of length 3.

Interest in sets of permutations that avoid a small set of "patterns" arose naturally in the study of stack-sorting (or equivalently stack-generating) algorithms. Knuth showed that a permutation p can be generated by passing the ordered sequence $12\dots|p|$ through an infinite stack if and only if p avoids 312, and that permutations of length n avoiding 312 are counted by the Catalan numbers [15].

Despite considerable effort, exact enumerative results about permutations passed through more than a single stack are still rare. The number of permutations sortable by 2 stacks in parallel was only recently solved by Albert and Bousquet-Mélou [3]. The full problem of two stacks in series remains unsolved, but a simplified version was studied by West [17] and solved by Zeilberger [19]. The problem we consider here is an attempt to "sneak up" on the full 2 stacks in series problem — we focus on the class of permutations generated by a stack of depth two and an infinite stack in series. We prove that the class is enumerated by a sequence that has an algebraic generating function.

If $\mathfrak{q}$ is a list of permutations, let $Av_n(\mathfrak{q})$ be the set of permutations of length n that avoid q for each $q \in \mathfrak{q}$. We call $Av(\mathfrak{q}) = \bigcup_{n=0}^{\infty} Av_n(\mathfrak{q})$ a *pattern-avoidance class*. A *basis* for a pattern avoidance class $Av(\mathfrak{q})$ is a set $\mathfrak{p}$ of pairwise avoiding permutations so that $Av(\mathfrak{p}) = Av(\mathfrak{q})$. A class is *finitely based* if it is equal to $Av(\mathfrak{p})$ for $\mathfrak{p}$ finite. The first author

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proved that the class of permutations generated by a stack of depth two and an infinite stack in series has a finite basis consisting of 20 permutations [11].

The list of pattern-avoidance classes for which a generating function for the sequence counting $Av_n(\mathfrak{q})$ has been computed, or shown to be rational, algebraic or non-algebraic, is limited. Classes avoiding a single pattern of length 3 are enumerated by the Catalan numbers [15, 16] and so have an algebraic generating function. For length four, $Av(\{1342\})$ has an algebraic generating function [7], $Av(\{1234\})$ has a generating function that is D-finite but not algebraic [13], and a closed form generating function for $Av(\{1324\})$ has not be found [2, 9]. It is known that for any pattern p of length four, $Av(\{p\})$ is in bijection with one of these three classes. For single patterns of length greater than four, and classes avoiding two or more patterns, various isolated results are known [4, 18].

Several authors have considered the language-theoretic complexity of pattern avoidance classes — see for example [1, 5, 6, 10]. Atkinson, Livesey, and Tulley [6] showed that the set of permutations generated by passing an ordered sequence through a finite *token-passing network* is in bijection with a regular language. Initially we applied this technique to the finite network consisting of a stack of depth 2 followed by a stack on depth k in series, constructing a sequence of languages and corresponding rational generating functions for small values of k. As k increased, the rational generating functions appeared to converge to the algebraic function given in Theorem 4.1 below. However, his method does not constitute a proof. To prove the result we instead follow another path — we establish a bijection between permutations generated and an unambiguous context-free language. The generating function is then guarenteed to be algebraic by a well known theorem of Chomsky and Schützenberger.

Of course there are a great many other variants of stack sorting that might be considered and we refer the reader to [8] for a survey.

2. Establishing a bijection

Let $\mathcal{P}$ be the set of permutations that can be generated by a stack of depth 2 and infinite stack in series, and fix ρ, λ, μ as the stack moves indicated in Figure 1.

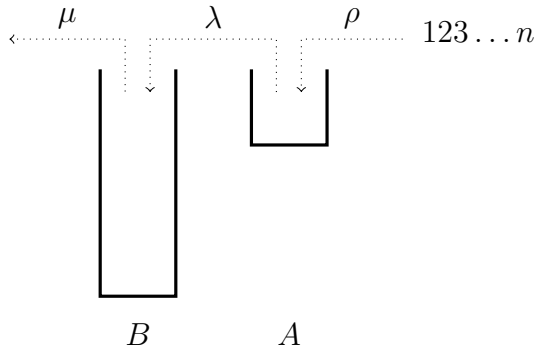

Figure 1. Token passing moves ρ, λ and μ for two stacks in series.

Definition 2.1 ($D_{a,b}(u)$)**.** *If u is a word over an alphabet that includes the letters a and b, define $D_{a,b}(u)$ to be the number of a letters minus the number of b letters contained in u.*

Definition 2.2 ($\mathcal{L}_{k,\infty}$)**.** *Let $k \in \mathbb{N}$. The language $\mathcal{L}_{k,\infty}$ is the set of words $w \in \{\rho, \lambda, \mu\}^*$ satisfying*

(1) $D_{\rho,\lambda}(u) \in [0, k]$ and $D_{\lambda,\mu}(u) \in [0, \infty)$ for all prefixes, u, of w,

(2) $D_{\rho,\lambda}(w) = D_{\lambda,\mu}(w) = 0$.

Lemma 2.3. *A word $w \in \{\rho, \lambda, \mu\}^*$ encodes a permutation in $\mathcal{P}$ if and only if $w \in \mathcal{L}_{2,\infty}$. Moreover, a word of length $3n$ in $\mathcal{L}_{2,\infty}$ encodes a permutation of length n.*

Proof. The first claim is clear from the definition. If $w \in \mathcal{L}_{2,\infty}$ has n ρ letters, then $D_{\rho,\lambda}(w) = 0$ implies w has n λ letters, and $D_{\lambda,\mu}(w) = 0$ then implies w has n μ letters, so the length of w is $3n$. □

The language $\mathcal{L}_{2,\infty}$ consists of all possible ways to pass tokens through the system of stacks as in Figure 1. We wish to find a sublanguage that is in bijection with $\mathcal{P}$. From the set of all words in $\mathcal{L}_{2,\infty}$ that generate the same permutation, we will try to choose the string that outputs tokens as soon as possible, that is, has more μ letters closer to the front. The next definition will help to formalise this.

Definition 2.4 (μ-ordering)**.** *Define an ordering, $\prec_\mu$, on words in $\{\rho, \lambda, \mu\}^*$ as follows. Let $\theta : \{\rho, \lambda, \mu\}^* \to \{\nu, \mu\}^*$ be a monoid homomorphism defined by $\theta(\mu) = \mu$ and $\theta(\rho) = \theta(\lambda) = \nu$. If $u \neq v$ as strings then $u \prec_\mu v$ if $|u| = |v|$ and $\theta(u)$ precedes $\theta(v)$ in lexographic ordering on $\{\mu, \nu\}^*$ where $\mu < \nu$.*

For example, if $u = \rho\lambda\mu\rho\lambda\mu$ and $v = \rho\lambda\rho\mu\lambda\mu$ then $u \prec_\mu v$. Note that both words generate the permutation 12, and u is obtained from v by replacing the subword $\rho\mu$ by $\mu\rho$, which has no affect on the permutation being produced. More generally we have the following.

Lemma 2.5. *Let $w \in \mathcal{L}_{2,\infty}$.*

(1) If $w = w_0\rho\mu w_1$ then $w' = w_0\mu\rho w_1$ generates the same permutation as w, and $w' \prec_\mu w$.
(2) If $w = w_0\rho\lambda w_1\lambda\mu w_2$ with $D_{\rho,\lambda}(w_0) = 1$ and $w_1 \in \mathcal{L}_{1,\infty}$, then $w' = w_0\lambda\rho w_1\mu\lambda w_2$ generates the same permutation as w, and $w' \prec_\mu w$.
(3) If $w = w_0\lambda\rho w_1\lambda\mu w_2$ with $D_{\rho,\lambda}(w_0) = 1$ and $w_1 \in \mathcal{L}_{1,\infty}$, then $w' = w_0\rho\lambda w_1\mu\lambda w_2$ generates the same permutation as w, and $w' \prec_\mu w$.

Proof. In each case it is clear that $w' \prec_\mu w$. We must show that in each case the two strings generate the same permutation. For case (1) this is clear since ρ and μ do not interact.

For case (2), since $D_{\rho,\lambda}(w_0) = 1$, there must be one token (say a) left in the first stack after reading w_0, and since the next letter to be read is ρ, there must be one token (say b) ready to enter the first stack. See Figure 2.

After reading $\rho\lambda$, b moves to the top of stack B and a stays in stack A. Reading w_1 leaves a and b in place and outputs some permutation of input tokens after b. Finally $\lambda\mu$ outputs a, leaving b on the top of stack B and stack A empty.

Starting from the initial configuration in Figure 2, the prefix $w_0\lambda\rho w_1\mu\lambda$ of w' moves a to the top of stack B and places b in stack A. The permutation generated by w_1 is then passed across as before, then a is output, and finally b is moved to stack B, leaving the stacks in the same configuration and the prefix of w.

A similar argument applies for Case (3) and is left to the reader. □

Definition 2.6 ($\mathcal{L}$)**.** *The language $\mathcal{L}$ is the set of words $w \in \mathcal{L}_{2,\infty}$ that do not*

(1) contain $\rho\mu$,
(2) have a prefix $w_0\rho\lambda w_1\lambda\mu$ with $w_1 \in \mathcal{L}_{1,\infty}$ and $D_{\rho,\lambda}(w_0) = 1$,
(3) have a prefix $w_0\lambda\rho w_1\lambda\mu$ with $w_1 \in \mathcal{L}_{1,\infty}$ and $D_{\rho,\lambda}(w_0) = 1$.

Lemma 2.7. *Let $w \in \mathcal{L}_{2,\infty}$. If either*

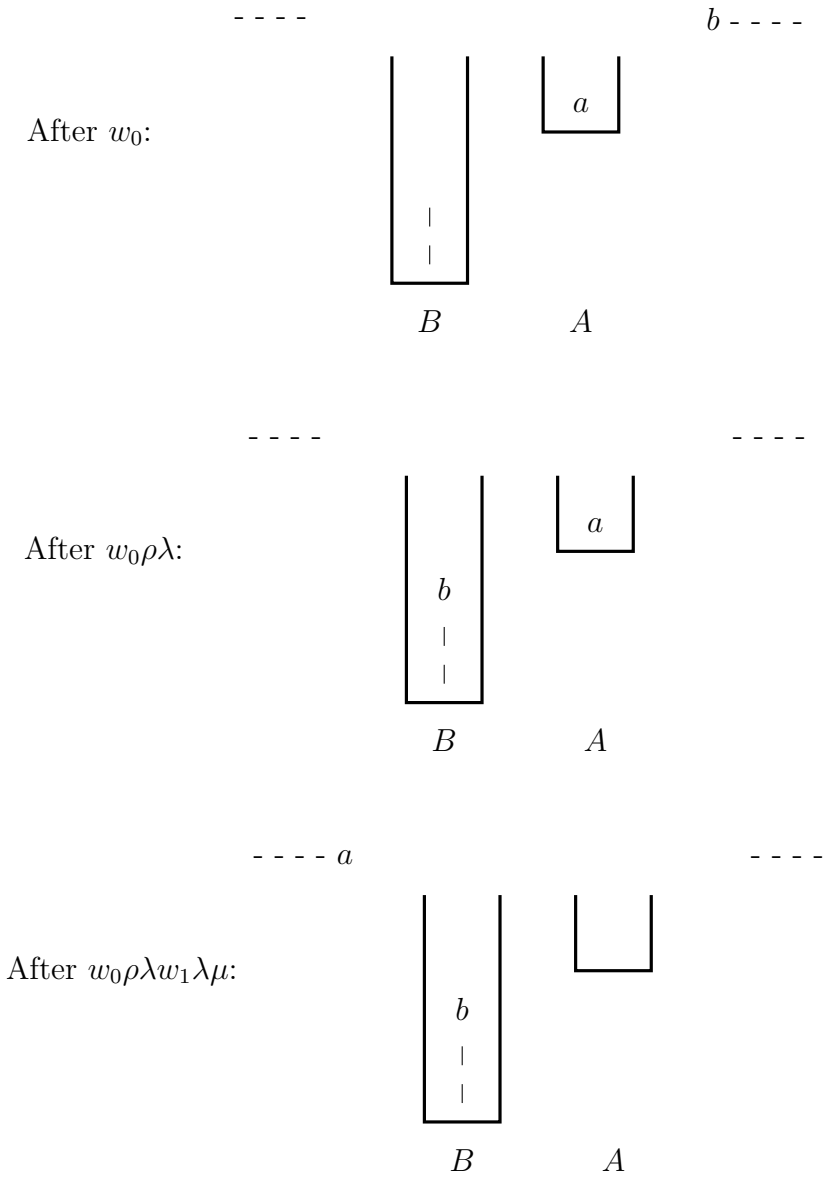

FIGURE 2. Stack configurations in the proof of Lemma 2.5.

(1) $w = w_0\rho\lambda w_1\lambda w_2\mu w_3$ *with* $D_{\rho,\lambda}(w_0) = 1, w_1 \in \mathcal{L}_{1,\infty}$, *and* $w_2 \in \mathcal{L}_{2,\infty}$ *generates a permutation that avoids* 312, *or*

(2) $w = w_0\lambda\rho w_1\lambda w_2\mu w_3$ *with* $D_{\rho,\lambda}(w_0) = 1, w_1 \in \mathcal{L}_{1,\infty}$, *and* $w_2 \in \mathcal{L}_{2,\infty}$ *generates a permutation that avoids* 312,

then $w \notin \mathcal{L}$.

Proof. Suppose for contradiction that $w \in \mathcal{L}$, $w = w_0vw_1\lambda w_2\mu w_3$ with $v \in \{\rho\lambda, \lambda\rho\}, D_{\rho,\lambda}(w_0) = 1, w_1 \in \mathcal{L}_{1,\infty}, w_2$ generates a permutation that avoids 312, and moreover that w_0 is the longest prefix of w with this property. That is, if $w = u_0vu_1\lambda u_2\mu w_3$ with $v \in \{\rho\lambda, \lambda\rho\}, D_{\rho,\lambda}(u_0) = 1, u_1 \in \mathcal{L}_{1,\infty}$ and u_2 generates a permutation that avoids 312, then $|u_0| \leq |w_0|$.

Since $D_{\rho,\lambda}(w_0vw_1) = 1$ and λ moves a token from stack A to stack B, after reading $w_0vw_1\lambda$ we have no tokens in stack A, and some token, say a, in stack B. See Figure 3.

Since $w \in \mathcal{L}$, w_2 cannot be empty, and since w_2 is a subword of $w \in \mathcal{L}$ we have $w_2 \in \mathcal{L}$. So w_2 moves some sequence of tokens completely through the stacks, leaving a in place.

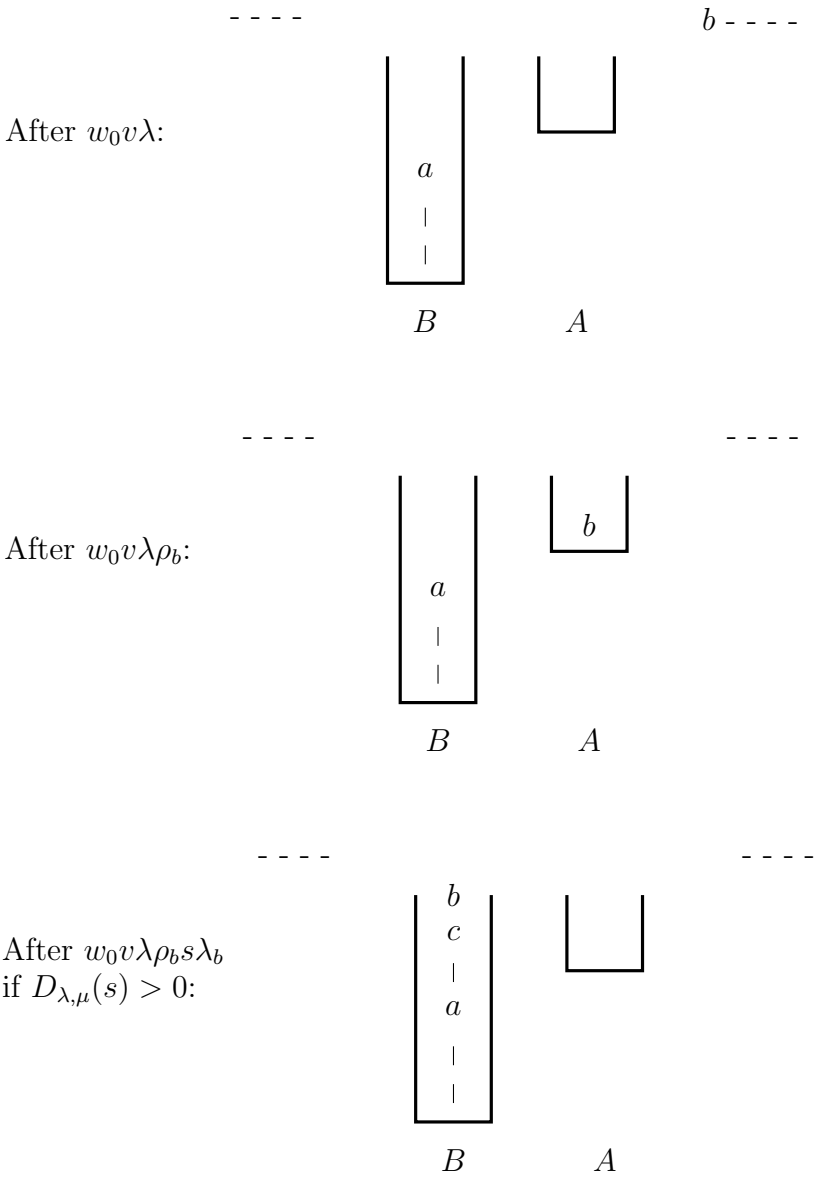

FIGURE 3. Stack configurations in the proof of Lemma 2.7.

The first letter of w_2 must be ρ, which moves some token, say b, onto stack A. Let ρ_b, λ_b, μ_b be the letters in w_2 that correspond to moving b through the stacks. Then w_2 has prefix $\rho_b s \lambda_b t \mu_b$ where s, t are subwords.

Since stack A contains b while s is read, if ρ occurs in s it must be immediately followed by λ, so $D_{\rho,\lambda}(u) \in [0, 1]$ for all prefixes u of s, and $D_{\rho,\lambda}(s) = 0$. Further, if $D_{\lambda,\mu}(u) < 0$ for any prefix u of s, then a would be output. Either $D_{\lambda,\mu}(s) = 0$ (and $s \in \mathcal{L}$) or $D_{\lambda,\mu}(s) > 0$.

If $s \in \mathcal{L}_{1,\infty}$ then $t \in \mathcal{L}_{2,\infty}$ and generates a permutation avoiding 312 since it is a subword of w_2. In this case w has prefix $w = w_0 v w_1 \lambda \rho_b s \lambda_b t \mu_b$ with $D_{\lambda,\mu}(w_0 v w_1) = 1$ and t generating a permutation avoiding 312, which contradicts the choice of w_0 as the longest such prefix.

Therefore we must have $D_{\lambda,\mu}(s) > 0$. In this case, after reading s at least one token, say c, remains on top of a in stack B when b is moved into it. After reading λ_b, the stack configuration is as in the third diagram shown in Figure 3.

Note that $a < b < c$ since they are input in this order. If $t \neq \varepsilon$ then it must contain at least one μ (it cannot leave a token covering b, and cannot just be ρ or $\rho\rho$) so it moves a token $d > c$ to the output. This means w_2 generates the subpermutation dbc which is order equivalent to 312, contradicting our assumption. Thus $t = \varepsilon$ and w_2 has prefix $\rho_b s \lambda_b \mu_b$, with $s \in \{\rho\lambda, \mu\}^*$. Either s ends with $\rho\lambda$, or $s = u\rho\lambda s'$ where $D_{\lambda,\mu}(u) = D_{\lambda,\mu}(s)$ since $D_{\lambda,\mu}$ starts at zero and increases to this value. Thus $s' \in \mathcal{L}$, and $w = w_0 v w_1 \lambda \rho_b u \rho \lambda s' \lambda_b \mu_b$ with $D_{\rho,\lambda}(w_0 v w_1 \lambda \rho_b u) = 1$, which contradicts $w \in \mathcal{L}$. □

Theorem 2.8. *There is a bijection between permutations in $\mathcal{P}$ of length n and words in $\mathcal{L}$ of length $3n$.*

Proof. Consider the map that sends a word of length $3n$ in $\mathcal{L} \subseteq \mathcal{L}_{2,\infty}$ to the permutation of length n it generates. If $\sigma \in \mathcal{P}$ then there is some word $w \in \mathcal{L}_{2,\infty}$ that generates it by Lemma 2.3. If $w \notin \mathcal{L}$, then w must either contain $\rho\mu$, or have prefix $w_0\rho\lambda w_1\lambda\mu$ or $w_0\lambda\rho w_1\lambda\mu$ with $D_{\rho,\lambda}(w_0) = 1$ and $w_1 \in \mathcal{L}_{1,\infty}$. We rewrite w as follows.

While w contains $\rho\mu$ or has prefix $w_0\rho\lambda w_1\lambda\mu$ or $w_0\lambda\rho w_1\lambda\mu$:

1. Replace $\rho\mu$ with $\mu\rho$
2. Replace $w_0\rho\lambda w_1\lambda\mu$ with $w_0\lambda\rho w_1\mu\lambda$
3. Replace $w_0\lambda\rho w_1\lambda\mu$ with $w_0\rho\lambda w_1\mu\lambda$

Each iteration replaces the current word by a word which generates the same permutation and is shorter in the μ-ordering by Lemma 2.5, so the procedure must terminate (there are finitely many words less than w in the μ-ordering). It follows that the map is surjective. We complete the proof by showing it is injective.

Suppose we have two words $u, v \in \mathcal{L}$ that generate the same permutation, and that $u \neq v$ as strings. Write

$$u = u_1 u_2 \dots u_n \text{ and } v = v_1 v_2 \dots v_n$$

where $u_i, v_i \in \{\rho, \lambda, \mu\}$.

Since $u, v \in \mathcal{L}$ we have $u_1 = v_1 = \rho$. Let $k \in [2, n]$ be such that $u_i = v_i$ for $i < k$ and $u_k \neq v_k$. Let $z = u_1 \dots u_{k-1} = v_1 \dots v_{k-1}$, so

$$u = z u_k \dots u_n \text{ and } v = z v_k \dots v_n.$$

First consider the case that one of u_k, v_k is μ. Without loss of generality assume $u = z\mu u_{k+1} \dots u_n$. Then z must leave some token, say a, at the top of stack B, and $u_k = \mu$ outputs this token.

If $v_k = \lambda$, then a will be covered and v will not be able to generate the same permutation. So we must have $v_k = \rho$. Then $v_{k+1} \neq \mu$. If $v_{k+1} = \lambda$ then a is covered. So $v_{k+1} = \rho$. Then $v_{k+2} \neq \mu$, if $v_{k+2} = \lambda$ then a is covered, and $v + k + 2 \neq \rho$ since stack A contains two tokens. So we have a contradiction, and it follows that neither u_k, v_k can be μ.

Without loss of generality assume $u_k = \rho$ and $v_k = \lambda$. Then z must leave at least one token in stack A to be followed by λ, and at most one token to be followed by ρ. Let a be the token in A, and b the token moved from the input by $u_k = \rho$. See Figure 4. Note that we have $D_{\rho,\lambda}(z) = 1$.

In u, $z\rho$ must be followed by λ since stack A is full after the ρ and ρ cannot be followed by a μ. So u has prefix $z\rho\lambda$ and we have the configuration shown in the second diagram in Figure 4.

In v, $z\lambda$ can be followed by either μ or ρ but not λ since stack A is empty after $v_k = \lambda$. Suppose $v_{k+1} = \mu$. Then after reading $z\lambda\mu$ we have the configuration shown in the third diagram in Figure 4. Since u and v are assumed to produce the same permutation, the

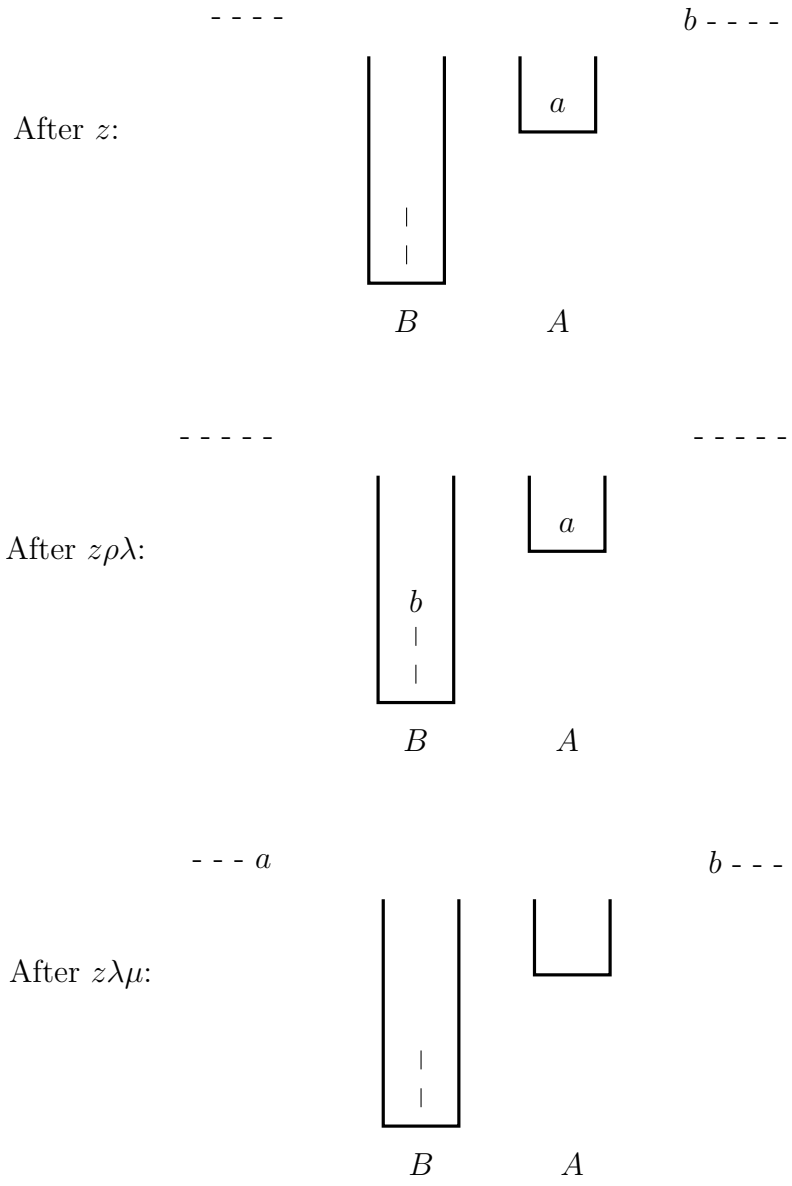

FIGURE 4. Stack configurations in Theorem 2.8 where $u_k = \rho$ and $v_k = \lambda$.

next μ letter appearing in u after the prefix $z\rho\lambda$ must move a to the output. Let λ_a, μ_a be the letters in u that move the token a. Then $u = z\rho\lambda u_1\lambda_a u_2\mu$ where $u_1, u_2 \in \{\rho, \lambda\}^*$. The subword u_2 cannot move tokens to cover a in stack B, so cannot contain any λ letters, and cannot contain any ρ letters since it is followed by μ, so it must be empty. The subword u_1 must be of the form $(\rho\lambda)^i$ for $i \geq 0$, since it cannot move a. Then $u = z(\rho\lambda)^i\rho\lambda\lambda_a\mu_a$ with $D_{\rho,\lambda}(z(\rho\lambda)^i) = 1$, so $u \notin \mathcal{L}$.

It follows that $v_{k+1} = \rho$, so we have

$$u = z\rho\lambda \dots u_n, v = z\lambda\rho \dots v_n.$$

The two configurations of the stacks after reading the length $k+1$ prefixes of u and v respectively are shown in Figure 5.

We now consider two possibilities: either a precedes b in the permutation generated by u and v, or b precedes a.

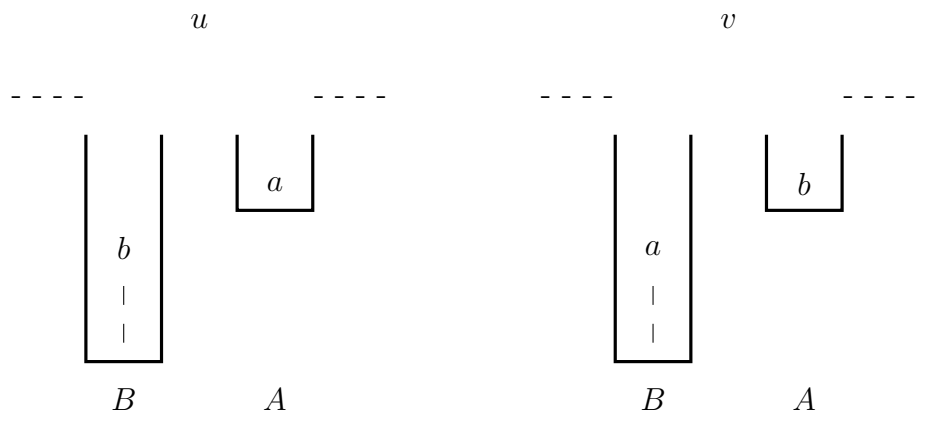

FIGURE 5. Stack configurations after $z\rho\lambda$ and $z\lambda\rho$ in Theorem 2.8.

Case 1: a **precedes** b

Mark the letters ρ, λ, μ in u and v that correspond to moving the token a, by appending the subscript a. So we have $u = z\rho\lambda w_1 \lambda_a w_2 \mu_a \dots u_n$ and $v = z\lambda_a \rho w \mu_a \dots v_n$ where $w, w_1, w_2 \in \{\rho, \lambda, \mu\}^*$.

First consider the word v. Since b must remain in stack A until a is output, w cannot end with ρ and w cannot leave any tokens covering a in stack B, we have $w \in \mathcal{L}_{1,\infty}$. If w is empty then v contains $\rho\mu_a$ which means $v \notin \mathcal{L}$. Thus w is nonempty, so moves some tokens, say $t_1, \dots, t_s$, from the input to the output.

Since u generates the same permutation as v, it must also move the tokens $t_1, \dots, t_s$ through the stacks and output them before a is output. The subword w_1 cannot leave any tokens covering a in stack A, so $w_1 \in \{\rho\lambda, \mu\}^*$.

If w_1 leaves some tokens in stack B, then these tokens must come after t_s in the input, and so w_1 must feed all the tokens $t_1, \dots, t_s$ into the input, so w_2 cannot output any tokens, so cannot contain μ, and cannot contain λ since a would be covered in stack B, and cannot be ρ or $\rho\rho$ since it is followed by μ_a, so w_2 is empty. If w_1 ends with $\rho\lambda$, then write $w_1 = p\rho\lambda$, and $z\rho\lambda w_1 \lambda_a \mu_a = z\rho\lambda p\rho\lambda\lambda_a\mu_a$ with $D_{\rho,\lambda}(z\rho\lambda p) = 1$, so $u \notin \mathcal{L}$. Otherwise w_1 ends in μ. Since w_1 has more ($\rho\lambda$) subwords than μ letters (it leaves tokens in stack B) then w_1 has some suffix $y \in \mathcal{L}_{1,\infty}$ and prefix p such that $z = p\rho\lambda y$. So we have $z\rho\lambda w_1 \lambda_a \mu_a = z\rho\lambda p\rho\lambda y \lambda_a \mu_a$ with $D_{\rho,\lambda}(z\rho\lambda p) = 1$ and $y \in \mathcal{L}_{1,\infty}$ so $u \notin \mathcal{L}$.

Thus w_1 does not leave any tokens in stack B, so $w_1 \in \mathcal{L}_{1,\infty}$. Let $t_1, \dots t_r$ with $r \le s$ be the tokens moved to the output by w_1. The situtation is shown in Figure 6.

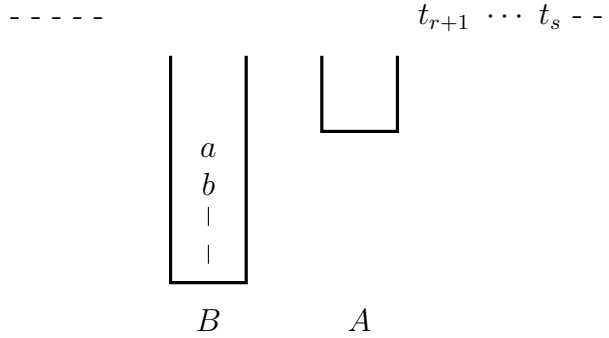

FIGURE 6. Stack configuration after $z\rho\lambda w_1 \lambda_a$ in Case 1 in Theorem 2.8.

If w_2 is empty then u has prefix $z\rho\lambda w_1\lambda_a\mu_a$ with $w_1 \in \mathcal{L}_{1,\infty}$ which is forbidden, so w_2 must move some tokens. The subword w_2 cannot leave any tokens in stack B. Either w_2 leaves some tokens in stack A, or not.

If w_2 leaves a token in stack A, this token cannot be one of $t_{r+1}, \ldots, t_s$ or else v would generate a different permutation to u. Therefore this token is moved into stack A after t_r by a letter ρ. This letter cannot be followed by μ, and since it remains in stack A it is not followed by λ. So this letter is either the last letter of w_2, or is followed by another ρ, which must also remain in stack A. Thus w_2 ends with ρ, but this is a contradiction since w_2 is followed by μ_a.

Thus w_2 does not leave any tokens in stacks A or B, so moves $t_{r+1}, \ldots, t_s$ from the input to the output, and $w_2 \in \mathcal{L}_{2,\infty}$. Note that w_1w_2 produces the same permutation of $t_1, \ldots, t_s$ as w does, and $w \in \mathcal{L}_{1,\infty}$ so generates a 312 avoiding permutation of $t_1, \ldots, t_s$. The subword w_1 permutes the first r tokens, and so w_2 must produce a permutation of $t_{r+1}, \ldots, t_s$ that avoids 312. In this case u has prefix $z\rho\lambda w_1\lambda_a w_2\mu_a$ where $D_{\rho,\lambda}(z\rho\lambda) = 1$, $w_1 \in \mathcal{L}_{1,\infty}$ and w_2 generates a 312-avoider, so by Lemma 2.7 u must also contain a prefix that is not allowed if $u \in \mathcal{L}$. This is a contradiction, so this case does not apply.

Case 2: b precedes a

We return to the situation shown in Figure 5 with $u = z\rho\lambda \ldots u_n$ and $v = z\lambda\rho \ldots v_n$. Mark the letters ρ, λ, μ in u and v that correspond to moving the token b, by appending a subscript. Then $u = z\rho_b\lambda_b w\mu_b \ldots u_n$ and $v = z\lambda\rho_b w_1\lambda_b w_2\mu_b \ldots v_n$ where $w, w_1, w_2 \in \{\rho, \lambda, \mu\}^*$.

First consider the word u. Since a must remain in stack A until b is output, w cannot end with ρ and w cannot leave any tokens covering b in stack B, we have $w \in \mathcal{L}_{1,\infty}$. If w is empty then u contains $\rho\mu_b$ which is forbidden, so w moves some tokens, say $t_1, \ldots, t_s$, from the input to the output.

Since v generates the same permutation as u, it must also move the tokens $t_1, \ldots, t_s$ through the stacks and output them before b is output. The subword w_1 cannot leave any tokens covering b in stack A, so $w_1 \in \{\rho\lambda, \mu\}^*$.

If w_1 leaves some tokens in stack B, then these tokens must appear after t_s in the input, and so w_1 must feed the tokens $t_1, \ldots, t_s$ into the input, so w_2 is empty (it cannot contain μ, λ and cannot end in ρ). If w_1 ends with $\rho\lambda$, then write $w_1 = p\rho\lambda$, and $z\lambda\rho_b w_1\lambda_b\mu_b = z\lambda\rho_b p\rho\lambda\lambda_b\mu_b$ with $D_{\rho,\lambda}(z\lambda\rho_b p) = 1$, so $v \notin \mathcal{L}$. Otherwise w_1 ends in μ. Since w_1 has more ($\rho\lambda$) subwords than μ letters (it leaves tokens in stack B) then w_1 has some suffix $y \in \mathcal{L}_{1,\infty}$ with $z = p\rho\lambda y$. So we have $z\lambda\rho_b w_1\lambda_b\mu_b = z\lambda\rho_b p\rho\lambda y\lambda_b\mu_b$ with $D_{\rho,\lambda}(z\lambda\rho_b p) = 1$ and $y \in \mathcal{L}_{1,\infty}$ so $v \notin \mathcal{L}$.

Thus w_1 does not leave any tokens in stack B, so $w_1 \in \mathcal{L}_{1,\infty}$. Let $t_1, \ldots t_r$ with $r \leq s$ be the tokens moved to the output by w_1. The situtation is shown in Figure 7.

If w_2 is empty then v has prefix $z\lambda\rho w_1\lambda_b\mu_b$ with $w_1 \in \mathcal{L}_{1,\infty}$ which is forbidden, so w_2 must move some tokens. The subword w_2 cannot leave any tokens in stack B. Either w_2 leaves some tokens in stack A, or not.

If w_2 leaves a token in stack A, this token cannot be one of $t_{r+1}, \ldots, t_s$ or else v would generate a different permutation to u. Therefore this token is moved into stack A after t_r by a letter ρ. This letter cannot be followed by μ, and since it remains in stack A it is not followed by λ. So this letter is either the last letter of w_2, or is followed by another ρ, which must also remain in stack A. Thus w_2 ends with ρ, but this is a contradiction since w_2 is followed by μ_b.

Thus w_2 does not leave any tokens in stacks A or B, so moves $t_{r+1}, \ldots, t_s$ from the input to the output, and $w_2 \in \mathcal{L}_{2,\infty}$. Note that w_1w_2 produces the same permutation of

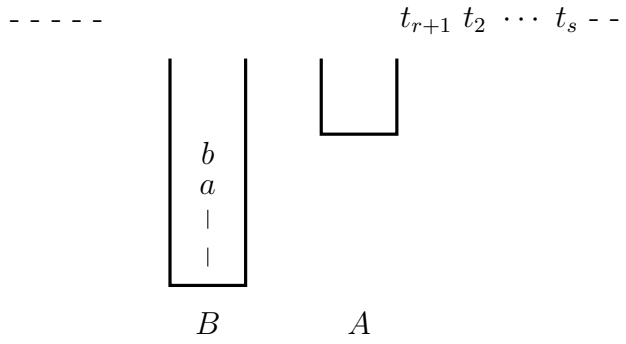

FIGURE 7. Stack configuration after $z\lambda\rho_b w_1 \lambda_b$ in Case 2 in Theorem 2.8.

$t_1, \ldots, t_s$ as w does, and $w \in \mathcal{L}_{1,\infty}$ so generates a 312 avoiding permutation of $t_1, \ldots, t_s$. The subword w_1 permutes the first r tokens, and so w_2 must produce a permutation of $t_{r+1}, \ldots, t_s$ that avoids 312. In this case v has prefix $z\lambda\rho_b w_1 \lambda_b w_2 \mu_b$ where $D_{\rho,\lambda}(z\lambda\rho_b) = 1$, $w_1 \in \mathcal{L}_{1,\infty}$ and w_2 generates a 312-avoider, so by Lemma 2.7 v must also contain a prefix that is not allowed if $v \in \mathcal{L}$. This is a contradiction, so we cannot have two such words u and v. □

2.1. **A related class of permutations.** A natural question to ask is whether switching the order of the stacks makes any difference to the problem. Let $\mathcal{Q}$ be the set of permutations that can be generated by passing an ordered sequence through an infinite stack followed by a depth 2 stack in series. Each word $w \in \mathcal{L}_{2,\infty}$ encodes a permutation in $\mathcal{Q}$ as follows: reading w from right to left, for each μ move a token from the input to the infinite stack, for each λ move a token from the infinite stack to the depth 2 stack, and for each ρ move a token from the depth 2 stack to the output. It follows that $\mathcal{P}$ and $\mathcal{Q}$ are in bijection.

3. Constructing a pushdown automaton

In this section we construct a deterministic pushdown automaton accepting on empty stack, which accepts the language

$$\mathcal{L}\$ = \{w\$ \mid w \in \mathcal{L}\}.$$

A *pushdown automaton accepting on empty stack* M is the following:

(1) Q a finite set of *states*,
(2) Σ a finite *input alphabet*,
(3) Γ a finite *stack alphabet*,
(4) $q_0 \in Q$ the *start state*,
(5) $0 \in \Gamma$ a special stack symbol,
(6) a map δ from $Q \times (\Sigma \cup \varepsilon) \times \Gamma$ to finite subsets of $Q \times (\Gamma^*)$,

which runs as follows. Before reading input, the stack contains a single 0. Input strings are accepted as soon as the stack becomes empty. A *configuration* of M is a pair (q, ω) where q is the current state and $\omega \in \Gamma^*$ is a string of stack symbols representing the contents of the stack (the first letter of ω is the top of the stack). The notation $\delta(q_i, a, k) = \{(q_{j_1}, \gamma_1), \ldots, (q_{j_s}, \gamma_s)\}$ means that if M has the configuration $(q_i, k\omega)$ and $a \in \Sigma \cup \{\varepsilon\}$ is the next input letter to be read, then M can move to the configuration $(q_{j_l}, \gamma_l \omega)$ for some $1 \leq l \leq s$, removing the token k from the top of the stack and replacing it by γ_l.

See [14] for more details.

A pushdown automaton is *deterministic* if for each state q and stack symbol i

(1) if $|\delta(q, \varepsilon, i)| = 1$ then $|\delta(q, a, i)| = 0$ for all $a \in \Sigma$,
(2) for each $a \in \Sigma \cup \{\varepsilon\}$ the set $\delta(q, a, i)$ has size at most one.

Note that a determistic pushdown automaton accepting on empty stack cannot accept the empty string (unless this is the only string it accepts) since there would have to be a transition $\delta(q_0, \varepsilon, 0)$ as well as a transition $\delta(q_0, a, 0)$ for some letter a.

Let M be the pushdown automaton shown in Figure 8, which accepts on empty stack.

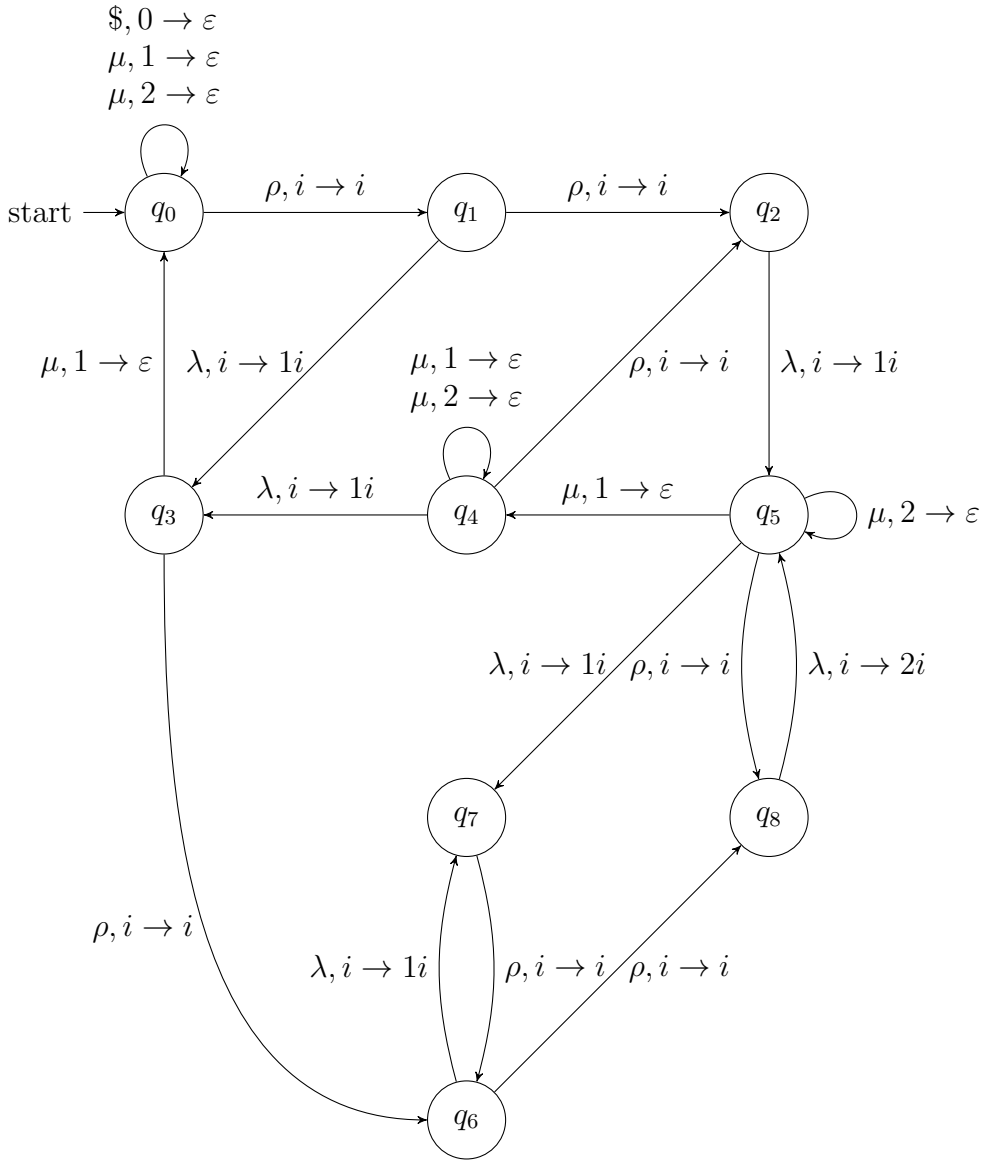

FIGURE 8. Pushdown automaton M accepting on empty stack, with start configuration $(q_0, 0)$. The symbol $i \in \{0, 1, 2\}$ represents a stack token that is kept in place by a transition.

The pushdown automaton uses its stack to keep track of $D_{\lambda,\mu}$ as it reads its input, and its states to keep track of $D_{\rho,\lambda}$. It uses the stack symbol 2 as a device to flag when the input has the potential to have a prefix of the form $w_0\rho\lambda$ or $w_0\lambda\rho$ with $D_{\rho,\lambda}(w_0) = 1$. Paths $\rho\mu$ are forbidden. We will prove that the language of this automaton is precisely the language $\mathcal{L}$.

Here is the formal description of M. Note that states q_3, q_6, q_7 are reached only when 1 is on top of the stack, and q_5, q_8 are reached when either 1 or 2 are on top of the stack, so we have omitted transitions from configurations that are not possible.

(1) states $Q = \{q_0, \ldots, q_8\}$,
(2) input alphabet $\Sigma = \{\rho, \lambda, \mu, \$\}$,
(3) stack alphabet $\Gamma = \{0, 1, 2\}$,
(4) start state q_0,
(5) transition function δ defined as follows.

$$
\begin{array}{lll}
\delta(q_0, \$, 0) = (q_0, \varepsilon) & \delta(q_0, \rho, 0) = (q_1, 0) & \delta(q_1, \lambda, 0) = (q_3, 10) \\
\delta(q_0, \mu, 1) = (q_0, \varepsilon) & \delta(q_0, \rho, 1) = (q_1, 1) & \delta(q_1, \lambda, 1) = (q_3, 11) \\
\delta(q_0, \mu, 2) = (q_0, \varepsilon) & \delta(q_0, \rho, 2) = (q_1, 2) & \delta(q_1, \lambda, 2) = (q_3, 12) \\
\delta(q_3, \mu, 1) = (q_0, \varepsilon) & \delta(q_1, \rho, 0) = (q_2, 0) & \delta(q_2, \lambda, 0) = (q_5, 10) \\
\delta(q_4, \mu, 1) = (q_4, \varepsilon) & \delta(q_1, \rho, 1) = (q_2, 1) & \delta(q_2, \lambda, 1) = (q_5, 11) \\
\delta(q_4, \mu, 2) = (q_4, \varepsilon) & \delta(q_1, \rho, 2) = (q_2, 2) & \delta(q_2, \lambda, 2) = (q_5, 12) \\
\delta(q_5, \mu, 1) = (q_4, \varepsilon) & \delta(q_3, \rho, 1) = (q_6, 1) & \delta(q_4, \lambda, 0) = (q_3, 10) \\
\delta(q_5, \mu, 2) = (q_5, \varepsilon) & \delta(q_4, \rho, 0) = (q_2, 0) & \delta(q_4, \lambda, 1) = (q_3, 11) \\
 & \delta(q_4, \rho, 1) = (q_2, 1) & \delta(q_4, \lambda, 2) = (q_3, 12) \\
 & \delta(q_4, \rho, 2) = (q_2, 2) & \delta(q_5, \lambda, 1) = (q_7, 11) \\
 & \delta(q_5, \rho, 1) = (q_8, 1) & \delta(q_5, \lambda, 2) = (q_7, 12) \\
 & \delta(q_5, \rho, 2) = (q_8, 2) & \delta(q_6, \lambda, 1) = (q_7, 11) \\
 & \delta(q_6, \rho, 1) = (q_8, 1) & \delta(q_8, \lambda, 1) = (q_5, 21) \\
 & \delta(q_7, \rho, 1) = (q_6, 1) & \delta(q_8, \lambda, 2) = (q_5, 22)
\end{array}
$$

To prove that M accepts precisely the language $\mathcal{L}$, we first show that M is deterministic. This allows us to identify input words with unique paths in M and simplify our arguments slightly.

Lemma 3.1. *The pushdown automaton M is deterministic.*

Proof. The claim is easily verified by considering the formal description for M. □

Proposition 3.2. *The pushdown automaton M accepts the language $\mathcal{L}\$ = \{w\$ \mid w \in \mathcal{L}\}$.*

Proof. Since M is deterministic, we identify input words with their corresponding unique path in M.

Let $w \in \{\rho, \lambda, \mu\}^*$. We must show that

(1) if w contains $\rho\mu$, then $w\$$ is rejected.
(2) if w fails to be in $\mathcal{L}_{2,\infty}$, then $w\$$ is rejected,
(3) if w has a bad prefix (conditions (2) and (3) in Definition 2.6), then $w\$$ rejected.
(4) if $w\$$ is rejected, then $w \notin \mathcal{L}$.

The only states that can be reached by a path $u\rho$ for $u \in \{\rho, \lambda, \mu\}^*$ from the start configuration are q_1, q_2, q_6 and q_8 and since none are the source of a μ transition, any word containing $\rho\mu$ will be rejected.

Next, we show that if w is not in $\mathcal{L}_{2,\infty}$, then $w\$$ is rejected by M. Each state represents the endpoint of a path labeling a prefix of an input string accepted by the automaton. One can verify the values of $D_{\rho,\lambda}(u)$ for each path labeled u ending at state q_i given by Table 1.

Let $h(u)$ be the height of the stack after reading $u \in \{\rho, \lambda, \mu\}^*$ starting from the start configuration $(q_0, 0)$. Then $h(\varepsilon) = 1$, $h(u\rho) = h(u), h(u\lambda) = h(u)+1$ and $h(u\mu) = h(u)-1$ since λ pushes a token to the stack, μ pops a token and ρ keeps the stack unchanged. It follows that $h(u) = D_{\lambda,\mu}(u) + 1$, and since 0 stays on the stack until \$ is read, $h(u) \geq 1$

state	$D_{\rho,\lambda}$
q_0	0
q_1	1
q_2	2
q_3	0
q_4	1
q_5	1
q_6	1
q_7	0
q_8	2

TABLE 1. Value of $D_{\rho,\lambda}$ for any prefix ending at each state.

for all prefixes $u \in \{\rho, \lambda, \mu\}^*$, so $D_{\lambda,\mu}(u) \geq 0$. If $w\$$ is accepted then the stack must contain only 0 after reading w, so $D_{\lambda,\mu}(w) = 0$.

It follows that if $D_{\rho,\lambda} > 2, D_{\rho,\lambda}, D_{\lambda,\mu}(u) < 0$ for some prefix u, or $D_{\lambda,\mu}(w) \neq 0$, then M will reject $w\$$.

Next, suppose $w \in \mathcal{L}_{2,\infty}$ has no $\rho\mu$ substring and a prefix of the form $w_0 v w_1 \lambda\mu$ where $D_{\rho,\lambda}(w_0) = 1, v \in \{\rho\lambda, \lambda\rho\}$ and $w_1 \in \mathcal{L}_{1,\infty}$. The string w_0 labels a path in the automaton starting at q_0 and ending at state q_1, q_4, q_5 or q_6 by Table 1. From each of these states, reading $v = \rho\lambda$ ends in state q_5, and reading $v = \lambda\rho$ ends in state q_6.

From q_5, the word w_1 labels a path that visits only states q_5 and q_8, since $D_{\lambda,\mu}(z) \geq 0$ for all prefixes z of w_1, so the 1 on top of the stack before reading w_1 remains (and is covered by 2s, which are removed by the μ loop at q_5), and ends at q_5 since $D_{\lambda,\mu}(w_1) = 0$. From here reading $\lambda\mu$ is rejected.

From q_6, if $w_1 = \varepsilon$ then $u\lambda\rho\lambda\mu$ is rejected. Otherwise w_1 labels a path from q_6 to q_8 and then moves between q_5 and q_8, and ends at q_5. From here reading $\lambda\mu$ is rejected.

We have now established that if $w \notin \mathcal{L}$ then $w\$$ is rejected by M. To complete the proof we must show that if $w\$$ is rejected, then $w \notin \mathcal{L}$. To show this, assume $w \in \mathcal{L}_{2,\infty}$ with no $\rho\mu$ substring, but $w\$$ is rejected by M. We will prove that w must have a bad prefix.

Let p be the longest prefix of $w\$$ labeling a path that is not rejected by M. Since $w \in \mathcal{L}_{2,\infty}$ we have $D_{\lambda,\mu}(w) = 0$, so if $w = p$, after reading w the stack contains just 0 so $w\$$ will be accepted, a contradiction. Thus p is strictly shorter than w. Let $w = pxw'$ where $x \in \{\rho\lambda, \mu\}$ is the next letter input after reading p.

We now consider the possible states where p can end.

(1) Suppose p ends at q_0. Then $D_{\rho,\lambda}(p) = 0$ so $x \neq \lambda$. If the top of stack is 0 then $D_{\lambda,\mu}(p) = 0$ so $x \neq \mu$. Otherwise M cannot reject on reading ρ, μ.
(2) Suppose p ends at q_1, so its last letter is ρ, and $D_{\rho,\lambda}(p) = 1$. Then $x \neq \mu$. Otherwise M cannot reject on reading ρ, λ.
(3) Suppose p ends at q_2, so its last letter is ρ, and $D_{\rho,\lambda}(p) = 2$. Then $x \neq \mu, \rho$. Otherwise M cannot reject on reading λ.
(4) Suppose p ends at q_3, so $D_{\rho,\lambda}(p) = 0$ and the top of stack is 1. Then $x \neq \lambda$. Otherwise M cannot reject on reading $\rho, (\mu, 1 \to \varepsilon)$.
(5) Suppose p ends at q_4, so $D_{\rho,\lambda}(p) = 1$. The only way M could reject is if the top of stack is 0 and $x = \mu$, which is not possible since $w \in \mathcal{L}_{2,\infty}$.

(6) Suppose p ends at q_5, so $D_{\rho,\lambda}(p) = 1$ and 1 is on top of the stack. Then no letter will cause M to reject.
(7) Suppose p ends at q_6, so $D_{\rho,\lambda}(p) = 1$ and p ends with $\lambda\rho$. Then x cannot be μ, and otherwise px is not rejected.
(8) Suppose p ends at q_8, so its last letter is ρ, and $D_{\rho,\lambda}(p) = 2$. Then $x \neq \mu, \rho$ and M cannot reject if $x = \lambda$.

These cases show that if p ends at any state except q_7, then M does not reject w on reading the next input letter. We finish the proof by showing that if p ends at q_7, then px is a bad prefix.

Since p ends at q_7, p ends with λ, $D_{\rho,\lambda}(p) = 2$, and $D_{\lambda,\mu}(p) > 0$. If $x = \rho$ then px is not rejected. If $x = \lambda$ then $w \notin \mathcal{L}_{2,\infty}$. So we must have $x = \mu$.

Let $p = p_1\lambda$. If p_1 ends at q_6, then $p_1 = p_2\lambda\rho$, and $px = p_2\lambda\rho\lambda\mu$ where $D_{\rho,\lambda}(p_2) = 1$ and so px is a bad prefix. The machine correctly rejects the string on reading $x = \mu$.

Otherwise p_1 ends at q_5. Either p_1 ends with $\rho\lambda$, or μ. If $p_1 = p_2\rho\lambda$ then $D_{\rho,\lambda}(p_2) = 1$ and $px = p_2\rho\lambda\lambda\mu$ is a bad prefix. Otherwise p_1 ends in μ, and must pop a token 2 from the stack. Let λ_* be the last λ letter in p_1 that pushed a 1 on top of the stack (which must exist, since all paths to q_5 must cross such an edge). Write $p_1 = p_2\lambda_* p_3\mu$.

The letter λ_* labels one of the following four edges:

(1) from q_2 to q_5,
(2) from q_1 to q_3,
(3) from q_4 to q_3,
(4) from q_5 to q_7,
(5) from q_6 to q_7.

In the first case, p_2 ends at q_2 so must have the form $p_2 = u\rho$ with $D_{\rho,\lambda}(u) = 1$. Then $p_3\mu$ labels a path that moves between states q_5 and q_8, reading $\rho\lambda$ and pushing a 2, or reading μ and popping a 2, so $p_3\mu \in \mathcal{L}_{1,\infty}$. It follows that w has the bad prefix $u\rho\lambda_*(p_3\mu)\lambda\mu$, and so M correctly rejects it.

In the other four cases we have that $D_{\rho,\lambda}(p_2) = 1$ since p_2 ends at state q_1, q_4, q_5 or q_6, λ_* must be immediately followed by a letter ρ, and $p_2\lambda_*\rho$ ends at state q_6. Let $p_3 = \rho p_4$. Then $p_4\mu$ labels a path that starts at q_6, goes to q_8, then moves between states q_5 and q_8, reading $\rho\lambda$ and pushing a 2, or reading μ and popping a 2. So $p_4\mu \in \mathcal{L}_{1,\infty}$. It follows that w has the bad prefix $p_2\lambda_*\rho(p_4\mu)\lambda\mu$, and so M correctly rejects it. □

4. Obtaining the generating function

Theorem 4.1. *The sequence counting the number of permutations of each length in $\mathcal{P}$ has an algebraic generating function:*

$$\sum_{n\geq 0} c_n z^n = \frac{(1+q)\left(1+5q-q^2-q^3-(1-q)\sqrt{(1-q^2)(1-4q-q^2)}\right)}{8q}$$

where c_n is the number of permutations in $\mathcal{P}$ of length n, and $q \equiv q(z) = \frac{1-2z-\sqrt{1-4z}}{2z}$.

Proof. We convert the pushdown automaton given in the previous section to an unambiguous context-free language, following the standard procedure as described in Hopcroft and Ullman [14]. Theorem 10.12 of Hopcroft and Ullman guarantees that the grammar obtained from a deterministic pushdown automaton accepting on empty stack is $LR(0)$ and hence unambiguous.

We then apply the Chomsky and Schützenberger theorem, as outlined for example in [12] I.5.4, to obtain an algebraic generating function. Since each step in this procedure is constructive, we can find the generating function explicitly.

We start by converting the pushdown automaton to a grammar. See Theorem 5.4 [14] for full details.

Define a grammar with nonterminals S and $[q_i, j, q_k] = N_{i,j,k}$ for each pair of states q_i, q_k and stack symbol j. The nonterminal $[q_i, j, q_k]$ represents a path in the configuration space of the pushdown automaton starting at q_i with j on top of the stack and ending at some state q_k. The productions "fill out" these paths with subpaths according to the transitions that are possible.

The production rules are then defined as follows:

(1) for each state q_i we have a production $S \to N_{00i}$,
(2) for each transition $\delta(q_i, a, j) = \{(q_k, \varepsilon)\}$ with $a \in \{\$, \mu\}$, add a production $N_{ijk} = a$,
(3) for each transition $\delta(q_i, \rho, j) = \{(q_k, l)\}$, add productions $N_{ijx} = \rho N_{klx}$ for $0 \leq x \leq 8$,
(4) for each transition $\delta(q_i, \lambda, j) = \{(q_k, lm)\}$, add productions $N_{ijx} = \lambda N_{kly} N_{ymx}$ $0 \leq x, y \leq 8$.

This gives the following set of productions, where $0 \leq x, y \leq 8$:

$$
\begin{array}{lll}
N_{000} \to \$ & N_{00x} \to \rho N_{10x} & N_{10x} \to \lambda N_{31y} N_{y0x} \\
N_{010} \to \mu & N_{01x} \to \rho N_{11x} & N_{11x} \to \lambda N_{31y} N_{y1x} \\
N_{020} \to \mu & N_{02x} \to \rho N_{12x} & N_{12x} \to \lambda N_{31y} N_{y2x} \\
N_{310} \to \mu & N_{10x} \to \rho N_{20x} & N_{20x} \to \lambda N_{51y} N_{y0x} \\
N_{414} \to \mu & N_{11x} \to \rho N_{21x} & N_{21x} \to \lambda N_{51y} N_{y1x} \\
N_{424} \to \mu & N_{12x} \to \rho N_{22x} & N_{22x} \to \lambda N_{51y} N_{y2x} \\
N_{514} \to \mu & N_{31x} \to \rho N_{61x} & N_{40x} \to \lambda N_{31y} N_{y0x} \\
N_{525} \to \mu & N_{40x} \to \rho N_{20x} & N_{41x} \to \lambda N_{31y} N_{y1x} \\
 & N_{41x} \to \rho N_{21x} & N_{42x} \to \lambda N_{31y} N_{y2x} \\
 & N_{42x} \to \rho N_{22x} & N_{51x} \to \lambda N_{71y} N_{y1x} \\
 & N_{51x} \to \rho N_{81x} & N_{52x} \to \lambda N_{71y} N_{y2x} \\
 & N_{52x} \to \rho N_{82x} & N_{61x} \to \lambda N_{71y} N_{y1x} \\
 & N_{61x} \to \rho N_{81x} & N_{81x} \to \lambda N_{52y} N_{y1x} \\
 & N_{71x} \to \rho N_{61x} & N_{82x} \to \lambda N_{52y} N_{y2x}
\end{array}
$$

We can reduce the size of the grammar description as follows. First, observe that the only productions that eliminate nonterminals (by generating \$ or μ) are of the form N_{*jk} for $k \in \{0, 4, 5\}$, and $j = 0$ implies $k = 0$. Since all productions with nonterminals on the right side have the form $N_{*ij} \to \rho N_{*ij}$ or $N_{*ij} \to \lambda N_{***} N_{*ij}$, it follows that any nonterminal N_{**k} with k not equal to $0, 4$ or 5 cannot be eliminated, so we can exclude them from the grammar.

Also, if we start a derivation with $S \to N_{00k}$ for $k \neq 0$, there will always be a nonterminal of the form N_{*0k} that cannot be eliminated. Therefore it suffices to make N_{000} the start nonterminal and remove all productions involving S.

Lastly, the resulting grammar contain nonterminals $N_{500}, N_{504}, N_{505}$ that will never produce a string of only terminals, since the configuration $(q_5, 0)$ is never realised (to reach q_5 the top of stack symbol is either 1 or 2. We modify the above grammar one step further by removing any production involving these nonterminals .

Taking these factors into consideration, and collecting productions with the same left side together we obtain the following grammar:

$$
\begin{aligned}
N_{000} &\rightarrow \$ \mid \rho N_{100},\\
N_{004} &\rightarrow \rho N_{104},\\
N_{005} &\rightarrow \rho N_{105},\\
N_{010} &\rightarrow \mu \mid \rho N_{110},\\
N_{014} &\rightarrow \rho N_{114},\\
N_{015} &\rightarrow \rho N_{115},\\
N_{020} &\rightarrow \mu \mid \rho N_{120},\\
N_{024} &\rightarrow \rho N_{124},\\
N_{025} &\rightarrow \rho N_{125},\\
N_{100} &\rightarrow \rho N_{200} \mid \lambda N_{310} N_{000} \mid \lambda N_{314} N_{400},\\
N_{104} &\rightarrow \rho N_{204} \mid \lambda N_{310} N_{004} \mid \lambda N_{314} N_{404},\\
N_{105} &\rightarrow \rho N_{205} \mid \lambda N_{310} N_{005} \mid \lambda N_{314} N_{405},\\
N_{110} &\rightarrow \rho N_{210} \mid \lambda N_{310} N_{010} \mid \lambda N_{314} N_{410} \mid \lambda N_{315} N_{510},\\
N_{114} &\rightarrow \rho N_{214} \mid \lambda N_{310} N_{014} \mid \lambda N_{314} N_{414} \mid \lambda N_{315} N_{514},\\
N_{115} &\rightarrow \rho N_{215} \mid \lambda N_{310} N_{015} \mid \lambda N_{314} N_{415} \mid \lambda N_{315} N_{515},\\
N_{120} &\rightarrow \rho N_{220} \mid \lambda N_{310} N_{020} \mid \lambda N_{314} N_{420} \mid \lambda N_{315} N_{520},\\
N_{124} &\rightarrow \rho N_{224} \mid \lambda N_{310} N_{024} \mid \lambda N_{314} N_{424} \mid \lambda N_{315} N_{524},\\
N_{125} &\rightarrow \rho N_{225} \mid \lambda N_{310} N_{025} \mid \lambda N_{314} N_{425} \mid \lambda N_{315} N_{525},\\
N_{200} &\rightarrow \lambda N_{510} N_{000} \mid \lambda N_{514} N_{400},\\
N_{204} &\rightarrow \lambda N_{510} N_{004} \mid \lambda N_{514} N_{404},\\
N_{205} &\rightarrow \lambda N_{510} N_{005} \mid \lambda N_{514} N_{405},\\
N_{210} &\rightarrow \lambda N_{510} N_{010} \mid \lambda N_{514} N_{410} \mid \lambda N_{515} N_{510},\\
N_{214} &\rightarrow \lambda N_{510} N_{014} \mid \lambda N_{514} N_{414} \mid \lambda N_{515} N_{514},\\
N_{215} &\rightarrow \lambda N_{510} N_{015} \mid \lambda N_{514} N_{415} \mid \lambda N_{515} N_{515},\\
N_{220} &\rightarrow \lambda N_{510} N_{020} \mid \lambda N_{514} N_{420} \mid \lambda N_{515} N_{520},\\
N_{224} &\rightarrow \lambda N_{510} N_{024} \mid \lambda N_{514} N_{424} \mid \lambda N_{515} N_{524},\\
N_{225} &\rightarrow \lambda N_{510} N_{025} \mid \lambda N_{514} N_{425} \mid \lambda N_{515} N_{525},\\
N_{310} &\rightarrow \mu \mid \rho N_{610},\\
N_{314} &\rightarrow \rho N_{614},\\
N_{315} &\rightarrow \rho N_{615},\\
N_{400} &\rightarrow \rho N_{200} \mid \lambda N_{310} N_{000} \mid \lambda N_{314} N_{400},\\
N_{404} &\rightarrow \rho N_{204} \mid \lambda N_{310} N_{004} \mid \lambda N_{314} N_{404},\\
N_{405} &\rightarrow \rho N_{205} \mid \lambda N_{310} N_{005} \mid \lambda N_{314} N_{405},\\
N_{410} &\rightarrow \rho N_{210} \mid \lambda N_{310} N_{010} \mid \lambda N_{314} N_{410} \mid \lambda N_{315} N_{510},\\
N_{414} &\rightarrow \mu \mid \rho N_{214} \mid \lambda N_{310} N_{014} \mid \lambda N_{314} N_{414} \mid \lambda N_{315} N_{514},\\
N_{415} &\rightarrow \rho N_{215} \mid \lambda N_{310} N_{015} \mid \lambda N_{314} N_{415} \mid \lambda N_{315} N_{515},\\
N_{420} &\rightarrow \rho N_{220} \mid \lambda N_{310} N_{020} \mid \lambda N_{314} N_{420} \mid \lambda N_{315} N_{520},\\
N_{424} &\rightarrow \mu \mid \rho N_{224} \mid \lambda N_{310} N_{024} \mid \lambda N_{314} N_{424} \mid \lambda N_{315} N_{524},\\
N_{425} &\rightarrow \rho N_{225} \mid \lambda N_{310} N_{025} \mid \lambda N_{314} N_{425} \mid \lambda N_{315} N_{525},\\
N_{510} &\rightarrow \rho N_{810} \mid \lambda N_{710} N_{010} \mid \lambda N_{714} N_{410} \mid \lambda N_{715} N_{510},\\
N_{514} &\rightarrow \mu \mid \rho N_{814} \mid \lambda N_{710} N_{014} \mid \lambda N_{714} N_{414} \mid \lambda N_{715} N_{514},\\
N_{515} &\rightarrow \rho N_{815} \mid \lambda N_{710} N_{015} \mid \lambda N_{714} N_{415} \mid \lambda N_{715} N_{515},\\
N_{520} &\rightarrow \rho N_{820} \mid \lambda N_{710} N_{020} \mid \lambda N_{714} N_{420} \mid \lambda N_{715} N_{520},\\
N_{524} &\rightarrow \rho N_{824} \mid \lambda N_{710} N_{024} \mid \lambda N_{714} N_{424} \mid \lambda N_{715} N_{524},\\
N_{525} &\rightarrow \mu \mid \rho N_{825} \mid \lambda N_{710} N_{025} \mid \lambda N_{714} N_{425} \mid \lambda N_{715} N_{525},
\end{aligned}
$$

$$
\begin{aligned}
N_{610} &\to \rho N_{810} \mid \lambda N_{710}N_{010} \mid \lambda N_{714}N_{410} \mid \lambda N_{715}N_{510},\\
N_{614} &\to \rho N_{814} \mid \lambda N_{710}N_{014} \mid \lambda N_{714}N_{414} \mid \lambda N_{715}N_{514},\\
N_{615} &\to \rho N_{815} \mid \lambda N_{710}N_{015} \mid \lambda N_{714}N_{415} \mid \lambda N_{715}N_{515},\\
N_{710} &\to \rho N_{610},\\
N_{714} &\to \rho N_{614},\\
N_{715} &\to \rho N_{615},\\
N_{810} &\to \lambda N_{520}N_{010} \mid \lambda N_{524}N_{410} \mid \lambda N_{525}N_{510},\\
N_{814} &\to \lambda N_{520}N_{014} \mid \lambda N_{524}N_{414} \mid \lambda N_{525}N_{514},\\
N_{815} &\to \lambda N_{520}N_{015} \mid \lambda N_{524}N_{415} \mid \lambda N_{525}N_{515},\\
N_{820} &\to \lambda N_{520}N_{020} \mid \lambda N_{524}N_{420} \mid \lambda N_{525}N_{520},\\
N_{824} &\to \lambda N_{520}N_{024} \mid \lambda N_{524}N_{424} \mid \lambda N_{525}N_{524},\\
N_{825} &\to \lambda N_{520}N_{025} \mid \lambda N_{524}N_{425} \mid \lambda N_{525}N_{525}.
\end{aligned}
$$

The next step is to convert nonterminals to generating functions, terminals to z and productions to equations, as described in [12] I.5.4.

$$
\begin{aligned}
f_{000} &= z + zf_{100},\\
f_{004} &= zf_{104},\\
f_{005} &= zf_{105},\\
f_{010} &= z + zf_{110},\\
f_{014} &= zf_{114},\\
f_{015} &= zf_{115},\\
f_{020} &= z + zf_{120},\\
f_{024} &= zf_{124},\\
f_{025} &= zf_{125},\\
f_{100} &= zf_{200} + zf_{310}f_{000} + zf_{314}f_{400},\\
f_{104} &= zf_{204} + zf_{310}f_{004} + zf_{314}f_{404},\\
f_{105} &= zf_{205} + zf_{310}f_{005} + zf_{314}f_{405},\\
f_{110} &= zf_{210} + zf_{310}f_{010} + zf_{314}f_{410} + zf_{315}f_{510},\\
f_{114} &= zf_{214} + zf_{310}f_{014} + zf_{314}f_{414} + zf_{315}f_{514},\\
f_{115} &= zf_{215} + zf_{310}f_{015} + zf_{314}f_{415} + zf_{315}f_{515},\\
f_{120} &= zf_{220} + zf_{310}f_{020} + zf_{314}f_{420} + zf_{315}f_{520},\\
f_{124} &= zf_{224} + zf_{310}f_{024} + zf_{314}f_{424} + zf_{315}f_{524},\\
f_{125} &= zf_{225} + zf_{310}f_{025} + zf_{314}f_{425} + zf_{315}f_{525},\\
f_{200} &= zf_{510}f_{000} + zf_{514}f_{400},\\
f_{204} &= zf_{510}f_{004} + zf_{514}f_{404},\\
f_{205} &= zf_{510}f_{005} + zf_{514}f_{405},\\
f_{210} &= zf_{510}f_{010} + zf_{514}f_{410} + zf_{515}f_{510},\\
f_{214} &= zf_{510}f_{014} + zf_{514}f_{414} + zf_{515}f_{514},\\
f_{215} &= zf_{510}f_{015} + zf_{514}f_{415} + zf_{515}f_{515},\\
f_{220} &= zf_{510}f_{020} + zf_{514}f_{420} + zf_{515}f_{520},\\
f_{224} &= zf_{510}f_{024} + zf_{514}f_{424} + zf_{515}f_{524},\\
f_{225} &= zf_{510}f_{025} + zf_{514}f_{425} + zf_{515}f_{525},\\
f_{310} &= z + zf_{610},\\
f_{314} &= zf_{614},\\
f_{315} &= zf_{615},
\end{aligned}
$$

$$
\begin{aligned}
f_{400} &= zf_{200} + zf_{310}f_{000} + zf_{314}f_{400},\\
f_{404} &= zf_{204} + zf_{310}f_{004} + zf_{314}f_{404},\\
f_{405} &= zf_{205} + zf_{310}f_{005} + zf_{314}f_{405},\\
f_{410} &= zf_{210} + zf_{310}f_{010} + zf_{314}f_{410} + zf_{315}f_{510},\\
f_{414} &= z + zf_{214} + zf_{310}f_{014} + zf_{314}f_{414} + zf_{315}f_{514},\\
f_{415} &= zf_{215} + zf_{310}f_{015} + zf_{314}f_{415} + zf_{315}f_{515},\\
f_{420} &= zf_{220} + zf_{310}f_{020} + zf_{314}f_{420} + zf_{315}f_{520},\\
f_{424} &= z + zf_{224} + zf_{310}f_{024} + zf_{314}f_{424} + zf_{315}f_{524},\\
f_{425} &= zf_{225} + zf_{310}f_{025} + zf_{314}f_{425} + zf_{315}f_{525},\\
f_{510} &= zf_{810} + zf_{710}f_{010} + zf_{714}f_{410} + zf_{715}f_{510},\\
f_{514} &= z + zf_{814} + zf_{710}f_{014} + zf_{714}f_{414} + zf_{715}f_{514},\\
f_{515} &= zf_{815} + zf_{710}f_{015} + zf_{714}f_{415} + zf_{715}f_{515},\\
f_{520} &= zf_{820} + zf_{710}f_{020} + zf_{714}f_{420} + zf_{715}f_{520},\\
f_{524} &= zf_{824} + zf_{710}f_{024} + zf_{714}f_{424} + zf_{715}f_{524},\\
f_{525} &= z + zf_{825} + zf_{710}f_{025} + zf_{714}f_{425} + zf_{715}f_{525},\\
f_{610} &= zf_{810} + zf_{710}f_{010} + zf_{714}f_{410} + zf_{715}f_{510},\\
f_{614} &= zf_{814} + zf_{710}f_{014} + zf_{714}f_{414} + zf_{715}f_{514},\\
f_{615} &= zf_{815} + zf_{710}f_{015} + zf_{714}f_{415} + zf_{715}f_{515},\\
f_{710} &= zf_{610},\\
f_{714} &= zf_{614},\\
f_{715} &= zf_{615},\\
f_{810} &= zf_{520}f_{010} + zf_{524}f_{410} + zf_{525}f_{510},\\
f_{814} &= zf_{520}f_{014} + zf_{524}f_{414} + zf_{525}f_{514},\\
f_{815} &= zf_{520}f_{015} + zf_{524}f_{415} + zf_{525}f_{515},\\
f_{820} &= zf_{520}f_{020} + zf_{524}f_{420} + zf_{525}f_{520},\\
f_{824} &= zf_{520}f_{024} + zf_{524}f_{424} + zf_{525}f_{524},\\
f_{825} &= zf_{520}f_{025} + zf_{524}f_{425} + zf_{525}f_{525}.
\end{aligned}
$$

Using Maple (version 14) we can solve to obtain an expression for the algebraic generating function $f_{000}(z)$, which counts the number of words in $\mathcal{L}\$$ of each length. Since words in $\mathcal{L}\$$ of length $3n+1$ are in bijection with permutations in $\mathcal{P}$ of length n, the generating function $\sum_{n\geq 0} c_n t^n$ where c_n is the number of permutations of length n in $\mathcal{P}$ is obtained by dividing f_{000} by z and substituting $z^3 = t$. □

From the expression for the generating function we can easily obtain the first few terms of the sequence:
$1 + z + 2z^2 + 6z^3 + 24z^4 + 114z^5 + 592z^6 + 3216z^7 + 17904z^8 + 101198z^9 + 578208z^{10} + 3332136z^{11} + 19343408z^{12} + \ldots$.

We can also use standard analytic combinatorial methods[12] to deduce the asymptotic growth of the number of such permutations:

$$c_n \sim \frac{\sqrt{25 - 11\sqrt{5}}}{2\sqrt{\pi n^3}} \cdot (2 + 2\sqrt{5})^n \cdot \left(1 + O(n^{-1})\right).$$

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