Partially directed paths in a wedge

E J Janse van Rensburg[1]*, T Prellberg[2] and A Rechnitzer[3,4].

[1] Department of Mathematics and Statistics,
York University, 4700 Keele Street, Toronto, Ontario, M3J 1P3, Canada
rensburg@yorku.ca

[2] School of Mathematical Sciences,
Queen Mary College, Univ of London, Mile End Road, London E1 4NS, UK
t.prellberg@qmul.ac.uk

[3] Department of Mathematics and Statistics,
The University of Melbourne, Victoria 3010, Australia.
a.rechnitzer@ms.unimelb.edu.au
[4] Department of Mathematics
University of British Columbia, Vancouver, BC V6T 1Z2, Canada.
andrewr@math.ubc.ca

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Abstract

The enumeration of lattice paths in wedges poses unique mathematical challenges. These models are not translationally invariant, and the absence of this symmetry complicates both the derivation of a functional recurrence for the generating function, and solving for it. In this paper we consider a model of partially directed walks from the origin in the square lattice confined to both a symmetric wedge defined by $Y = \pm pX$, and an asymmetric wedge defined by the lines $Y = pX$ and $Y = 0$, where $p > 0$ is an integer. We prove that the growth constant for all these models is equal to $1+\sqrt{2}$, independent of the angle of the wedge. We derive functional recursions for both models, and obtain explicit expressions for the generating functions when $p = 1$. From these we find asymptotic formulas for the number of partially directed paths of length n in a wedge when $p = 1$.

The functional recurrences are solved by a variation of the kernel method, which we call the "iterated kernel method". This method appears to be similar to the obstinate kernel method used by Bousquet-Mélou (see, for example, references [5, 6]). This method requires us to consider iterated compositions of the roots of the kernel. These compositions turn out to be surprisingly tractable, and we are able to find simple explicit expressions for them. However, in spite of this, the generating functions turn out to be similar in form to Jacobi θ-functions, and have natural boundaries on the unit circle.

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*Please address correspondence to rensburg@yorku.ca

1 Introduction

The problem of counting random walks on lattices with various restrictions is perhaps one of the oldest problems in enumerative combinatorics, with a history that dates back at least 100 years [1]. It has also seen a great deal of recent activity, particularly surrounding problems of counting random walks on the slit-plane and quarter-plane [15, 22, 13, 12, 4, 7, 5].

These models pose interesting mathematical problems, and powerful methods have been developed in recent years to solve for the generating functions of path problems. Normally, these methods are a three step process: First a recurrence is determined, this is solved in the second step, and lastly, the asymptotics of the number of paths are extracted.

Perhaps the simplest and most studied model is Dyck paths. While there are numerous techniques for enumerating Dyck paths, the most powerful technique involves a recurrence for the generating function which is solved and expanded to determine an explicit expression. If d_n is the number of Dyck paths of half-length n, then the generating function $g_t = \sum_{n\geq 0} d_n t^n$ satisfies the recurrence

$$g_t = 1 + t g_t^2 \tag{1.1}$$

with solution

$$g_t = \frac{2}{1+\sqrt{1-4t}} = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{t^n}{n+1} \tag{1.2}$$

so that d_n is given by Catalan's number.

In this paper we follow a similar strategy to determine the generating function and asymptotic expressions for the number of partially directed paths confined to a wedge. The wedge destroys translational invariance in the model, and both the derivation of a recurrence for the generating function, and solving the generating function, poses difficult mathematical problems. Our strategy is in principle no different from the above for Dyck paths - we shall derive functional recurrences for the generating functions, solve those in special cases, and then find the asymptotics for the number of paths. Unfortunately, the problem for general wedges appears intractable, and even in the cases that we do solve we encountered significant difficulties.

Models of paths and walks frequently appear as simple models of polymers in dilute solution in the physics literature [29]. The properties of polymers are in part determined by their conformational entropy, and models of walks and paths contributes to our understanding of the significance of the conformational entropy contributions in the free energy of polymers. These entropic contributions are important when polymers are in confined geometries. For example, the steric stabilisation of colloids by polymers results when polymers are confined to the spaces between colloidal particles [21]. This situation have been modelled by studying paths confined to the slab between two planes, see for example [9, 23].

Lattice random walk models of polymers in confined geometries are generally more tractable. These models can generally be solved, at least in principle, by a Bethe ansatz or constant term formulation. This technique have been used to solve for random walks in a half-space and which interacts with the boundary of the space [20]. Such random walk models, however, do not take into account the volume exclusion of monomers in a polymer. A more realistic model is the self-avoiding walk [19]. This model is non-Markovian, and while much is known about it from constructive [17, 18] and conformal invariance techniques (in two dimensions) [8], solving it remains beyond the current techniques in combinatorics.

Self-avoiding walk models of polymers in confined spaces have not been solved (except in the most trivial of cases), but there are some results in the literature. For example, the exponential growth constant of self-avoiding walks in a wedge geometry is independent of the angle of the wedge [16]. Additionally, for self-avoiding walks in wedges, conformal field theory have been used to examine the dependence of scaling exponents on the wedge angle [8, 11].

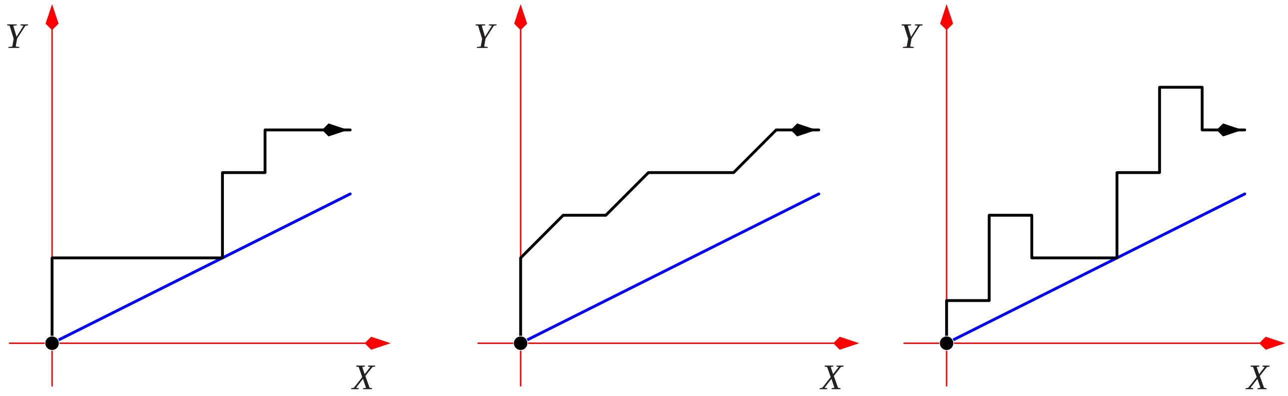

Figure 1: Models of directed walks above a line $Y = +pX$. The walks are constrained to take only north and east steps (left); north, east and north-east steps (middle); and north, east and south steps (right).

The introduction of directedness in a self-avoiding walk in a wedge may give models of directed or partially directed walks in a wedge which are both self-avoiding and which may in some cases be solvable. Models of directed and partially directed walks in wedge geometries (see Figure 1) have been studied previously [10, 28, 25]. These models include directed paths in a wedge; a model which is related to Dyck paths. It is interesting that the radius of convergence of the generating function is known in this model, even for wedges with wedge-angles of irrational cotangent [28, 25].

In this paper we consider models of a partially directed path confined in wedges (see Figure 2). These models are similar to the directed path models in Figure 1, however, they are also substantially more challenging, since the path interacts with the wedge on two sides, rather than on only one side. As a result, it is much harder to find their generating functions and analyse their asymptotics.

1.1 Directed and partially directed paths

A directed walk on the square lattice is a path taking unit steps only in the north and east directions. Such objects are necessarily self-avoiding; they cannot revisit the same vertex. Partially directed paths may take unit steps only in the north, south and east directions with the further condition that no vertex is visited twice — *ie* they are self-avoiding. Hence, north steps cannot be followed by south steps and *vice-versa*. The generating function of such walks can be derived using standard techniques:

$$W(t) = \sum_{n\geq 0} c_n t^n = \frac{1+t}{1-2t-t^2}, \tag{1.3}$$

where c_n is the number of walks of length n and t is the length generating variable. An expansion of $W(t)$ of $W(t)$ in t produces an explicit expression for c_n:

$$c_n = \frac{1}{2}\left((1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}\right). \tag{1.4}$$

The exponential growth constant is the exponential rate at which c_n increases with n. This is given by

$$\mu = \lim_{n\to\infty} c_n^{1/n} = 1+\sqrt{2}. \tag{1.5}$$

This is the most fundamental quantity in this model from a statistical mechanics point of view. The radius of convergence of $W(t)$ is μ^{-1}, and the limiting free energy is the logarithm of

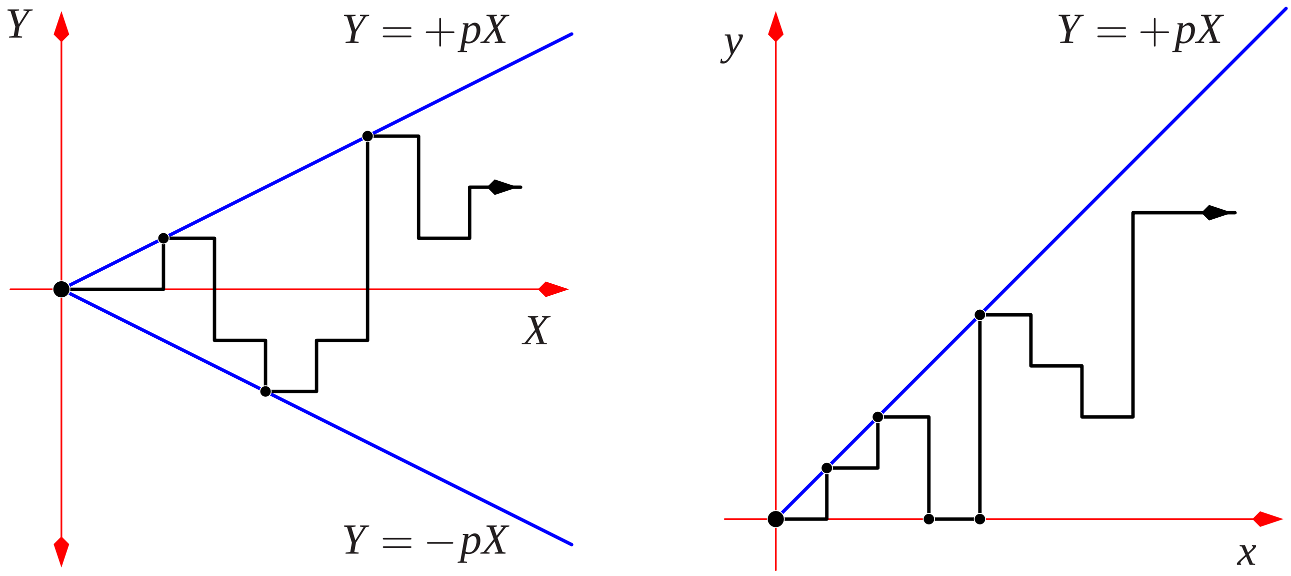

Figure 2: (left) The symmetric model of a partially directed path in a p-wedge formed by the lines $Y = \pm pX$. (right) The asymmetric model of a partially directed path in a p-wedge formed by the line $Y = +pX$ and the X-axis.

the growth constant, $\kappa = \log \mu$, this defines the explicit connection between the combinatorial properties of the model and its thermodynamic properties.

We show some models of directed and partially directed paths in wedge geometries in Figure 1. The model in Figure 1 (left) was considered in [10, 27]. In general the growth constant is a (non-trivial) function of the wedge angle. The derivative of the free-energy with respect to the wedge angle gives the moment of the entropic force exerted by the polymer on the wedge and this was computed in [27]. This model may be also generalised by introducing an interaction between the line $Y = +pX$ and the path, or by considering partially directed paths or Motzkin paths instead [26, 25].

In Figure 1 (right) a partially directed path confined to the wedge above the line $Y = pX$ and the Y-axis is proposed instead. This model was considered in reference [26]. If the partially directed path is instead confined to the wedge between the X-axis and the line $Y = pX$, then the model in Figure 2 (right) is obtained, which is the subject of this paper.

In particular, we consider the variants illustrated in Figure 2 - firstly a model of a partially directed path in a wedge formed by the lines $Y = \pm pX$ (we call this the *symmetric model* - see Figure 2 (left)), and secondly a model of a partially directed path in a wedge formed by the X-axis and the line $Y = +pX$ (this is the *asymmetric model* - see Figure 2 (right)).

The related model of a partially directed path in a wedge with last vertex in the line $y = pX$ is illustrated in Figure 3. This is a *bargraph path* above the line $Y = +pX$. This model was examined in reference [26], and while the generating function $g_p(t)$ is not known explicitly, it is given by

$$g_p(t) = \frac{h(t)}{1 - t^2(1 + h(t))} \tag{1.6}$$

where $h(t)$ is an appropriate solution of the equation

$$h(t) = t^{p+1} \left(1 + h(t)\right)^p \left(1 + \frac{h(t)}{1 - t^2(1 + h(t))}\right), \tag{1.7}$$

where (as above), t is conjugate to the number of edges in the path.

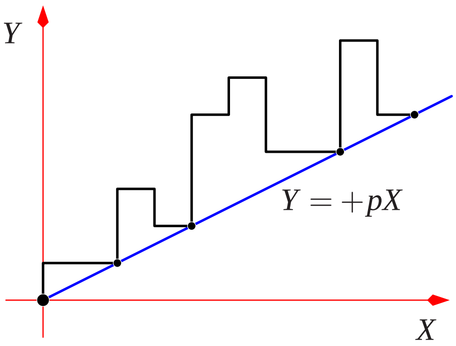

Figure 3: A bargraph path above the line $Y = +pX$. The generating function of this model is not known explicitly. However, a set of equations derived in reference [26] can be solved to determine the radius of convergence of the generating function for integer values of p.

The model in Figure 3 was also used as a model of *adsorbing bargraphs* which interact with the line $Y = +pX$ [10], and an asymptotic expression for the adsorption critical point has been estimated in reference [26] (the location of the singular point on the radius of convergence of the generating function).

1.2 Partially Directed Paths in Wedges

Consider the square lattice $\mathbb{Z}^2$ of points in the plane with integer coordinates. Let $p > 0$ be an integer. The symmetric p-wedge $\mathcal{V}_p$ is defined by

$$\mathcal{V}_p = \left\{ (n,m) \in \mathbb{Z}^2 \,\middle|\, \text{where } n \geq 0 \text{ and } -pn \leq m \leq pn \right\}. \tag{1.8}$$

The (asymmetric) *p-wedge* is defined by

$$\mathcal{W}_p = \left\{ (n,m) \in \mathbb{Z}^2 \,\middle|\, \text{where } n \geq 0 \text{ and } 0 \leq m \leq pn \right\}. \tag{1.9}$$

Let $v_{p,n}$ (resp. $w_{p,n}$) be the number of partially directed walks in $\mathcal{V}_p$ (resp. $\mathcal{W}_p$) of length n.

In the next section we establish some basic facts about the asymptotic growth of $v_{n,p}$ and $w_{n,p}$ as $n \to \infty$. In Section 3 we find functional equations satisfied by the corresponding generating functions, which we solve in Sections 4 and 5. We then analyse the generating functions to determine $v_{n,1}$ and $w_{n,1}$ to leading order. We show in particular that

$$v_{n,1} = A_0(1+\sqrt{2})^n + \frac{\sqrt{5}^n}{\sqrt{(n+1)^3}} \left(A_1 + (-1)^n A_2 + O(1/n) \right) \tag{1.10}$$

for the number of paths in a symmetric wedge when $p = 1$ where A_0, A_1 and A_2 are constants. The asymmetric wedge poses more difficult mathematical problems, and we were only able to show that

$$w_{n,1} = \frac{(1+\sqrt{2})^n}{\sqrt{n+1}} \left(B_0 + o(1) \right), \tag{1.11}$$

for some constant B_0.

2 Growth constants in wedges

In this section we prove that the growth constant for partially directed walks is independent of the angle of the wedge and is equal to that of unrestricted partially directed walks. First, let b_n be the number of partially directed walks in the wedge defined by the lines $X = 0$ and $Y = 0$, whose last vertex lies in the line $Y = 0$. These paths are counted by the generating function $g_0(t)$ defined above, and singularity analysis gives the following lemma.

Lemma 2.1. *The growth constant of partially directed paths in the wedge defined by $X = 0$ and $Y = 0$ is*

$$\lim_{n\to\infty} b_n^{1/n} = (1 + \sqrt{2}) = \mu. \tag{2.1}$$

This result can be used to determine the growth constants of partially directed walks in the wedges $\mathcal{V}_p$ and $\mathcal{W}_p$. We first prove existence of the growth constants.

Lemma 2.2. *For any given $p \in (0, \infty)$ the following limits exist:*

$$\lim_{n\to\infty} v_{n,p}^{1/n} = \mu_p^v \qquad \text{and} \qquad \lim_{n\to\infty} w_{n,p}^{1/n} = \mu_p^w. \tag{2.2}$$

The limits satisfy

$$\mu_p^w \le \mu_p^v \le \mu = (1 + \sqrt{(2)}) \tag{2.3}$$

Proof. We have that $w_{n,p} \le v_{n,p} \le c_n$. Hence, if the above limits exist, we must have $\mu_p^w \le \mu_p^v \le \mu = (1 + \sqrt{2})$.

To show existence, we prove that the sequences are super-multiplicative. Take any walk counted by $v_{n,p}$ and append a horizontal step, and any walk counted by $v_{m,p}$. This gives a walk of $n+m+1$ steps that lies within $\mathcal{V}_p$, and so is counted by $v_{n+m+1,p}$. Hence $v_{n,p}v_{m,p} \le v_{n+m+1,p}$. A standard result (Fekete's lemma) on super-additive sequences (which we can apply by taking logarithms) then implies that μ_p^v exists. The proof for walks in $\mathcal{W}_p$ is identical. □

Next, we show that $\mu_p^w = \mu_p^v$, and we show that they are equal to $1 + \sqrt{2}$.

Lemma 2.3. *For any given $p \in (0, \infty)$ we have*

$$b_n^N \le w_{(\lceil np\rceil + nN + N),p} \tag{2.4}$$

And hence $\lim_{n\to\infty} b_n^{1/n} \le \mu_p^w$.

Proof. Take any walk counted by b_n. By prepending $\lceil np\rceil + 1$ horizontal steps, this walk will fit inside the wedge, $\mathcal{W}_p$. Now append another horizontal step and a walk counted by b_n — repeat this until there are N walks counted by b_n. This gives a walk counted by $w_{(\lceil np\rceil + nN + N),p}$. Thus we have the first inequality. Taking logs and dividing by $(\lceil np\rceil + nN + N)$ gives

$$\frac{N}{(\lceil np\rceil + nN + N)} \log b_n \le \frac{1}{(\lceil np\rceil + nN + N)} \log w_{(\lceil np\rceil + nN + N),p} \tag{2.5}$$

Take the limit as $N \to \infty$ to obtain

$$\frac{1}{n} \log b_n \le \log \mu_p^w. \tag{2.6}$$

Next, take the limit as $n \to \infty$ to complete the proof. □

By combining the above lemmas we can prove that the growth constant for partially directed paths is independent of the wedge angle.

Theorem 2.4. *For any given $p \in (0, \infty)$*

$$\mu_p^v = \mu_p^w = \mu = (1 + \sqrt{2}). \tag{2.7}$$

This shows that the dominant asymptotic behaviour of the number of walks is independent of the wedge angle. Below, we show that the leading sub-dominant behaviour is also independent of the wedge angle (*ie* for $p \ge 1$).

3 Functional equations for walks in wedges

3.1 The symmetric wedge model

Consider a model of partially directed paths in a symmetric wedge as illustrated in Figure 2 (right). If p is an integer or a rational number, then the path may touch vertices in the lines $Y = \pm pX$. These vertices are *visits* in the lines $Y = \pm pX$. In the event that p is an irrational number such visits cannot occur, however the path may approach arbitrarily close to the adsorbing lines (for large enough X-ordinate). In this paper we shall only consider the simplest version of this model, and we assume that p is a positive integer. Even in this case the model is apparently intractable, and we have only found the generating functions when $p = 1$.

We will derive a functional equation satisfied by the generating function of partially directed paths in $\mathcal{V}_p$ (those illustrated in Figure 2 (left)), by finding a recursive construction, similar to those in [3, 5] (and elsewhere).

Let x be the generating variable for horizontal edges in the path and let y be the generating variable for vertical edges in the path. Introduce generating variables a and b to be conjugate to the distances between the last vertex in the path and the line $Y = -pX$ and the line $Y = +pX$ respectively. The generating function of the paths are now denoted by $g_p(a, b; x, y) \equiv g_p(a, b)$ where the variables x and y are suppressed.

It turns out that the construction and resulting functional equation is simplified by considering only those partially directed walks that are either a single vertex (no edges) or end in a horizontal step. Let $f_p(a, b; x, y) \equiv f_p(a, b)$ be the generating function of such paths. It is simply related to $g_p(a, b)$ via

$$f_p(a, b) = 1 + x(ab)^p g_p(a, b). \tag{3.1}$$

We now obtain a functional equation satisfied by f_p by recursively constructing the paths column-by-column. Each path is either a single vertex, or can be constructed from a shorter path by appending either a horizontal step, or a sequence of up steps followed by a horizontal step, or a sequence of down steps followed by a horizontal step. See Figure 4.

Consider a path counted by $f_p(a, b)$, and see Figure 4.

- Appending a single horizontal step to its end increases the distance of the end point from both wedge boundary lines by p. Hence the generating function of paths with a horizontal edge appended is $x(ab)^p f_p(a, b)$.
- Appending an up step to the end of such a path increases the number of vertical steps by 1, increases the distance from the line $Y = -pX$ by 1 and decreases the distances from the line $Y = +pX$ by 1. Hence such a path has generating function $y(b/a)f_p(a, b)$. Hence appending some positive number of up steps gives $\frac{yb/a}{1-yb/a} f_p(a, b)$. Appending a horizontal step to the end of such a path gives (by the above reasoning) $x(ab)^p \frac{yb/a}{1-yb/a} f_p(a, b)$.
- Similarly appending some positive number of down steps followed by a horizontal step gives $x(ab)^p \frac{ya/b}{1-ya/b} f_p(a, b)$.

Unfortunately, when appending up or down steps it is possible that the resulting path will step outside of the wedge. Hence we must subtract off the contributions from such paths (Figure 4 right-top and -bottom).

- Consider a path that ends at a distance h_+ from the line $Y = +pX$. It we append more than h_+ up steps to the path then it will leave the wedge. We can decompose the resulting path into the original path with exactly h_+ up steps appended, and an "overhanging" Γ shaped path which is a sequence of some positive number of up steps and a horizontal step (see Figure 4 top-right).

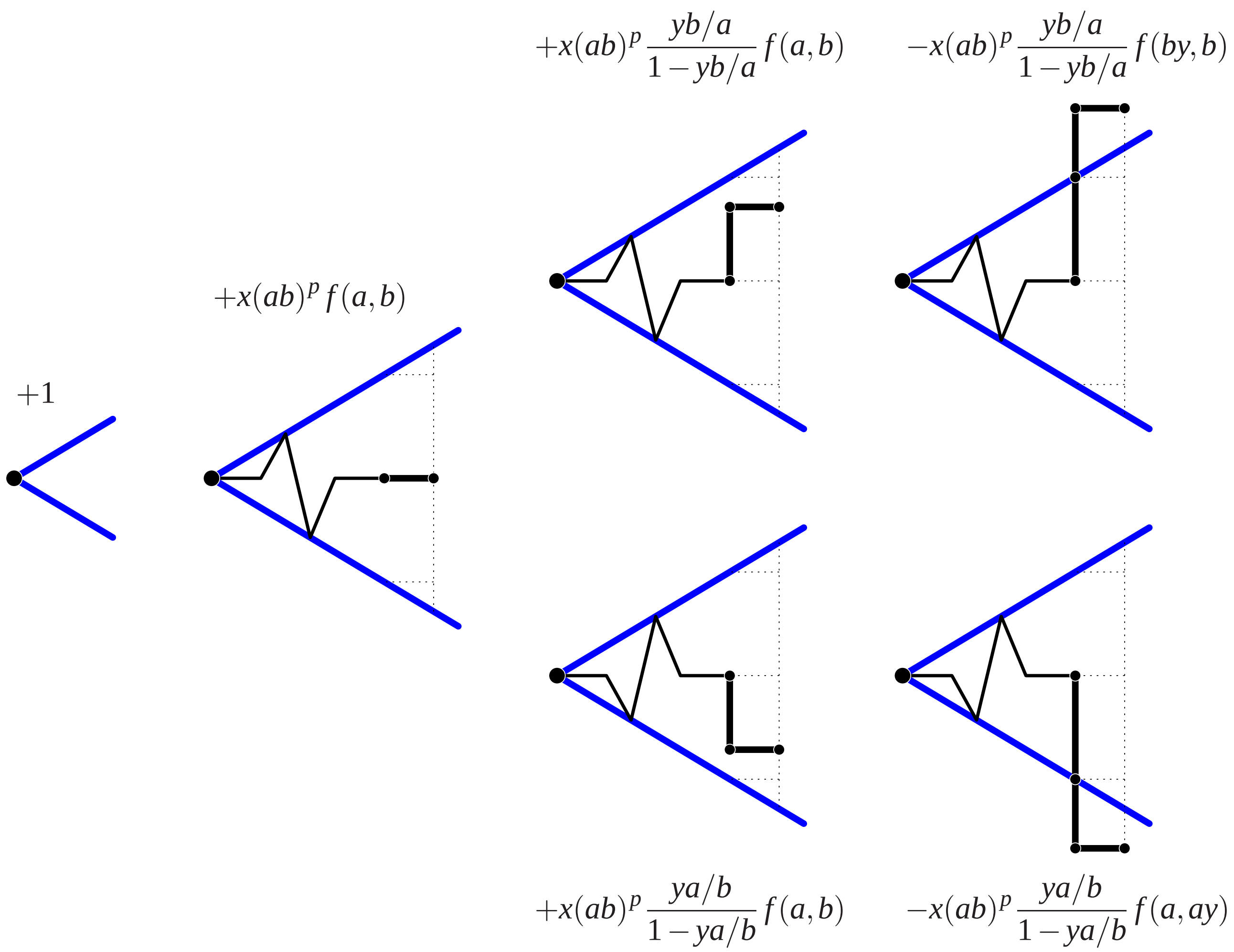

Figure 4: Constructing partially directed walks in the wedge $\mathcal{V}_p$. Every walk is either a single vertex, or can be obtained from a shorter walk by appending a horizontal edge (left), or a run of north steps and a horizontal edge or a run of south steps and a horizontal edge (centre-top and -bottom). Care must be taken to not step outside the wedge when appending north or south steps (right-top and -bottom).

Appending exactly h_+ up steps to the path increases the distance from $Y = -pX$ by h_+, decreases the distance from $Y = -pX$ to zero. This gives the generating function $f_p(by, b)$. The overhanging piece is (by the reasoning above) enumerated by $x(ab)^p \frac{yb/a}{1-yb/a}$.

Hence the g.f. of walks that leave the wedge is given by $x(ab)^p \frac{yb/a}{1-yb/a} f(by, b)$.

- Similarly when appending too many down steps we obtain configurations counted by $x(ab)^p \frac{ya/b}{1-ya/b} f_p(a, ay)$.

Using the above construction we arrive at the following theorem

Proposition 3.1. *The generating function $f_p(a, b; x, y) \equiv f_p(a, b)$ of partially directed walks* *ending in a horizontal step in the wedge $\mathcal{V}_p$ satisfies the following functional equation:*

$$\begin{aligned} f_p(a,b) = 1 &+ x(ab)^p f_p(a,b) \\ &+ x(ab)^p \frac{yb/a}{1-yb/a}\left(f_p(a,b) - f_p(by,b)\right) \\ &+ x(ab)^p \frac{ya/b}{1-ya/b}\left(f_p(a,b) - f_p(a,ay)\right) \end{aligned} \tag{3.2}$$

The generating function of all partially directed walks in $\mathcal{V}_p$ is given by

$$g_p(a,b) = x^{-1}(ab)^{-p}\left(f_p(a,b) - 1\right) \tag{3.3}$$

In the next section we turn to the problem of solving this functional equation.

3.2 The asymmetric wedge model

Let us now turn to the construction of partially directed paths in the asymmetric wedge $\mathcal{W}_p$. Let the generating function of all partially directed walks in this wedge be denoted $k_p(a,b;x,y) \equiv k_p(a,b)$ where the variables x and y are suppressed.

As above, the resulting functional equation satisfied by the generating function is simpler if we consider only those walks that are either a single vertex or end in a horizontal step. Let this generating function be denoted $h_p(a,b;x,y)$. This is simply related back to k_p by

$$h_p(a,b) = 1 + xa^p k_p(a,b). \tag{3.4}$$

We now use the same construction as was used above for the symmetric case — each walk is either a single vertex, or can be constructed from a shorter walk by appending either a horizontal step, or a run of up steps and a horizontal step, or a run of down steps and a horizontal step — see Figure 5. Again care must be taken not to step outside the wedge, and so those walks that do step outside the wedge must be removed. Indeed the argument is *almost* identical to that used above, except that a horizontal step contributes xa^p instead of $x(ab)^p$, since a horizontal step increases the distance from the line $Y = +pX$ by p, but does not change the distance from the line $Y = 0$.

The above construction gives the following theorem:

Proposition 3.2. *The generating function $h_p(a,b;x,y) \equiv f_p(a,b)$ of partially directed walks ending in a horizontal step in the wedge $\mathcal{W}_p$ satisfies the following functional equation:*

$$\begin{aligned} h_p(a,b) = 1 &+ xa^p h_p(a,b) \\ &+ xa^p \frac{yb/a}{1-yb/a}\left(h_p(a,b) - h_p(by,b)\right) \\ &+ xa^p \frac{ya/b}{1-ya/b}\left(h_p(a,b) - h_p(a,ay)\right) \end{aligned} \tag{3.5}$$

The generating function of all partially directed walks in $\mathcal{V}_p$ is given by

$$k_p(a,b) = x^{-1}a^{-p}\left(h_p(a,b) - 1\right) \tag{3.6}$$

We solve this equation in Section 5.

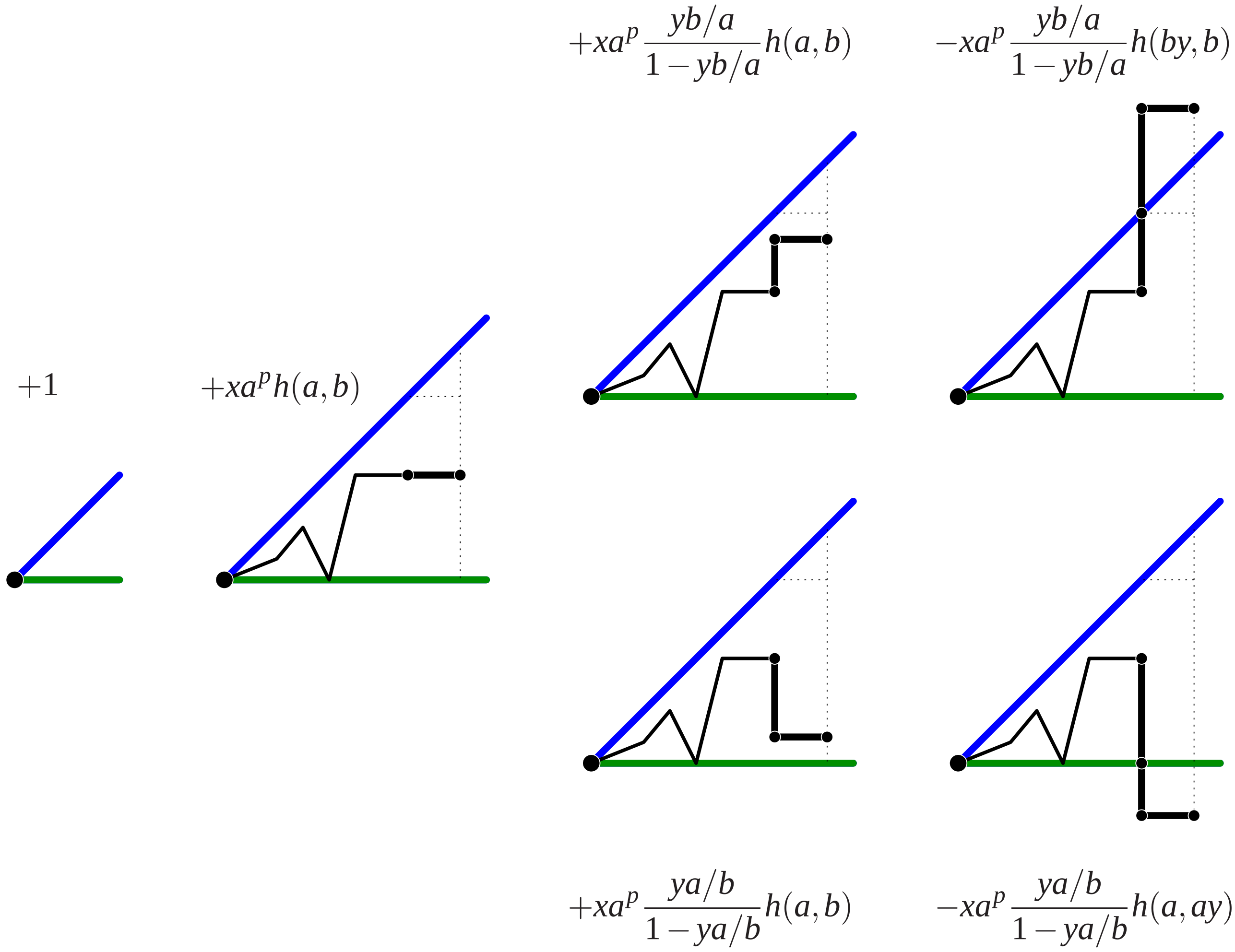

Figure 5: Constructing partially directed walks in the asymmetric wedge $\mathcal{W}_p$. Every walk is either a single vertex, or can be obtained from a shorter walk by appending a horizontal edge (left), or a run of north steps and a horizontal edge or a run of south steps and a horizontal edge (centre-top and -bottom). Care must be taken to not step outside the wedge when appending north or south steps (right-top and -bottom).

4 Solving the symmetric case

At first sight, one might try to solve equation (3.2) by the iteration method used in [3], however the coefficients of the equation are singular when $a = by$ and $b = ay$. Multiplying both sides of the equation by $(a - by)(b - ay)$ gives a non-singular equation, however when we set $a = by$ or $b = ay$ the equation reduces to a tautology.

Instead we apply a variation of the kernel method, which we call *the iterated kernel method.* This appears to be similar in flavour to the "obstinate kernel method" used by Bousquet-Mélou [5, 6]. We start by collecting all the $f(a, b)$ terms together on the left-hand side of the equation — this gives the *kernel form* of the equation:

$$K(a, b)\, f_p(a, b) = X(a, b) + Y(a, b)\, f_p(a, ya) + Z(a, b)\, f_p(yb, b), \tag{4.1}$$

where the functions $K(a,b)$, $X(a,b)$, $Y(a,b)$ and $Z(a,b)$ are given by

$$K(a,b) = (b - ya)(a - yb)(1 - x(ab)^p) - xy(ab)^p(a^2 + b^2 - 2yab), \tag{4.2a}$$
$$X(a,b) = (b - ya)(a - yb) \tag{4.2b}$$
$$Y(a,b) = -xya^{p+1}b^p(a - yb) \tag{4.2c}$$
$$Z(a,b) = -xya^pb^{p+1}(b - ya) \tag{4.2d}$$

for each integer $p \geq 1$. The function $K(a,b)$ is called the *kernel* of the equation. Note that the equation is symmetric under interchange of a and b:

$$f_p(a,b) = f_p(b,a) \qquad K(a,b) = K(b,a) \qquad X(a,b) = X(b,a) \qquad Y(a,b) = Z(b,a). \tag{4.3}$$

We solve equation (4.1) by substituting an infinite number of pairs of a and b values that set the kernel $K(a,b)$ to zero. While the method we describe below should work for general p, the resulting expressions are so complex that the process becomes intractable. This will become clear even in the case that $p = 1$, which we simplified only after significant effort.

4.1 Iterated kernel method for $\mathcal{V}_1$

When $p = 1$, the kernel becomes a quadratic function of a and b and we can explicitly write down (simple) expressions for its zeros. This is not generally true for larger values of p, and since simplifying our expressions requires that compositions of the zeros of the kernel must simplify as well - the general p case appears intractable from a practical point of view.

Thus, we restrict ourselves to $p = 1$. We write $f_1(a,b) \equiv F(a,b)$ and the coefficients in equation (4.1) become

$$K(a,b) = (xy^2a^2 - xa^2 - y)b^2 + (1 + y^2)ab - ya^2 \tag{4.4a}$$
$$X(a,b) = (b - ya)(a - yb) \tag{4.4b}$$
$$Y(a,b) = -xya^2b(a - yb) \tag{4.4c}$$
$$Z(a,b) = -xyab^2(b - ya) \tag{4.4d}$$

Let $\beta_\pm(a; x, y) \equiv \beta_\pm(a)$ be the zeros of $K(a,b)$ with respect to b. Hence

$$K(a, \beta_\pm(a)) = 0. \tag{4.5}$$

Thus, setting $b = \beta_\pm(a)$ removes $F(a,b)$ from equation (4.1). This is the key idea behind the "kernel method" which has been used to solve equations of this type (see [2] for example).

Unfortunately in this case, removing the kernel reduces the recurrence to an equation containing terms $F(a, ya)$ and $F(y\beta_\pm(a), \beta_\pm(a))$, which we cannot use immediately to solve for $F(a,b)$. Similar situations have been studied before using the "obstinate kernel method" ([5, 6] for example).

The method we use appears to be similar to the obstinate kernel method, except that instead of finding a finite number of pairs of values of a and b to set the kernel to zero we must use an infinite sequence of pairs. In this way, our "iterated kernel method" is related both to the kernel method and perhaps also to the iterative scheme used in [3].

The roots $\beta_\pm(a)$ can be determined explicitly:

$$\beta_\pm(a) = \frac{a}{2}\left(\frac{1 + y^2 \pm \sqrt{(1 - y^2)(1 - 4xya^2 - y^2)}}{y + xa^2 - xy^2a^2}\right). \tag{4.6}$$

Define the two roots:

$$\beta_1(a) \equiv \beta_-(a) = ya + O(xy^2a^3), \tag{4.7}$$
$$\beta_{-1}(a) \equiv \beta_+(a) = a/y + O(xy^{-2}a). \tag{4.8}$$

as power series in a. Later we require our solution to be a formal power series in t (after setting $x = y = t$) and one can confirm that $\beta_1(a)$ defines a formal power series in t.

Since a is a variable, we are able to substitute something else for it; substituting $a \mapsto \beta_1(a)$ into equation (4.5) gives

$$K(\beta_1(a), \beta_1(\beta_1(a)) = 0. \tag{4.9}$$

Hence the pair $(a, b) = \big(\beta_1(a), \beta_1(\beta_1(a))\big)$ also sets the kernel to zero. We can continue in this way. Hence we need to define the repeated composition of $\beta_1(a)$ with itself:

$$\beta_n(a) = \beta_1^{(n)}(a) = \underbrace{\beta_1 \circ \beta_1 \circ \ldots \circ \beta_1}_{n}(a). \tag{4.10}$$

Note that

$$\beta_{-1} \circ \beta_1(a) = \beta_1 \circ \beta_{-1}(a) = a, \qquad \text{and} \tag{4.11}$$
$$\beta_n(a) = ay^n + O(xy^{n+1}a^3). \tag{4.12}$$

There is no finite value of n such that $\beta_n = \beta_0$. If we further define $\beta_0(a) = a$ and $\beta_{-n}(a)$ by composition of $\beta_{-1}(a)$, then the functions $\{\beta_n \,|\, n \in \mathbb{Z}\}$ form an infinite group with identity β_0 and inverses $\beta_n \circ \beta_{-n} = \beta_0$.

These observations are enough to iterate the functional equation to find a solution. Set $b = \beta_1(a)$ in equation (4.1), and set $a = \beta_n(a)$ for any finite $n \geq 0$. Then since $K(\beta_n(a), \beta_{n+1}(a)) = 0$, we have

$$\begin{aligned} X(\beta_n(a), \beta_{n+1}(a)) + Y(\beta_n(a), \beta_{n+1}(a))\, F(\beta_n(a), y\beta_n(a)) \\ + Z(\beta_n(a), \beta_{n+1}(a))\, F(y\beta_{n+1}(a), \beta_{n+1}(a)) = 0. \end{aligned} \tag{4.13}$$

We can then solve this equation for $F(\beta_n(a), y\beta_n(a))$:

$$\begin{aligned} F(\beta_n(a), y\beta_n(a)) = &- \left[\frac{X(\beta_n(a), \beta_{n+1}(a))}{Y(\beta_n(a), \beta_{n+1}(a))}\right] \\ &- \left[\frac{Z(\beta_n(a), \beta_{n+1}(a))}{Y(\beta_n(a), \beta_{n+1}(a))}\right] F(y\beta_{n+1}(a), \beta_{n+1}(a)) \end{aligned} \tag{4.14}$$

We can simplify the above by defining

$$\mathcal{F}_n(a) = F(\beta_n(a), y\beta_n(a)) = F(y\beta_n(a), \beta_n(a)),$$
$$\mathcal{X}_n(a) = -\left[\frac{X(\beta_n(a), \beta_{n+1}(a))}{Y(\beta_n(a), \beta_{n+1}(a))}\right], \tag{4.15}$$
$$\mathcal{Z}_n(a) = -\left[\frac{Z(\beta_n(a), \beta_{n+1}(a))}{Y(\beta_n(a), \beta_{n+1}(a))}\right], \tag{4.16}$$

where we have made use of the symmetry $F(a, b) = F(b, a)$. While this symmetry is not essential, it does make the solution substantially simpler. Instead of exploiting this symmetry we could iterate again to find $F(\beta_n(a), y\beta_n(a))$ in terms of $F(\beta_{n+2}(a), y\beta_{n+2}(a))$. Indeed this is what is required to solve walks in the asymmetric wedge $\mathcal{W}_1$ (see Section 5 below).

Equation (4.14) may be written as

$$\mathcal{F}_n(a) = \mathcal{X}_n(a) + \mathcal{Z}_n(a)\mathcal{F}_{n+1}(a). \tag{4.17}$$

Starting at $n = 0$, this can be iterated to get a series solution for $\mathcal{F}_0(a)$:

$$F(a, ya) = \mathcal{F}_0(a) = \sum_{n=0}^{\infty} \mathcal{X}_n(a) \prod_{k=0}^{n-1} \mathcal{Z}_k(a), \tag{4.18}$$

where we have *assumed* that the above sum converges (we will show that this is the case). This also gives $F(yb, b)$:

$$F(yb, b) = F(b, yb) = \mathcal{F}_0(b) = \sum_{n=0}^{\infty} \mathcal{X}_n(b) \prod_{k=0}^{n-1} \mathcal{Z}_k(b). \tag{4.19}$$

This allows us to write down the solution for $F(a, b)$:

$$f_p(a, b) = \frac{X(a, b)}{K(a, b)} + \frac{Y(a, b)}{K(a, b)} \sum_{n=0}^{\infty} \mathcal{X}_n(a) \prod_{k=0}^{n-1} \mathcal{Z}_k(a) + \frac{Z(a, b)}{K(a, b)} \sum_{n=0}^{\infty} \mathcal{X}_n(b) \prod_{k=0}^{n-1} \mathcal{Z}_k(b). \tag{4.20}$$

Of course, the above "solution" still contains many complicated algebraic functions in the form of the $\beta_n(a)$. It is quite surprising (at least to the authors!) is that these functions can be drastically simplified.

4.2 An explicit expression for $f_1(1, 1)$

We have outlined above the iterated kernel method that we shall use to write down the generating function $f_1(a, b) = F(a, b)$. We are primarily interested in the number of paths (and not the location of their endpoints), so we will actually focus on the function $F(1, 1)$.

We start by considering the $\beta_n(a)$ functions. It is quite surprising that while $\beta_n(a)$ is (upon superficial inspection for small n) very complicated, its reciprocal appears relatively simple. Examining equations (4.4a) and (4.6) one obtains

$$\frac{1}{\beta_1(a)} + \frac{1}{\beta_{-1}(a)} = \frac{1 + y^2}{y} \frac{1}{a}. \tag{4.21}$$

Substituting $a = \beta_{n-1}(a)$ in the above, and using the group properties of β_n leads to the following three term recurrence for β_n:

$$\frac{1}{\beta_n} = \frac{1 + y^2}{y} \frac{1}{\beta_{n-1}} - \frac{1}{\beta_{n-2}}. \tag{4.22}$$

Since β_0 is the identity, and β_1 is given explicitly by $\beta_-(a)$ in equation (4.6), the recurrence above can be iterated to get a solution for $\beta_n(a)$:

$$\frac{1}{\beta_n(a)} = \frac{y(1 - y^{2n})}{y^n(1 - y^2)} \frac{1}{\beta_1(a)} - \frac{y^2(1 - y^{2n-2})}{y^n(1 - y^2)} \frac{1}{a}. \tag{4.23}$$

By using the expressions for $X(a, b)$, $Y(a, b)$ and $Z(a, b)$ in equation (4.4) to determine $\mathcal{X}(a, b)$ and $\mathcal{Z}(a, b)$, one obtains

$$F(a, ya) = \sum_{n=0}^{\infty} (-1)^n \left[\frac{\beta_{n+1} - y\beta_n}{xya\beta_n\beta_{n+1}} \right] \prod_{k=0}^{n-1} \left(\frac{\beta_{k+1} - y\beta_k}{\beta_k - y\beta_{k+1}} \right). \tag{4.24}$$

Substituting the expression for $\beta_n(a)$ given in equation (4.23) and simplifying gives:

$$\frac{\beta_{n+1} - y\beta_n}{xya\beta_n\beta_{n+1}} = y^n \left[\frac{1}{a} - \frac{y}{\beta_1}\right], \qquad \text{and} \tag{4.25}$$

$$\frac{\beta_{k+1} - y\beta_k}{\beta_k - y\beta_{k+1}} = y^{2k+1} \left[\frac{1}{xya^2} - \frac{1}{xa\beta_1} - 1\right]. \tag{4.26}$$

Using these, one can get an explicit expression for the generating function $F(a, ya)$:

$$F(a, ya) = \left[\frac{1}{xya^2} - \frac{1}{xa\beta_1}\right] \sum_{n=0}^{\infty} (-1)^n y^{n(n+1)} \left(\frac{1}{xya^2} - \frac{1}{xa\beta_1} - 1\right)^n. \tag{4.27}$$

By defining

$$Q(a; x, y) = \left(\frac{1}{xa^2} - \frac{y}{xa\beta_1} - y\right). \tag{4.28}$$

the above expression for $F(a, ya)$ can be further simplified to

$$F(a, ya) = \left[1 + \frac{Q(a; x, y)}{y}\right] \sum_{n=0}^{\infty} (-1)^n y^{n^2} Q(a; x, y)^n. \tag{4.29}$$

Using the $a \leftrightarrow b$ symmetry of $F(a, b)$, we can get a similar expression for $F(yb, b)$, and so finally $F(a, b)$.

$$f(a, b) = \frac{X(a, b)}{K(a, b)} + \frac{Y(a, b)}{K(a, b)} \left(1 + \frac{Q(a)}{y}\right) \sum_{n \geq 0} (-1)^n Q(a)^n y^{n^2} + \frac{Z(a, b)}{K(a, b)} \left(1 + \frac{Q(b)}{y}\right) \sum_{n \geq 0} (-1)^n Q(b)^n y^{n^2} \tag{4.30}$$

We can reduce the above equation by considering only the number of walks of length n (by setting $a = b = 1, x = y = t$):

Proposition 4.1. *The generating function of partially directed walks ending in a horizontal step in the wedge $\mathcal{V}_1$ is*

$$f_1(1, 1) = \frac{1 - t}{1 - 2t - t^2} - \frac{1 - t^2 - \sqrt{(1 - t^2)(1 - 5t^2)}}{1 - 2t - t^2} \sum_{n=0}^{\infty} (-1)^n t^{n^2} Q(1; t, t)^n, \tag{4.31}$$

where t counts the number of edges and

$$Q(1; t, t) = (1 - 3t^2 - \sqrt{(1 - t^2)(1 - 5t^2)})/2t. \tag{4.32}$$

The generating function of all paths in $\mathcal{V}_1$ is then found using equation (3.1):

$$g_1(1, 1) = \frac{1 + t}{1 - 2t - t^2} - \frac{1 - t^2 - \sqrt{(1 - t^2)(1 - 5t^2)}}{t(1 - 2t - t^2)} \sum_{n=0}^{\infty} (-1)^n t^{n^2} Q(1; t, t)^n. \tag{4.33}$$

Firstly we note that $F(a, ya)$ counts all partially directed paths in the wedge $\mathcal{V}_1$ whose last vertex ends in the line $Y = -pX$. Additionally we note that the generating function $Q(a; x, y)/y$ counts the number of partially directed paths starting at the origin, lying on or above the line $Y = -pX$ and whose last vertex lies in the line $Y = -pX$. Hence $Q(a)/y$ counts a very similar set of paths to $F(a, ya)$, except that the paths counted by Q are not confined by the line $Y = pX$.

In light of the above interpretation of the function Q, we expended considerable effort to uncover a more direct combinatorial derivation of the alternating sum in equation (4.29). There appears to be some inclusion-exclusion process underlying this, but unfortunately we have not made progress in this respect.

4.3 Asymptotics for $p = 1$

The asymptotics of the number of partially directed paths in the symmetric wedge with $p = 1$ can be analysed by examining the singularities of the generating function $g_1(1,1)$ in equation (4.33). Singularities arise either as zeros of the factor $(1 - 2t - t^2)$ in equation (4.33), or as singularities in $\sqrt{(1-t^2)(1-5t^2)}$, or as singularities in the series $\sum_{n=0}^{\infty}(-1)^n t^{n^2} Q(1;t,t)^n$.

An examination of $g_1(1,1)$ shows that it has simple poles at the solution of $(1 - 2t - t^2) = 0$, or when $t = -1 \pm \sqrt{2}$. We note that $\sqrt{(1-t^2)(1-5t^2)}$ has branch-points (square root singularities) at $t = \pm 1$ and again at $t = \pm 1/\sqrt{5}$. The series $\sum_{n=0}^{\infty}(-1)^n t^{n^2} Q(1;t,t)^n$ is a Jacobi θ-function and it is convergent inside the unit circle except at singularities of $Q(1;t,t)$; that is, when $t = \pm 1/\sqrt{5}$.

The dominant singularity is the simple pole at $\sqrt{2} - 1$, while the next sub-dominant contributions to the asymptotics will be given by the singularities at $t = \pm 1/\sqrt{5}$. These two sub-dominant singularities will give a parity effect. The contributions from these singularities allow us to write down the asymptotic form of $v_{n,1}$.

Proposition 4.2. *The number of paths in the wedge $\mathcal{V}_1$ is asymptotic to*

$$v_{n,1} = A_0\left(1+\sqrt{2}\right)^n + \frac{5^{n/2}}{(n+1)^{3/2}}\Big(A_1 + (-1)^n A_2 + O(1/n)\Big). \tag{4.34}$$

Where the constants are

$$A_0 = 0.27730985348603118827\ldots, \tag{4.35a}$$
$$A_1 = 3.71410486533662324953\ldots, \tag{4.35b}$$
$$A_2 = 0.20697997020804157910\ldots. \tag{4.35c}$$

We note that the constants were derived by expanding the expression for $g_1(1,1)$ about $t = \sqrt{2} - 1$ and $t = \pm 1/\sqrt{5}$ (or rather the first 40 or so terms of the sum). These were then checked using both Bruno Salvy's *gdev* package for Maple [24] and by direct examination of $v_{n,1}$ for $n \le 1000$. The above formula is quite precise and it correctly estimates $v_{10,1}, v_{20,1}, v_{30,1}$ and $v_{40,1}$ to within $7\%, 1\%, 0.2\%$ and 0.06% respectively.

Note that the above result implies that walks in the wedge $\mathcal{V}_p$ have the same dominant asymptotic behaviour as walks with no bounding wedge (see equation (1.4)). Since the number of walks in any wedge $\mathcal{V}_p$ for $1 \le p < \infty$ is bounded between the number of walks in $\mathcal{V}_1$ and partially directed walks with no bounding wedge, we have the following result:

Corollary 4.3. *The number of partially directed walks in the wedge $\mathcal{V}_p$, $c_n^{(p)}$ obeys the following inequality*

$$0.2773\ldots \le \lim_{n\to\infty} \frac{c_n^{(p)}}{(1+\sqrt{2})^n} \le (1+\sqrt{2})/2 = 1.2071\ldots \tag{4.36}$$

for any $1 \le p < \infty$.

5 Partially Directed Paths in the Asymmetric Wedge

In this section we turn our attention to the model in Figure 2 (right). The partially directed path is confined to an asymmetric wedge, $\mathcal{W}_p$, and its generating function does not have the $a \leftrightarrow b$ symmetry we have exploited in solving for $f_1(a,b)$ in the previous section.

We proceed by examining the generating function of walks that end in a horizontal step. The functional equation for these walks is given in Proposition 3.2 and we can arrange equation (3.5) in kernel form:

$$K(a,b)\,h_p(a,b) = X(a,b) + Y(a,b)\,h_p(a,ya) + Z(a,b)\,h_p(yb,b) \tag{5.1}$$

where

$$K(a,b) = (b-ya)(a-yb)(1-xa^p) - xya^p(a^2+b^2-2yab), \tag{5.2a}$$
$$X(a,b) = (b-ya)(a-yb), \tag{5.2b}$$
$$Y(a,b) = -xya^{p+1}(a-yb), \tag{5.2c}$$
$$Z(a,b) = -xya^p b(b-ya). \tag{5.2d}$$

This functional equation is very similar to that of the symmetric wedge given in equation (4.1). However we no longer have $a \leftrightarrow b$ symmetry and this means that we have to work quite a bit harder and we concentrate only on the case $p=1$. We will write $h_1(a,b) \equiv H(a,b)$ for the remainder of this section.

5.1 Solving for $H(a,b)$ when $p=1$

For the remainder of this section we concentrate on the case $p=1$ and walks in the 45° wedge $\mathcal{W}_1$. The kernel $K(a,b)$ (given in equation (5.2)) is no longer symmetric in a and b, nor is the desired generating function $H(a,b)$. In order to repeat the iterated kernel method as described in Section 4.1 we must now consider the solutions of the kernel as functions of a and b. These solutions are defined by $K(a,\beta(a))=0$ and $K(\alpha(b),b)=0$:

$$\beta_\pm(a) = \frac{a}{2y}\left[1+y^2-x(1-y^2)a \pm \sqrt{(1-y^2)((1-xa)^2-y^2(1+xa)^2)}\right] \tag{5.3}$$

and

$$\alpha_\pm(b) = \frac{b}{2}\left[\frac{1+y^2 \pm \sqrt{(1-y^2)(1-y^2-4xyb)}}{y+x(1-y^2)b}\right]. \tag{5.4}$$

One can confirm that $\alpha_-(b)$ and $\beta_-(a)$ both define formal power series in b and a (respectively). Additionally these same choices (when $x=y=t$) also define formal power series in t — which we will require for our solution. Write these as $\alpha_1(b)$ and $\beta_1(b)$, and the other roots as $\alpha_{-1}(b)$ and $\beta_{-1}(a)$.

In Section 4.1 we considered composing $\beta(a)$ with itself, however due to the asymmetry of the kernel we now need to consider mixed compositions $\beta(\alpha(b))$ and $\alpha(\beta(a))$. Indeed, we find that

$$\alpha_{\pm 1}(\beta_{\mp 1}(a)) = a, \tag{5.5a}$$
$$\beta_{\pm 1}(\alpha_{\mp 1}(b)) = b. \tag{5.5b}$$

We will need the function $\gamma(a) = \alpha_1(\beta_1(a))$, and define its nested composition by $\gamma_n(a) = \gamma(\gamma_{n-1}(a))$ with $\gamma_0(a)=a$. Note that

$$\gamma_n(a) = y^{2n}a + O(xy^{2n}a^2). \tag{5.6}$$

We can now repeat the iterated kernel method in the new asymmetric setting. Setting $b=\beta_1(a)$ in equation (5.1) gives

$$0 = X(a,\beta_1(a)) + Y(a,\beta_1(a))H(a,ya) + Z(a,\beta_1(a))H(y\beta_1(a),\beta_1(a)). \tag{5.7}$$

Since there is apparently not a simple relation between $H(by,b)$ and $H(b,by)$, this equation cannot be iterated to find a solution. Instead, it turns out that the other roots of the kernel must be considered as well.

Setting $a=\alpha_1(b)$ gives:

$$0 = X(\alpha_1(b),b) + Y(\alpha_1(b),b)H(\alpha_1(b),y\alpha_1(b)) + Z(\alpha_1(b),b)H(yb,b). \tag{5.8}$$

Now set $b = \beta_1(a)$ in the above equation

$$0 = X(\gamma(a), \beta_1(a)) + Y(\gamma_1(a), \beta_1(a))H(\gamma_1(a), y\gamma_1(a)) + Z(\gamma_1(a), \beta_1(a))H(y\beta_1(a), \beta_1(a)). \quad (5.9)$$

We can now eliminate $H(y\beta_1(a), \beta_1(a))$ between equations (5.7) and (5.9) and solve for $H(a, ya)$. This gives

$$H(a, ya) = -\left[\frac{X(a, \beta_1(a))}{Y(a, \beta_1(a))}\right] + \left[\frac{Z(a, \beta_1(a))}{Y(a, \beta_1(a))}\right]\left[\frac{X(\gamma_1(a), \beta_1(a))}{Z(\gamma_1(a), \beta_1(a))}\right] + \left[\frac{Z(a, \beta_1(a))}{Y(a, \beta_1(a))}\right]\left[\frac{Y(\gamma_1(a), \beta_1(a))}{Z(\gamma_1(a), \beta_1(a))}\right] H(\gamma_1(a), y\gamma_1(a)). \quad (5.10)$$

We can now iterate the above equation by substituting $a = \gamma_{n-1}(a)$. Define

$$\mathcal{X}_n(a) = -\frac{X(\gamma_n(a), \beta_1(\gamma_n(a)))}{Y(\gamma_n(a), \beta_1(\gamma_n(a)))} = \frac{\beta(\gamma_n) - y\gamma_n}{xy\gamma_n^2}, \quad (5.11a)$$

$$\mathcal{Y}_n(a) = \frac{Z(\gamma_n(a), \beta_1(\gamma_n(a)))}{Y(\gamma_n(a), \beta_1(\gamma_n(a)))} = \frac{\beta(\gamma_n)}{\gamma_n}\left(\frac{\beta(\gamma_n) - y\gamma_n}{\gamma_n - y\beta(\gamma_n)}\right), \quad (5.11b)$$

$$\mathcal{Z}_n(a) = \frac{X(\gamma_{n+1}(a), \beta_1(\gamma_n(a)))}{Z(\gamma_{n+1}(a), \beta_1(\gamma_n(a)))} = -\frac{\gamma_{n+1} - y\beta(\gamma_n)}{xy\gamma_{n+1}\beta(\gamma_n)}, \quad (5.11c)$$

$$\mathcal{A}_n(a) = \frac{Y(\gamma_{n+1}(a), \beta_1(\gamma_n(a)))}{Z(\gamma_{n+1}(a)), \beta_1(\gamma_n(a)))} = \frac{\gamma_{n+1}}{\beta(\gamma_n)}\left(\frac{\gamma_{n+1} - y\beta(\gamma_n)}{\beta(\gamma_n) - y\gamma_{n+1}}\right). \quad (5.11d)$$

And further define

$$\mathcal{B}_n(a) = \mathcal{X}_n(a) + \mathcal{Y}_n(a)\mathcal{Z}_n(a), \qquad \mathcal{C}_n(a) = \mathcal{Y}_n(a)\mathcal{A}_n(a). \quad (5.12)$$

Equation (5.10) now becomes:

$$H(\gamma_n(a), y\gamma_n(a)) = \mathcal{B}_n + \mathcal{C}_n H(\gamma_{n+1}(a), y\gamma_{n+1}(a)). \quad (5.13)$$

We obtain a solution for $H(a, ya)$ by iterating the above equation

$$H(a, ya) = \mathcal{B}_0 + \mathcal{C}_0\mathcal{B}_1 + \mathcal{C}_0\mathcal{C}_1\mathcal{B}_2 + \cdots = \sum_{n=0}^{\infty} \mathcal{B}_n(a) \prod_{m=0}^{n-1} \mathcal{C}_m(a). \quad (5.14)$$

As was the case for the symmetric wedge, we are able to simplify the above expression by rewriting $\gamma_n(a)$ in terms of the original kernel roots and thereby rewrite the expressions for $\mathcal{B}_n$ and $\mathcal{C}_n$.

In order to find $H(a, b)$ from equation (5.1), we need both $H(a, ya)$ *and*$H(yb, b)$. Equation (5.8) gives $H(b, yb)$ in terms of $H(\alpha_1(b), y\alpha_1(b))$:

$$H(yb, b) = -\frac{X(\alpha_1(b), b)}{Z(\alpha_1(b), b)} - \frac{Y(\alpha_1(b), b)}{Z(\alpha_1(b), b)} H(\alpha_1(b), y\alpha_1(b)) \quad (5.15)$$

So using the above expressions for $H(a, ya)$ and $H(yb, b)$ we have, at least in principle, a solution for $H(a, b)$:

$$H(a, b) = \frac{X(a, b)}{K(a, b)} - \frac{Z(a, b)X(\alpha_1(b), b)}{Z(\alpha_1(b), b)K(a, b)} + \frac{Y(a, b)}{K(a, b)} H(a, ya) - \frac{Z(a, b)Y(\alpha_1(b), b)}{Z(\alpha_1(b), b)K(a, b)} H(\alpha_1(b), y\alpha_1(b)) \quad (5.16)$$

Of course, we would like to be able to simplify the above expression. In particular we would like to rewrite $\gamma_n(a)$ and $\gamma_n(\alpha_1(b))$ in terms of simpler functions, as we did for $\beta_n(a)$ in Section 4.2.

Interestingly enough, it is possible to determine $H(a, ya)$ in equation (5.14) by inspection of the terms in this expression. Putting $x = y = t$, one obtains

$$\begin{aligned} H(a,ta) &= -\sum_{n=0}^{\infty}\left[\frac{t^{2(n+1)^2-3}}{a}\right]\left[\frac{a-\beta t-a\beta t^2+a\beta t^{2n+2}}{a(1+\beta)t^{2n}-\beta(a+t^{2n-1})}\right] \\ &\quad\times\left[\frac{a-\beta t-a\beta t^2-\beta t^{4n+1}+a(1+\beta)t^{4n+2}}{a+\left[\frac{1-t^{2n}}{1-t^2}\right](a(1-\beta)t^2+a(1+\beta)t^{2n+2})-\left[\frac{1-t^{4n}}{1-t^2}\right]\beta t}\right] \\ &\quad\times\prod_{m=0}^{n}\left[\frac{a(1+\beta)t^{2m}-\beta(a+t^{2m-1})}{a-\beta t-a\beta t^2+a\beta t^{2m+2}}\right]. \end{aligned} \tag{5.17}$$

From this expression one may determine $H(bt, t)$ from equation (5.8), and thus an expression for $H(a, b)$. While the resulting expression gives a series expansion for the numbers of paths, it is not very useful because it is so complex. In the next section we proceed by simplifying expressions for the compositions of the α's and β's above; ultimately this will lead to a simpler expression for $H(a, b)$.

5.2 Simplifying things

In much the same way as for the symmetric case, we can find simple expressions for the $1/\gamma_n(a)$ in terms of the original kernel roots. Consideration of the kernel and its roots gives:

$$\frac{1}{\alpha_{-1}(b)}+\frac{1}{\alpha_{+1}(b)}=\frac{1+y^2}{yb}, \tag{5.18a}$$

$$\frac{1}{\beta_{-1}(a)}+\frac{1}{\beta_{+1}(a)}=\frac{1+y^2}{ya}-\frac{x(1-y^2)}{y}. \tag{5.18b}$$

Since certain compositions of α and β give the identity (see equations (5.5)), we have the additional relations:

$$\frac{1}{\alpha_1(\beta_1(a))}=\frac{1}{\gamma_1(a)}=\frac{1+y^2}{y\beta_1(a)}-\frac{1}{a}, \tag{5.19a}$$

$$\frac{1}{\beta_1(\alpha_1(b))}=\frac{1+y^2}{y\alpha_1(b)}-\frac{1}{b}-\frac{x(1-y^2)}{y}. \tag{5.19b}$$

We note that the last term in equation (5.19b) means that the resulting expressions for $\gamma_n(a)$ are more complicated than those for $\beta_n(a)$ for the symmetric case (see equation (4.23)); this in turn leads to a significantly more complicated solution.

Setting $a = \gamma_{n-1}(a)$ and $b = \beta_1(\gamma_{n-1}(a))$ in the above two equations give:

$$\frac{1}{\gamma_n(a)}=\frac{1+y^2}{y\beta_1(\gamma_{n-1}(a))}-\frac{1}{\gamma_{n-1}(a)}, \tag{5.20a}$$

$$\frac{1}{\beta_1(\gamma_n(a))}=\frac{1+y^2}{y\gamma_n(a)}-\frac{1}{\beta_1(\gamma_{n-1}(a))}-\frac{x(1-y^2)}{y}. \tag{5.20b}$$

These equations can be solved:

$$\begin{aligned} \frac{1}{\gamma_n(a)} &= \frac{1-y^{4n}}{y^{2n-1}(1-y^2)\beta_1(a)} - \frac{1-y^{4n-2}}{y^{2n-2}(1-y^2)a} - \frac{x(1-y^{2n})(1-y^{2n-2})}{y^{2n-2}(1-y^2)}, \\ &= \frac{1}{1-y^2}\left(x(1+y^2) + Q(a)\, y^{2n} + y\bar{Q}(a)\, y^{-2n}\right), \end{aligned} \tag{5.21a}$$

$$\begin{aligned} \frac{1}{\beta_1(\gamma_n(a))} &= \frac{1-y^{4n+2}}{y^{2n}(1-y^2)\beta_1(a)} - \frac{1-y^{4n}}{y^{2n-1}(1-y^2)a} - \frac{x(1-y^{2n})^2}{y^{2n-1}(1-y^2)}, \\ &= \frac{1}{1-y^2}\left(2xy + yQ(a)\, y^{2n} + \bar{Q}(a)\, y^{-2n}\right). \end{aligned} \tag{5.21b}$$

where we have used

$$Q(a) = \frac{1}{a} - \frac{y}{\beta_1(a)} - x \qquad \bar{Q}(a) = \frac{1}{\beta_1(a)} - \frac{y}{a} - xy \tag{5.22}$$

Note that $\bar{Q}(a)Q(a) = x^2y$. In fact we can reduce the above expressions for γ_n and $\beta(\gamma_n)$ even further using this fact:

$$\frac{1}{\gamma_n(a)} = \frac{(x+y^{2n-2}Q)(x+y^{2n}Q)}{y^{2n-2}(1-y^2)Q} \tag{5.23a}$$

$$\frac{1}{\beta(\gamma_n(a))} = \frac{(x+y^{2n}Q)^2}{y^{2n-1}(1-y^2)Q} \tag{5.23b}$$

The above then lead to the following expressions that will be useful in writing down our solution:

$$\frac{1}{\gamma_n(a)} - \frac{y}{\beta_1(\gamma_n(a))} = (x+y^{2n}Q), \tag{5.24a}$$

$$\frac{1}{\beta_1(\gamma_n(a))} - \frac{y}{\gamma_n(a)} = \frac{x}{y^{2n-1}Q}\left(x+y^{2n}Q\right), \tag{5.24b}$$

$$\frac{1}{\beta_1(\gamma_n(a))} - \frac{y}{\gamma_{n+1}(a)} = y\left(x+y^{2n}Q\right), \tag{5.24c}$$

$$\frac{1}{\gamma_{n+1}(a)} - \frac{y}{\beta_1(\gamma_n(a))} = \frac{x}{y^{2n}Q}\left(x+y^{2n}Q\right) \tag{5.24d}$$

This in turn lets us write

$$\frac{\beta(\gamma_n) - y\gamma_n}{\gamma_n - y\beta(\gamma_n)} = \frac{y^{2n-1}}{x}Q \tag{5.25a}$$

$$\frac{\gamma_{n+1} - y\beta(\gamma_n)}{\beta(\gamma_n) - y\gamma_{n+1}} = \frac{y^{2n+1}}{x}Q \tag{5.25b}$$

$$\tag{5.25c}$$

where we have made use of the fact that $\bar{Q} = x^2y/Q$. Hence $\mathcal{C}_n = \mathcal{Y}_n\mathcal{A}_n$ can now be written as

$$\mathcal{C}_n = \frac{\gamma_{n+1}}{\gamma_n}\left(\frac{\beta(\gamma_n) - y\gamma_n}{\gamma_n - y\beta(\gamma_n)}\right)\left(\frac{\gamma_{n+1} - y\beta(\gamma_n)}{\beta(\gamma_n) - y\gamma_{n+1}}\right) = \frac{\gamma_{n+1}}{\gamma_n}\cdot\frac{y^{4n}}{x^2}Q^2 \tag{5.26}$$

In a similar way we find that

$$\frac{\beta(\gamma_n) - y\gamma_n}{\gamma_n^2} = y(x+y^{2n-2}Q) \tag{5.27a}$$

$$\frac{\gamma_{n+1} - y\beta(\gamma_n)}{\gamma_n\gamma_{n+1}} = y^2(x+y^{2n-2}Q) \tag{5.27b}$$

which allows us to also simplify the expression for $\mathcal{B}_n = \mathcal{X}_n + \mathcal{Y}_n\mathcal{Z}_n$:

$$\mathcal{X}_n = \frac{1}{xy}\left(\frac{\beta(\gamma_n) - y\gamma_n}{\gamma_n^2}\right) \qquad = \frac{x + y^{2n-2}Q}{x} \tag{5.28a}$$

$$\mathcal{Y}_n\mathcal{Z}_n = -\frac{1}{xy}\left(\frac{\gamma_{n+1} - y\beta(\gamma_n)}{\gamma_n\gamma_{n+1}}\right)\left(\frac{\beta(\gamma_n) - y\gamma_n}{\gamma_n - y\beta(\gamma_n)}\right) \qquad = -\frac{y^{2n}Q}{x^2}\left(x + y^{2n-2}Q\right) \tag{5.28b}$$

$$\mathcal{B}_n = \mathcal{X}_n + \mathcal{Y}_n\mathcal{Z}_n \qquad = \frac{1}{x^2}\left(x + y^{2n-2}Q\right)\left(x - y^{2n}Q\right) \tag{5.28c}$$

Substituting the above into the expression for $H(a, ya)$ in equation (5.14) gives:

$$\begin{aligned} H(a, ya) &= \sum_{n=0}^{\infty} \mathcal{B}_n(a) \prod_{m=0}^{n-1} \mathcal{C}_m(a). \\ &= \sum_{n=0}^{\infty} \frac{1}{x^2}\left(x + y^{2n-2}Q\right)\left(x - y^{2n}Q\right)\frac{\gamma_n(a)}{a}\left(\frac{Q}{x}\right)^{2n} y^{2n(n-1)} \\ &= \frac{(1-y^2)Q}{ax^2y^2} \sum_{n=0}^{\infty} \frac{(x - y^{2n}Q)}{(x + y^{2n}Q)}\left(\frac{Q}{x}\right)^{2n} y^{2n^2} \end{aligned} \tag{5.29}$$

which is a significant simplification of equation (5.17). We can now substitute this into equation (5.16) to obtain $H(a, b)$. This requires us to compute $H(\alpha_1(b), y\alpha_1(b))$ from the above expression. Let

$$P(b) = Q(\alpha_1(b)) = \frac{y}{2b}\left(1 - 2xyb - y^2 + \sqrt{(1-y^2)(1-4xyb-y^2)}\right) \tag{5.30}$$

then we have

$$H(\alpha, y\alpha) = \frac{(1-y^2)P}{\alpha_1(b)x^2y^2} \sum_{n=0}^{\infty} \frac{(x - y^{2n}P)}{(x + y^{2n}P)}\left(\frac{P}{x}\right)^{2n} y^{2n^2} \tag{5.31}$$

Below we give the length generating function (when $x = t, y = t, a = 1, b = 1$).

Proposition 5.1. *The generating function of partially directed walks ending in a horizontal step in the wedge $\mathcal{W}_1$ is*

$$\begin{aligned} h_1(1,1) = &\frac{(1-t)^2 - \sqrt{(1-t^2)(1-5t^2)}}{2(1-2t-t^2)} \\ &- Q\frac{(1-t^2)}{t^2(1-2t-t^2)} \sum_{n=0}^{\infty} \frac{(1 - t^{2n-1}Q)}{(1 + t^{2n-1}Q)}\left(\frac{Q}{t}\right)^{2n} t^{2n^2} \\ &+ \frac{(1-t^2)}{1-2t-t^2} \sum_{n=0}^{\infty} \frac{(1 - t^{2n-1}P)}{(1 + t^{2n-1}P)}\left(\frac{P}{t}\right)^{2n} t^{2n^2} \end{aligned} \tag{5.32}$$

where t counts the number of edges and

$$Q(1; t, t) = (1 - t - t^2 - t^3 - \sqrt{(1-t^4)(1-2t-t^2)})/2 \tag{5.33a}$$

$$P(1; t, t) = (1 - 3t^2 - \sqrt{(1-t^2)(1-5t^2)})/2t \tag{5.33b}$$

The generating function of all paths in $\mathcal{W}_1$ is then $k_1(1, 1; t, t) = (h_1(1, 1; t, t) - 1)/t$.

As was the case for the symmetric wedge, the functions P and Q that make up our expression for H have combinatorial interpretations in terms of partially directed walks bounded by a single line. Let $B_-(x,y)$ be the generating function of walks that end with a horizontal step, start and end on the line $Y=0$ and stay on or above that same line. Then

$$Q(a;x,y) = xy^2(B_-(ax,y)-1). \tag{5.34}$$

Similarly let $B_/(x,y)$ be the generating function of walks that end with a horizontal step, start and end on the line $Y=X$ and stay on or above that same line. Then

$$P(b;x,y) = xy^2(B_/(bx,y)-1). \tag{5.35}$$

Again we would like to find a more direct combinatorial derivation of the generating functions $H(1,1)$ and $H(1,t)$. We have been unable to do so.

5.3 Asymptotics for $p=1$

Before we study the asymptotics of walks in the wedge $\mathcal{W}_1$, let us compute the number of walks lying on or above the line $y=0$.

Lemma 5.2. *The generating function of partially directed walks lying on or above the line $y=0$ is*

$$\frac{-1+z+3z^2+z^3-\sqrt{(1-z^4)(1-2z-z^2)}}{2z^2(z^2-2z-1)} \tag{5.36}$$

The number of these walks is asymptotic to

$$\sqrt{\frac{7+5\sqrt{2}}{2\pi}}\frac{(1+\sqrt{2})^n}{\sqrt{n}}\,(1+O(1/n)) \tag{5.37}$$

Hence the number of walks in the wedge $\mathcal{W}_1$ must also be $O((\sqrt{2}+1)^n/\sqrt{n})$.

Proof. One can derive a functional equation for the generating function of such walks which can be solved using the kernel method. The asymptotics can then be computed by analysing the dominant singularity at $t=\sqrt{2}-1$. The last result follows since the number of walks in the wedge $\mathcal{W}_1$ cannot exceed the number of walks lying above $y=0$. □

As was the case for walks in the symmetric wedge, we analyse the asymptotics of partially directed walks in the asymmetric wedge $\mathcal{W}_1$ by singularity analysis. Let us split the expression given in equation (5.32) into 3 pieces and study their dominant singularities:

$$p_1 = \frac{(1-t)^2-\sqrt{(1-t^2)(1-5t^2)}}{2(1-2t-t^2)} \tag{5.38a}$$

$$p_2 = -Q\frac{(1-t^2)}{t^2(1-2t-t^2)}\sum_{n=0}^{\infty}\frac{(1-t^{2n-1}Q)}{(1+t^{2n-1}Q)}\left(\frac{Q}{t}\right)^{2n}t^{2n^2} \tag{5.38b}$$

$$p_3 = \frac{(1-t^2)}{1-2t-t^2}\sum_{n=0}^{\infty}\frac{(1-t^{2n-1}P)}{(1+t^{2n-1}P)}\left(\frac{P}{t}\right)^{2n}t^{2n^2} \tag{5.38c}$$

Let us treat the asymptotics of each of these functions separately.

Lemma 5.3. *The coefficients of $p_1(t)$ are asymptotic to*

$$[t^n]p_1 = -\sqrt{\frac{5}{8\pi}}\cdot\left((2+\sqrt{5})-(-1)^n(\sqrt{5}-2)\right)\cdot\frac{(\sqrt{5})^n}{\sqrt{n^3}}\cdot\left(1+O(n^{-1})\right) \tag{5.39}$$

Proof. The generating function p_1 appears to have 6 singularities: 2 simple poles from the zeros of the denominators and four square-root singularities at $t = \pm 1, \pm 1/\sqrt{5}$. Closer analysis shows that there are no singularities at the zeros of the denominator and that generating function is dominated by the singularities at $t = \pm 1/\sqrt{5}$. Analysis (by the techniques in [14]) of these singularities leads to the above expression. □

Before we can study the asymptotics of $p_2(t)$ and $p_3(t)$ we need the following lemma about the location of the zeros of $1 + Qt^k$ and $1 + Pt^k$.

Lemma 5.4. *For $k = -1, 0, 1, 2, \dots$, the functions $1 + Q(t)t^k$ and $1 + P(t)t^k$ are not zero within the disk $|t| < 1/2$.*

Proof. The function $Q(t)$ satisfies $Q^2 - (1 - t - t^2 - t^3)Q + t^4 = 0$, and so $h = Qt^k$ satisfies:

$$h^2 - (1 - t - t^2 - t^3)ht^k + t^{2k+4}. \tag{5.40}$$

Now if $1 + Qt^k = 0$ we have $h = -1$ and so

$$1 + (1 - t - t^2 - t^3)t^k + t^{2k+4} = 0. \tag{5.41}$$

For $|t| \leq 1/2$ we can bound $|1 - t - t^2 - t^3| \leq (1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}) < 2$, and therefore

$$|(1 - t - t^2 - t^3)t^k + t^{2k+4}| \leq |2t^k + t^{2k+4}| \leq 3|t|^k < 3 \cdot 2^{-k} \tag{5.42}$$

Hence for $k \geq 2$, the above quantity is less than 1 and so equation (5.41) cannot be satisfied. It remains to check the cases $k = -1, 0, 1$. In these cases we can solve equation (5.41) directly and verify that t lies outside $|t| \leq 1/2$.

The argument for $P(t)$ follows a similar line. The function $h = Pt^k$ satisfies:

$$h^2 - t^{k-1}(1 - 3t^2)h + t^{2k+2} = 0. \tag{5.43}$$

Hence if $h = -1$ we have

$$1 + (1 - 3t^2)t^{k-1} + t^{2k+2} = 0 \tag{5.44}$$

For $|t| < 1/2$ we can bound $|1 - 3t^2| < 2$ and so

$$|(1 - 3t^2)t^{k-1} + t^{2k+2}| \leq 3|t|^{k-1} \tag{5.45}$$

Hence for $k \geq 3$, equation (5.44) cannot be satisfied for $|t| \leq 1/2$. For $k = -1, 0, 1, 2$, we can check equation (5.44) directly and verify that the zeros do not lie inside $|t| < 1/2$.

Note that when $k = 0$, equation (5.44) has a solution $t = \sqrt{2} - 1$, however this point corresponds to the other branch of P being $+1$.

□

We can now move onto the asymptotics of $p_2(t)$ and $p_3(t)$.

Lemma 5.5. *The coefficients of $p_3(t)$ are asymptotic to*

$$\begin{aligned}[t^n]p_3(t) &= \frac{(\sqrt{2}+1)^n}{\sqrt{2}} \sum_{k=0}^{\infty} \frac{1 - (\sqrt{2}-1)^{2k+1}}{1 + (\sqrt{2}-1)^{2k+1}} (\sqrt{2}-1)^{2k^2+2k} + O\left(\left(\sqrt{5}\right)^n\right) \\ &= (0.31096381899209832\dots)(\sqrt{2}+1)^n + O\left(\left(\sqrt{5}\right)^n\right)\end{aligned} \tag{5.46}$$

Proof. The function p_3 has a simple pole from the zero of the denominator of prefactor. There are also square-root singularities in $P(t)$ at $t = \pm 1, \pm 1/\sqrt{5}$, simple poles when $1 + Pt^{2n-1} = 0$ and a natural boundary at $|t| = 1$ coming from the θ-function like structure of the sum. Of these singularities, the simple pole dominates, followed by the singularities at $\pm 1/\sqrt{5}$. The simple pole and its residue give the dominant asymptotics and the square-root singularities give the $O(5^{n/2})$ corrections. □

Lemma 5.6. *The coefficients of $p_2(t)$ are asymptotic to*

$$[t^n]p_2(t) = -\frac{(\sqrt{2}+1)^n}{\sqrt{2}} \sum_{k=0}^{\infty} \frac{1-(\sqrt{2}-1)^{2k+1}}{1+(\sqrt{2}-1)^{2k+1}} (\sqrt{2}-1)^{2k^2+2k} (1+o(1))$$
$$= -(0.31096381899209832\ldots)(\sqrt{2}+1)^n (1+o(1)) \tag{5.47}$$

The dominant term is equal in magnitude, but opposite in sign, to the dominant term in the asymptotics of $p_3(t)$.

Proof. The singularities of the function $p_2(t)$ arise from singularities of the prefactor, the singularities of $Q(t)$, the zeros of $1+Qt^{2n-1}$ and the natural boundary at $|t| = 1$ from the θ-function structure of the sum. Of these, the simple pole of the prefactor at $t = \sqrt{2}-1$ and the singularities arising from $Q(t)$ at the same point, dominate the asymptotics.

The contribution from the simple pole may be computed by finding its residue. We note that $P(\sqrt{2}-1) = Q(\sqrt{2}-1) = 3-2\sqrt{2}$, and this means that the residue is in fact equal in magnitude, but opposite in sign, to that computed for $p_3(t)$.

□

We note that if the dominant asymptotics of $p_2(t)$ and $p_3(t)$ must cancel each other. Otherwise, the number of walks in this wedge is $\sim (\sqrt{2}-1)^n$ which would contradict Lemma 5.2.

Lemma 5.7. *The k^{th} summand of $p_2(t)$ is*

$$p_{2,k}(t) = \left(-Q \frac{(1-t^2)}{t^2(1-2t-t^2)} \frac{(1-t^{2k-1}Q)}{(1+t^{2k-1}Q)} \left(\frac{Q}{t}\right)^{2k} t^{2k^2} \right) \tag{5.48}$$

so that $p_2(t) = \sum p_{2,k}(t)$. The coefficient of t^n in $p_{2,k}(t)$ is asymptotic to

$$[t^n]p_{2,k}(t) = -(1+\sqrt{2})^n \cdot \frac{1}{\sqrt{2}} \left(\frac{1-(\sqrt{2}-1)^{2k+1}}{1+(\sqrt{2}-1)^{2k+1}} \right) (1+\sqrt{2})^{-2k^2-2k}$$
$$+ (1+\sqrt{2})^n \cdot \sqrt{\frac{2}{\pi n}} \left[\frac{(2k+1)(1-(\sqrt{2}-1)^{4k+2}) - (\sqrt{2}-1)^{2k+1}}{(1+(\sqrt{2}-1)^{2k+1})^2} + O\left(\frac{1}{\sqrt{n}}\right) \right] (1+\sqrt{2})^{-2k^2-2k-5/2} \tag{5.49}$$

Proof. The result follows from standard singularity analysis ([14]) of $p_{2,k}(t)$. □

Remark. It is unfortunately the case that we have been unable to proceed completely rigorously from this point. In particular, we have been unable to obtain uniform bounds on the error terms in the above asymptotic expressions. On the basis of numerical testing, we think that the that the error term is $O(k^3/\sqrt{n})$. If this is the case, then one can sum the contributions of the individual $p_{2,k}(t)$ to obtain the asymptotics of coefficients of $p_2(t)$.

We believe that the expressions that follow are indeed *exact*, if not completely rigorous.

Assuming that the asymptotic expression in the previous lemma has a uniform error bound, so that we may sum the contribution to the individual $p_{2_k}(t)$, we find that

$$[t^n]p_2(t) = (1+\sqrt{2})^n \Big(-0.31096381899209832\ldots + \frac{0.090584741026764287\ldots}{\sqrt{n}} + O\big(1/\sqrt{n^3}\big) \Big) \tag{5.50}$$

where the constant $0.31\ldots$ is the constant that appears in both Lemmas 5.5 and 5.6. So adding together the contributions from the p_i we obtain

$$[t^n]h_1(1,1) = (1+\sqrt{2})^n \left(\frac{0.0905847410267642 87\ldots}{\sqrt{n}} + O\big(1/\sqrt{n^3}\big) \right) \tag{5.51}$$

The generating function $h_1(1,1)$ enumerates walks that end in a horizontal step, and so we obtain the number of walks in $\mathcal{W}_1$ ending with any step by multiplying this expression by a factor of $(1+\sqrt(2))$ (since the generating functions differ by a factor of t):

$$[t^n]k_1(1,1) = (1+\sqrt{2})^n \left(\frac{0.2186939166943031 77\ldots}{\sqrt{n}} + O\big(1/\sqrt{n^3}\big) \right) \tag{5.52}$$

We have confirmed this numerically using the first 1000 terms in the series expansion of $k_1(1,1)$. Additionally we have check the above expression using Bruno Salvy's *gdev* package for Maple [24].

Remark. We note that if the above result is indeed made rigorous then we have a result analogous to Corollary 4.3. The number of walks in any wedge $\mathcal{W}_p$ for $1 \le p < \infty$ is bounded between the number of walks in $\mathcal{W}_1$ and partially directed walks inside the first quadrant (see Lemma 5.2. Hence the number of partially directed walks in the wedge $\mathcal{W}_p$, $c_n^{(p)}$ obeys the following inequality

$$0.21869\ldots \le \lim_{n\to\infty} \frac{c_n^{(p)} n^{1/2}}{(1+\sqrt{2})^n} \le \sqrt{\frac{7+5\sqrt{2}}{2\pi}} = 1.496489\ldots \tag{5.53}$$

for any $1 \le p < \infty$.

6 Conclusions

In this paper we have proved that partially directed paths in the wedges $\mathcal{V}_p$ and $\mathcal{W}_p$ all grow with the same exponential growth rate $1+\sqrt{2}$ independent of p. Additionally we have found generating functions for partially directed paths in the symmetric wedge $\mathcal{V}_1$ and the asymmetric wedge $\mathcal{W}_1$, using a variation of the kernel method. From these generating functions we have computed the asymptotics of the number of paths in both of these wedges.

Curiously the number of paths in the symmetric wedge, $\mathcal{V}_1$, has the same leading asymptotic behaviour as partially directed paths with no bounding wedge. Similarly the number of paths in the asymmetric wedge, $\mathcal{W}_1$, has the same leading asymptotic behaviour as partially directed paths above the line $Y = 0$. Because of this, we are able to determine the leading asymptotic behaviour of paths in the wedges $\mathcal{V}_p$ and $\mathcal{W}_p$ for all $p \ge 1$.

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References

[1] D. André. Solution directe du problème résolu par M. Bertrand. *Comptes Rendus Acad. Sci. Paris*, 105:436–437, 1887.

[2] C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, D. Gardy, and D. Gouyou-Beauchamps. Generating functions for generating trees. *Discrete Math.*, 246:29–55, 2002.

[3] M. Bousquet-Mélou. A method for the enumeration of various classes of column-convex polygons. *Discrete Math.*, 154:1–25, 1996.

[4] M. Bousquet-Mélou. Walks on the slit plane: other approaches. *Adv. in Appl. Math.*, 27:243–288, 2001.

[5] M. Bousquet-Mélou. Counting walks in the quarter plane. In *Mathematics and Computer Science: Algorithms, trees, combinatorics and probabilities*, Trends in Mathematics, pages 49–67. Birkhauser, 2002.

[6] M. Bousquet-Mélou. Four classes of pattern-avoiding permutations under one roof: generating trees with two labels. *Electronic J. Combinatorics*, 9(2):R19, 2003.

[7] M. Bousquet-Mélou and G. Schaeffer. Walks on the slit plane. *Probab. Theory Relat. Fields*, 124:305–344, 2002.

[8] J.L Cardy. Conformal invariance and surface critical behavior. *Nuc. Phys. B.*, 240:514–532, 1984.

[9] E.A. DiMarzio and R.J. Rubin. Adsorption of a chain polymer between two plates. *J. Chem. Phys.*, 55:4318–4336, 1971.

[10] P. Duchon. On the enumeration of generation of generalized dyck words. *Discrete Math.*, 225:121–135, 2000.

[11] B. Duplantier and H. Saleur. Exact surface and wedge exponents for polymers in two dimensions. *Phys. Rev. Lett.*, 57:3179–3182, 1986.

[12] R. Pinzani E. Barcucci, E. Pergola and S. Rinaldi. A bijection for some paths on the slit plane. *Adv. in Appl. Math.*, 26:89–96, 2001.

[13] G. Fayolle, R. Iasnogorodski, and V. Malyshev. *Random walks in the quarter plane: algebraic methods, boundary value problems and applications.* Number 40 in Applications of Mathematics. Springer Verlag, Berlin, 1999.

[14] P. Flajolet and A. M. Odlyzko. Singularity analysis of generating functions. *SIAM J. Discrete Math.*, 3:216–240, 1990.

[15] I. M. Gessel and D. Zeilberger. Random walk in a weyl chamber. *Proc. Amer. Math. Soc.*, 115:27–31, 1992.

[16] J. M. Hammersley, , and S. G. Whittington. Self-avoiding walks in wedges. *J. Phys. A: Math. Gen.*, 18:101–111, 1985.

[17] H. Kesten. On the number of self-avoiding walks. *J. Math. Phys.*, 4:960–959, 1963.

[18] H. Kesten. On the number of self-avoiding walks ii. *J. Math. Phys.*, 5:1128–1137, 1964.

[19] N. Madras and G. Slade. *The Self-Avoiding Walk*. Birkhauser, London, 1993.

[20] W. K. McCrae and F. J. W. Whipple. Random paths in two and three dimensions. *Proc. Roy. Soc. Edinburgh*, 60:281–298, 1940.

[21] D. H. Napper. *Polymeric Stabilization of Colloidal Dispersions*. Academic Press, London, 1983.

[22] H. Niederhausen. Lattice paths between diagonal boundaries. *Electronic J. Combinatorics*, 5:R30, 1998.

[23] A. Rechnitzer R. Brak, A.L. Owczarek and S.G. Whittington. A directed walk model of a long chain polymer in a slit with attractive walls. *J. Phys. A: Math. Gen.*, 38:4309–4325, 2005.

[24] B. Salvy. Examples of automatic asymptotic expansions. *SIGSAM Bulletin*, 25(2):4–17, April 1991.

[25] E. J. Janse van Rensburg. Forces in motzkin paths in a wedge. To appear in J. Phys. A: Math. Gen.

[26] E. J. Janse van Rensburg. Adsorbing bargraph paths in a q-wedge. *J. Phys. A: Math. Gen.*, 38:8505–8525, 2005.

[27] E. J. Janse van Rensburg. Square lattice directed paths adsorbing on the line $y = qx$. *J. Stat. Mech.: Theo. and Exp.*, page P09010, 2005.

[28] E. J. Janse van Rensburg and Y. Le. Forces in square lattice directed paths in a wedge. *J. Phys. A: Math. Gen.*, 38:8493–8503, 2005.

[29] C. Vanderzande. *Lattice Models of Polymers*, volume 11 of *Cambride Lecture Notes in Physics*. Cambridge University Press, 1998.