Long Answer Questions

Quiz 6: Find the global maximum and the global minimum for $f(x) = x^3 - 6x^2 + 2$ on the interval $[3, 5]$.

Quiz 5: Two particles move in the cartesian plane. Particle A travels on the x-axis starting at $(10, 0)$ and moving towards the origin with a speed of 2 units per second. Particle B travels on the y-axis starting at $(12, 0)$ and moving towards the origin with a speed of 3 units per second. What is the rate of change of the distance between the two particles when particle A reaches the point $(4, 0)$?

Quiz 4: If $x^2 \cos(y) + 2xe^y = 8$, then find y' at the points where $y = 0$. You must justify your answer.

Quiz 3: Determine whether the derivative of following function exists at $x = 0$

$$f(x) = \begin{cases} 2x^3 - x^2 & \text{if } x \leq 0 \\ x^2 \sin\left(\frac{1}{x}\right) & \text{if } x > 0 \end{cases}$$

You must justify your answer using the definition of a derivative.

Quiz 2: Show that there exists at least one real number c such that $2\tan(c) = c + 1$.

Quiz 1: Compute the limit $\displaystyle\lim_{x\to1} \frac{\sqrt{x+2} - \sqrt{4-x}}{x-1}$.

Quizzes 4-6 Short Answer Questions

Quiz 6:
Find the intervals where $f(x) = \frac{\sqrt{x}}{x+6}$ is increasing.

Let $f(x) = x^2 - 2\pi x - \sin(x)$. Show that there exists a real number c such that $f'(c) = 0$.

Quiz 5:
Estimate $\sqrt{35}$ using a linear approximation

Consider a function $f(x)$ which has $f'''(x) = \dfrac{x^3}{10 - x^2}$. Show that when we approximate $f(1)$ using its second Maclaurin polynomial, the absolute error is less than $\frac{1}{50} = 0.02$.

Quiz 4:
Find $f'(x)$ if $f(x) = (x^2 + 1)^{\sin(x)}$.

Consider a function of the form $f(x) = Ae^{kx}$ where A and k are constants. If $f(0) = 3$ and $f(2) = 5$, find the constants A and k.

Quizzes 1-3 Short Answer Questions

Quiz 3: Find the equation of the tangent line to the graph of $y = \cos(x)$ at $x = \frac{\pi}{4}$.

For what values of x does the derivative of $\frac{\sin(x)}{x^2 + 6x + 5}$ exist?

Quiz 2: Compute

$$\lim_{x \to -\infty} \frac{3x + 5}{\sqrt{x^2 + 5} - x}$$

Find all values of c such that the following function is continuous:

$$f(x) = \begin{cases} 8 - cx & \text{if} \quad x \leq c \\ x^2 & \text{if} \quad x > c \end{cases}$$

Use the definition of continuity to justify your answer.

Quiz 1: Find all solutions to $x^3 - 3x^2 - x + 3 = 0$

Compute the limit $\lim_{x \to 2} \frac{x - 2}{x^2 - 4}$

Solutions

Quiz 6 Long Answer

Find the global maximum and the global minimum for $f(x) = x^3 - 6x^2 + 2$ on the interval $[3, 5]$.

We compute $f'(x) = 3x^2 - 12x$, which means that $f(x)$ has no singular points (i.e., it is differentiable for all values of x), but it has two critical points obtained by solving $f'(x) = 0$, i.e. $3x(x-4) = 0$ which yields the two critical points $x = 0$ and $x = 4$. In order to compute the global maximum and the global minimum for $f(x)$ on the interval $[3, 5]$, we compute

$$f(3) = -25,\ f(4) = -30 \text{ and } f(5) = -23.$$

So, the global maximum is $f(5) = -23$ while the global minimum is $f(4) = -30$.

Quiz 5 Long Answer

Two particles move in the cartesian plane. Particle A travels on the x-axis starting at $(10, 0)$ and moving towards the origin with a speed of 2 units per second. Particle B travels on the y-axis starting at $(12, 0)$ and moving towards the origin with a speed of 3 units per second. What is the rate of change of the distance between the two particles when particle A reaches the point $(4, 0)$?

The position of particle A along the x axis starts at 10, and decreases 2 units per second, so its position is given by $10 - 2t$, where t is measured in seconds. Similarly, the position of B along the y axis is given by $y = 12 - 3t$. The distance z between the two particles satisfies $z^2 = x^2 + y^2$. When $x = 4$, we solve $4 = 10 - 2t$ for t and find $t = 3$, so $y = 12 - 3(3) = 3$. Then $z = 5$ when $t = 2$.
Differentiating implicitly, $z^2 = x^2 + y^2$ tells us

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

so, when $t = 3$,

$$2(5)\frac{dz}{dt} = 2(4)(-2) + 2(3)(-3)$$

Then the distance between the two particles is changing at $\boxed{-\frac{17}{5}}$ units per second.

Quiz 4 Long Answer

If $x^2\cos(y) + 2xe^y = 8$, then find y' at the points where $y = 0$.
You must justify your answer.

- First we find the x-ordinates where $y = 0$.

$$\begin{aligned} x^2\cos(0) + 2xe^0 &= 8 \\ x^2 + 2x - 8 &= 0 \\ (x+4)(x-2) &= 0 \end{aligned}$$

  So $x = 2, -4$.
- Now we use implicit differentiation to get y' in terms of x, y:

$$\begin{aligned} x^2\cos(y) + 2xe^y &= 8 && \text{differentiate both sides} \\ x^2 \cdot (-\sin y) \cdot y' + 2x\cos y + 2xe^y \cdot y' + 2e^y &= 0 \end{aligned}$$

- Now set $y = 0$ to get

$$\begin{aligned} x^2 \cdot (-\sin 0) \cdot y' + 2x\cos 0 + 2xe^0 \cdot y' + 2e^0 &= 0 \\ 0 + 2x + 2xy' + 2 &= 0 \\ y' = -\frac{2+2x}{2x} &= -\frac{1+x}{x} \end{aligned}$$

- So at $(x, y) = (2, 0)$ we have $y' = -\frac{3}{2}$,
- and at $(x, y) = (-4, 0)$ we have $y' = -\frac{3}{4}$.

Quiz 3 Long Answer

Determine whether the derivative of following function exists at $x = 0$

$$f(x) = \begin{cases} 2x^3 - x^2 & \text{if } x \leq 0 \\ x^2 \sin\left(\dfrac{1}{x}\right) & \text{if } x > 0 \end{cases}$$

You must justify your answer using the definition of a derivative.

The function is differentiable at $x = 0$ if the following limit:

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f(x) - 0}{x} = \lim_{x\to 0} \frac{f(x)}{x}$$

exists (note that we used the fact that $f(0) = 0$ as per the definition of the first branch which includes the point $x = 0$). We compute left and right limits; so

$$\lim_{x\to 0^-} \frac{f(x)}{x} = \lim_{x\to 0^-} \frac{2x^3 - x^2}{x} = \lim_{x\to 0^-} 2x^2 - x = 0$$

and

$$\lim_{x\to 0^+} \frac{x^2 \sin\left(\frac{1}{x}\right)}{x} = \lim_{x\to 0^+} x \cdot \sin\left(\frac{1}{x}\right).$$

This last limit equals 0 by Squeeze Theorem since

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

and so,

$$-x \leq x \cdot \sin\left(\frac{1}{x}\right) \leq x,$$

where in these inequalities we used the fact that $x \to 0^+$ yields positive values for x. Finally, since $\lim_{x\to 0^+} -x = \lim_{x\to 0^+} x = 0$, Squeeze Theorem yields that also $\lim_{x\to 0^+} x \sin\left(\frac{1}{x}\right) = 0$, as claimed. Since the left and right limits match (they're both equal to 0), we conclude that indeed $f(x)$ is differentiable at $x = 0$ (and its derivative at $x = 0$ is actually equal to 0).

Quiz 2 Long Answer

Show that there exists at least one real number c such that $2\tan(c) = c + 1$.

We let $f(x) = 2\tan(x) - x - 1$. Then $f(x)$ is a continuous function on the interval $(-\pi/2, \pi/2)$ since $\tan(x) = \sin(x)/\cos(x)$ is continuous on this interval, while $x + 1$ is a polynomial and therefore continuous for all real numbers.

We find a value $a \in (-\pi/2, \pi/2)$ such that $f(a) < 0$. We observe immediately that $a = 0$ works since

$$f(0) = 2\tan(0) - 0 - 1 = 0 - 1 = -1 < 0.$$

We find a value $b \in (-\pi/2, \pi/2)$ such that $f(b) > 0$. We see that $b = \pi/4$ works since

$$f(\pi/4) = 2\tan(\pi/4) - \pi/4 - 1 = 2 - \pi/4 - 1 = 1 - \pi/4 = (4 - \pi)/4 > 0,$$

because $3 < \pi < 4$.

So, because $f(x)$ is continuous on $[0, \pi/4]$ and $f(0) < 0$ while $f(\pi/4) > 0$, then the Intermediate Value Theorem guarantees the existence of a real number $c \in (0, \pi/4)$ such that $f(c) = 0$.

Quiz 1 Long Answer

Compute the limit $\lim_{x\to 1} \frac{\sqrt{x+2}-\sqrt{4-x}}{x-1}$.

If we try to do the limit naively we get 0/0. Hence we must simplify.

$$\begin{aligned}
\frac{\sqrt{x+2}-\sqrt{4-x}}{x-1} &= \frac{\sqrt{x+2}-\sqrt{4-x}}{x-1}\cdot\frac{\sqrt{x+2}+\sqrt{4-x}}{\sqrt{x+2}+\sqrt{4-x}} \\
&= \frac{(x+2)-(4-x)}{(x-1)(\sqrt{x+2}+\sqrt{4-x})} \\
&= \frac{2x-2}{(x-1)(\sqrt{x+2}+\sqrt{4-x})} \\
&= \frac{2}{\sqrt{x+2}+\sqrt{4-x}}
\end{aligned}$$

Quiz 6 Short Answer

Find the intervals where $f(x) = \frac{\sqrt{x}}{x+6}$ is increasing.

$(0, \infty)$

Let $f(x) = x^2 - 2\pi x - \sin(x)$. Show that there exists a real number c such that $f'(c) = 0$.

We note that $f(x)$ is continuous and differentiable over all real numbers. Since $f(0) = f(2\pi) = 0$, by Rolle's Theorem (also by the Mean Value Theorem) there exist ssome c between 0 and 2π such that $f'(c) = 0$.

Quiz 5 Short Answer

Estimate $\sqrt{35}$ using a linear approximation

$L(x) = f(a) + f'(a)(x - a)$. If $f(x) = \sqrt{x}$ and $a = 36$, then $f(a) = 6$ and $f'(a) = \frac{1}{12}$. So, $L(x) = 6 + \frac{1}{12}(x - 36)$. Then: $\sqrt{35} = f(35) \approx L(35) = 6 + \frac{1}{12}(35 - 36) = 6 - \frac{1}{12} = \boxed{\frac{71}{12}}$

Consider a function $f(x)$ which has $f'''(x) = \dfrac{x^3}{10 - x^2}$. Show that when we approximate $f(1)$ using its second Maclaurin polynomial, the absolute error is less than $\frac{1}{50} = 0.02$.

For some c between 0 and 1:

$$|f(1) - T_2(1)| = \left|\frac{f'''(c)}{3!}(1-0)^3\right| = \frac{1}{6}\left|\frac{c^3}{10 - c^2}\right|$$

Since c is between 0 and 1, we note $0 < c^3 < 1$ and $9 < 10 - c^2 < 10$, so:

$$|f(1) - T_2(1)| < \frac{1}{6}\left|\frac{1}{9}\right| = \frac{1}{54} < \frac{1}{50}$$

Quiz 4 Short Answer

Find $f'(x)$ if $f(x) = (x^2+1)^{\sin(x)}$.

$$f'(x) = (x^2+1)^{\sin x}\left[\frac{2x\sin x}{x^2+1} + \cos x \cdot \log(x^2+1)\right]$$

Consider a function of the form $f(x) = Ae^{kx}$ where A and k are constants. If $f(0) = 3$ and $f(2) = 5$, find the constants A and k.

$A = 3,\ k = \frac{\log(5/3)}{2}$

Quiz 3 Short Answer

Find the equation of the tangent line to the graph of $y = \cos(x)$ at $x = \dfrac{\pi}{4}$.

$$y = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right)$$

For what values of x does the derivative of $\dfrac{\sin(x)}{x^2 + 6x + 5}$ exist? Explain your answer.

$x \neq -1, -5$

Quiz 2 Short Answer

Compute

$$\lim_{x\to-\infty} \frac{3x+5}{\sqrt{x^2+5}-x}$$

$-\frac{3}{2}$ (Remember: when x is negative, $\sqrt{x^2} = |x| = -x$.)

Find all values of c such that the following function is continuous:

$$f(x) = \begin{cases} 8 - cx & \text{if} \quad x \le c \\ x^2 & \text{if} \quad x > c \end{cases}$$

Use the definition of continuity to justify your answer.

When $x \neq c$, $f(x)$ is locally a polynomial, so it is continuous. The only difficult spot is when $x = c$. Note:
- $f(c) = 8 - c^2$
- $\lim\limits_{x\to c^-} f(x) = \lim\limits_{x\to c^-} (8 - cx) = 8 - c^2$
- $\lim\limits_{x\to c^+} f(x) = \lim\limits_{x\to c^+} (x^2) = c^2$

Since $f(x)$ is continuous at c only if $f(c) = \lim\limits_{x\to c} f(x)$, we see the only values of c that make f continuous are those that satisfy $c^2 = 8 - c^2$. That is, $\boxed{\pm 2}$.

Quiz 1 Short Answer

Find all solutions to $x^3 - 3x^2 - x + 3 = 0$

$x^3 - 3x^2 - x + 3 = x^2(x-3) - (x-3) = (x^2-1)(x-3) = (x+1)(x-1)(x-3)$
So, the solutions are $x = 1$, $x = 3$, and $x = -1$.

Compute the limit $\lim_{x\to 2} \frac{x-2}{x^2-4}$

$$\lim_{x\to 2} \frac{x-2}{x^2-4} = \lim_{x\to 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{1}{x+2} = \boxed{\frac{1}{4}}$$