1 Orbits — summary

Definition. A discrete dynamical system consists of:

- a function $f : X \mapsto X$, where X is some arbitary set (such as $\mathbb{R}$ or the unit interval $[0, 1]$ or the unit circle S^1).
- iteration of the function (repeated application of the function) which defines the dynamics of the system. Since f is a function, the value of $f(x)$ is completely determine and so the system is *deterministic*.

We call:

- X the *state space* of the system and
- x the *state variable*.

A given initial point x_0 will define all the x_n by iteration of the function F:

$$\begin{aligned} x_1 &= f(x_0) \\ x_2 &= f(x_1) = f(f(x_0)) = f^2(x_0) \\ x_3 &= f(x_2) = f^3(x_0) \\ &\vdots \\ x_n &= f^n(x_0) \\ &\vdots \end{aligned}$$

Note: The symbol $f^n(x)$ is used to mean the function obtained by applying f n times, *not* the n^{th} derivative of f — which we shall write as $\frac{\mathrm{d}^n}{\mathrm{d}x^n} f$

Definition. Given a point $x_0 \in X$ we define its orbit to be the set of points

$$\{x_0, x_1 = f(x_0), x_2 = f(f(x_0)), \ldots, x_n = f^n(x), \ldots\}$$

i.e. the set of future images of x_0 under f.

Definition. A *fixed point*, x^* of a function $f(x)$, is any point that is unchanged under application of f:

$$f(x^*) = x^*$$

Definition. A point x_0 is a *periodic point*, of *period* n if

$$f^n(x_0) = x_0$$

We also say that x_0 lies on an n-cycle. The smallest n for which $f^n(x_0) = x_0$ is true, defines the *prime period* of the point. A fixed point has prime period 1.

Definition. A point x_0 is *eventually fixed*, of it itself is not fixed, but some future iterate of it is fixed. *i.e.* $f(x_0) \neq x_0$ but $\exists k > 0$ such that $x_{k+1} = f^{k+1}(x_0) = f^k(x_0) = x_k$.

The definition of an *eventually periodic point* is similar.