Assignment 2

1. Using the java applets on the course homepage, calculate the locations of the first 6 *superstable* orbits of:

   - The logistic map $F_\mu(x) = \mu x(1-x)$, and
   - The sine map $S_\mu(x) = \mu \sin(\pi x)$

   A point x_0 is a superstable stable n-cycle if it has $(F^n)'(x_0) = 0$. Since both these maps have a unique maximum at $x = 1/2$, the superstable n-cycles occur at μ values where the n-cycle contains the point $x = 1/2$. See next page for an example.

   Remember — these maps undergo period doubling bifurcations, so the first 6 superstable orbits will have periods $1, 2, 2^2, \dots 2^5$. It is important that your estimates are *very* accurate — as many decimal places as you can get! Take your time and be careful. You will need to set "Discard first" and "Show next" to around 100,000 in order to get good results.

   Call these μ values for the logistic and sine maps $l_0, l_1, \dots l_5$ and $s_0, s_1, \dots s_5$ (respectively). To check you have the right idea — $l_0 = 2$ and $s_0 = 1/2$. These numbers are expected to have the following behaviour:

$$\begin{aligned} l_n &\approx l_\infty - A\delta_l^n \\ s_n &\approx s_\infty - B\delta_s^n \end{aligned}$$

   Tabulate your data and estimate δ_l and δ_s. What do your results suggest?

   Hint: If $l_n \approx l_\infty - A\delta_l^n$ then

$$\begin{aligned} l_n - l_{n-1} &\approx A\delta^{n-1}(\delta - 1) \\ \frac{l_n - l_{n-1}}{l_{n-1} - l_{n-2}} &\approx \delta \end{aligned}$$

Total = 5 marks

- Here is a table of the data:

Logistic	l_n	$l_n - l_{n-1}$	$\frac{l_n - l_{n-1}}{l_{n-1} - l_{n-2}} \approx \delta$	$1/\delta$
l_0	2	◦	◦	◦
l_1	3.2360679	1.2360679	◦	◦
l_2	3.4985616	0.2624937	0.212361878	4.70894311
l_3	3.5546406	0.056079	0.213639413	4.680784251
l_4	3.5666678	0.0120272	0.214468874	4.662681256
l_5	3.56924347	0.00257567	0.214153751	4.669542294
Sine				
s_0	0.5	◦	◦	◦
s_1	0.77773396	0.27773396	◦	◦
s_2	0.84638216	0.0686482	0.24717251	4.045757354
s_3	0.86145038	0.01506822	0.219499127	4.555826767
s_4	0.86469419	0.00324381	0.21527493	4.645222747
s_5	0.86538966	0.00069547	0.214399117	4.664198312

4 marks

- For both of these maps the number $1/\delta$ appears to be converging to around $4.67\ldots$. It seems to be the same for both the Logistic and Sine maps.

1 mark

- This number is called a *Feigenvalue* and is thought to be the same for all unimodal maps with a quadratic maximum.

$$1/\delta = 4.669201609102990671853203820466201617258185577475768632\ldots$$

2. Let Σ_N denote the space of sequences whose entries are the integers $0, 1, \ldots, N-1$.

   (a) For all $s, t \in \Sigma_N$ define
   $$d[s,t] = \sum_{k=0}^{\infty} \frac{|s_k - t_k|}{N^k}$$
   Prove that this function is a metric on Σ_N, and find the maximum distance between any two points in Σ_N.

   (b) Let σ_N be the shift map on Σ_N (defined in the usual way). How many fixed points does σ_N have? How many points are fixed by σ_N^k?

Total = 5 marks

- In order for this function to be a metric, it must be positive, reflexive and obey the triangle inequality.

- We require $d[s,t] = d[t,s]$. This follows because $|a-b| = |b-a|$:

$$\begin{aligned} d[s,t] &= \sum_{k=0}^{\infty} \frac{|s_k - t_k|}{N^k} \\ &= \sum_{k=0}^{\infty} \frac{|t_k - s_k|}{N^k} \\ &= d[t,s] \end{aligned}$$

  as required.
- We need $d[s,t] \geq 0$ — Since $d[s,t]$ is the sum of non-negative terms, it cannot be negative.
- We need $d[s,t] = 0 \leftrightarrow s = t$. A sum of non-negative terms is equal to zero if and only if each term in the sum is equal to zero. This means that $d[s,t] = 0$ if and only if $s_k = t_k$ for all $k \geq 0$. And $s_k = t_k$ for all $k \geq 0$ if an only if $s = t$.
- Finally for all $s, t, u \in \Sigma_N$, we need that $d[s,t] \leq d[s,u] + d[u,t]$. For any three real numbers, s_k, t_k and u_k the following is true:

$$|s_k - t_k| \leq |s_k - u_k| + |u_k - t_k|$$

  Dividing this statement by N_k and then summing over k gives:

$$\sum_{k=0}^{\infty} \frac{|s_k - t_k|}{N^k} \leq \sum_{k=0}^{\infty} \frac{|s_k - u_k|}{N^k} + \sum_{k=0}^{\infty} \frac{|u_k - t_k|}{N^k}$$

  And we are done.

2 marks

- We need to maximise the distance function:

$$d[s,t] = \sum_{k=0}^{\infty} \frac{|s_k - t_k|}{N^k}.$$

  The maximum value $|s_k - t_k|$ can take is $(N-1)$, so the distance function must be bounded above by:

$$d[s,t] \leq \sum_{k=0}^{\infty} \frac{N-1}{N^k} = (N-1) \sum_{k=0}^{\infty} N^{-k} = (N-1) \frac{1}{1-1/N} = N.$$

  We now need to show that we can actually obtain this maximum. That is, there are s and t such that $d[s,t] = N$. An obvious choice is:

$$s = (0,0,0,0,\dots) \qquad \text{and} \qquad t = (N-1, N-1, N-1, \dots)$$

1 mark

- If a point $s = (s_0 s_1 s_2 \dots) \in \Sigma_N$ is fixed by σ_N then:

  $$(s_0 s_1 s_2 \dots) = \sigma_n(s) = (s_1 s_2 s_3 \dots)$$

  which implies that $s_k = s_{k-1} = s_{k-2} = \cdots = s_1 = s_0$. Hence the only points that are fixed under σ_N are those that consist only of a single repeating digit — there are N such sequences:

  $$(0,0,0,\dots) \qquad (1,1,1,\dots) \qquad \dots \qquad (N-1, N-1, N-1\dots)$$

1 mark

- If a point $s = (s_0 s_1, \dots) \in \Sigma_N$ is fixed by σ_N^k, then

  $$(s_0 s_1 s_2 \dots) = \sigma_n(s) = (s_k s_{k+1} s_{k+2} \dots)$$

  Hence we must have

  $$\begin{array}{rlll} s_k = s_0 & s_{k+1} = s_1 & \dots & s_{2k-1} = s_{k-1} \\ s_{2k} = s_k = s_0 & s_{2k+1} = s_{k+1} = s_1 & \dots & \end{array}$$

  And so s must consist of a repeating block of k digits:

  $$s = (s_0 s_1 \dots s_{k-1} s_0 s_1 \dots s_{k-1} \dots) = (\overline{s_0 s_1 \dots s_{k-1}})$$

  There are N^k such sequences.

1 mark