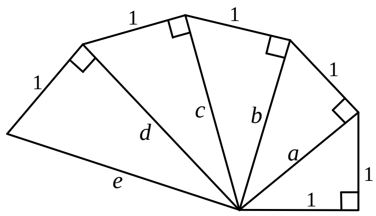
UBC Grade 8–10 Workshop Problems, 2008–2009

1. Tom Sawyer has four colours of paint and three fence posts, which are in a row, to paint. In how many ways can he paint the posts so that no two adjacent posts have the same colour?

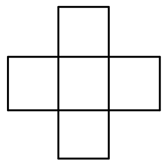
2. In the diagram at the right, all the triangles are right-angle triangles and a, b, c, d, and e are the lengths of the indicated hypotenuses. Find the value of e.



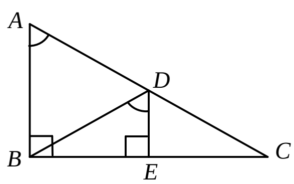
3. Deborah and James are on a road trip in a rental car which they want to return with its 60-litre gas tank completely empty, since they prepaid for a full tank of gas and won't get a refund for any unused gas. They have driven 2/3 of the distance on a 900-km highway trip and the tank is 1/4 full. If they stop for gas, how many litres of fuel should they put in the tank?

4. In a class of 40 students, 60% are boys and 40% are girls. As their math activity one day, they decide to measure their weights and compute some averages. They find that the average weight of all the boys is 40 kg, and by coincidence that the average weight of *all* members of the class, including the 80-kg teacher, also equals 40 kg. What is the average weight of the girls in the class?

5. In how many ways can the numbers 2008, 2009, 2010, 2011, 2012 be put into the figure at the right so that the sum of the three numbers in the middle vertical column equals the sum of the numbers in the middle horizontal row?



6. In the diagram, triangles ABC and DEB are right-angle triangles and ABC is similar to triangle DEB. Also, the length of BC is 13 1/3 cm and the length of AB is 10 cm. Find the length of BD.



7. An unknown positive integer has two digits. If the two digits are reversed, the new integer is 27 greater than the original one. How many such integers are there? Are there any such integers if 27 is replaced by 26?

8. In the x-y plane, A is the origin, (0,0), B is the point (3,4), and $C = (x,10)$. Find all values of x for which the distance from B to C is twice the distance from A to B.

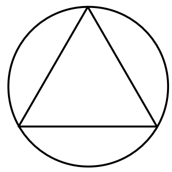
9. If Riana drives to the theatre at 100 km/h she will arrive 5 minutes before the movie starts. If she drives at 60 km/h she will arrive 5 minutes after the movie starts. How fast should Riana drive to arrive exactly when the movie starts?

10. How many positive integer divisors does the number 2008^{2009} have?

11. A hotel owner is very superstitious and avoids room numbers that include the string "13." For example, he doesn't number any room 213 or 1395. The first room is numbered 101, and the other rooms are given increasing consecutive numbers, with the unlucky numbers containing 13 skipped. The last room has is numbered 2009. How many rooms are in the hotel?

12. A lottery is created to help fund the construction of a new school, which will be built on a prime piece of Vancouver real estate and whose base will be in the shape of a perfect square. The prize money for the lottery is taken from the money raised by the purchase of tickets. There are 1000 tickets sold, and 20 of these tickets are randomly drawn. The owner of each winning ticket then draws one of the numbers 101, 102,…, 1001 from a bag, without replacing it. If the number drawn is not a perfect square then the winner's prize is $0; if it is the square of a prime number then the prize is $400, and if it is the square of another number then the prize is $50. If $10001 must be raised, what is the lowest price at which the tickets can be sold?

13. In the picture at the right, an equilateral triangle is inscribed in a circle. If the length of a side of the triangle is $\sqrt{12}$, find the area of the circle.



14. An ice cream cone has height 15 cm, and the diameter of its circular opening is 10 cm. A single scoop of ice cream, in the shape of a sphere, is placed in the cone so that the centre of the scoop coincides with the centre of the circular opening of the cone. What is the radius of the scoop?

15. You have a standard deck of 52 cards. Each card has a certain value, and there are four cards of each value. An ace has value 1, a jack has value 11, a queen has value 12, and a king has value 13. The other 36 cards have value 2, 3, 4, 5, 6, 7, 8, 9, or 10. How many different ways are there of picking two cards from the deck so that the sum of the values is less than 10? Note that the order of the two cards doesn't matter. For example, picking the 3 of diamonds first and the 2 of hearts second is the same as picking the 2 of hearts first and the 3 of diamonds second.

16. Eight marbles all look alike, but one of them is actually slightly heavier than the other seven, which all have the same weight. Using a balance scale, which lets you determine which of two chosen collections of marbles, if any, has greater total weight, explain how you can find the heavier marble with just three weighings. Can you find the heavier marble with just two weighings?

SOLUTIONS

Note: These concise solutions are meant for workshop leaders and teachers. Presentations to Grade 8–10 students should include additional detail and motivation. The solutions outlined here are considered appropriate for school students at this level; alternate solutions are often possible.

1. Answer: 36. Start with one of the posts at the end of the row. That post can be painted any one of the 4 colours. Now move toward the other end. For each unpainted post encountered, there are 3 choices for the colour, since it can't be the same colour as the previous post. There are total of 4 x 3 x 3 = 36 ways to colour the posts.

2. Answer: $6^{1/2}$. Using Pythagoras repeatedly gives $a^2 = 1^2 + 1^2 = 2$, $b^2 = a^2+1^2 = 3$, $c^2 = b^2+ 1^2 = 4$. The pattern is clear, and $e^2 = 6$.

3. Answer: 7.5 litres. 600 km has been driven on 3/4 of a tank, so a full tank has a range of 4/3 x 600 = 800 km. The remaining gas, when they stop, can take them another 200 km. Since there is still 300 km to go, they need to buy enough gas for 100 km, i.e. 1/8 x 60 = 15/2 litres.

4. Answer: 37.5 kg. There are 0.6 x 40 = 24 boys and 0.4 x 40 = 16 girls in the class. Since the average of some quantities is the sum of those quantities divided by the number of quantities, the sum is the average multiplied by the number. So, the sum of the weights of the boys is 40 x 24 = 960 kg. Now, for the average of all the members of the class to be 40 also, we need the total weight of the members of the class to equal 40 x 41 = 1640 kg. Since the teacher weighs 80 kg, the total weight of the girls is 1640 – 960 – 80 = 600 kg. The average weight of the girls is 600/16 = 75/2 = 37.5 kg.

5. Answer: 24. We may as well subtract 2007 from each of the numbers and use the numbers 1, 2, 3, 4, and 5, since that won't affect the equality of the two sums. The given condition is really just saying that the sum of the left and right entries in the middle row equals that of the top and bottom entries in the middle column. The only sums of pairs of that work are 1 + 4 = 2 + 3, 1 + 5 = 2 + 4, and 2 + 5 = 3 + 4. Each equality gives 8 ways of placing the numbers; the missing number is placed in the middle and the other numbers at the extremities. There are a total of 24 ways.

6. Answer: 25/3. Triangle *ABC* has sides of length 10 = 30/3 and 40/3; these are in the familiar ratio of 3 to 4 so this triangle is a scaled version of the standard 3-4-5 right triangle. Hence, so is the triangle *DEB*. Using the isosceles triangle *BDC*, the length of *DE* is 10/2 = 5. Since *DEB* is a scaled 3-4-5 right triangle, the length of *BD* is 5 x 5/3 = 25/3.

7. Answers: 6, no. Write the integer at $10a + b$, where a is the tens digit and b is the ones digit. Rearrange the equation $10b + a = 10a + b + 27$ to get $9(b – a) = 27$. So, $b – a = 3$. There are 6 such integers, namely 14, 25, 36, 47, 58, and 69. There are no such integers if 27 is replaced by 26 since 26 is not a multiple of 9.

8. Answer: 11 and –5. The right triangle with hypotenuse *AB* and one side along the *x*-axis is a standard 3-4-5 triangle. The right triangle with hypotenuse BC, which has length 10, and sides parallel to the axes has vertical side of length 10 – 4 = 6, and so is a 6-8-10 triangle. That triangle can lie to the right or left of B, which gives two values of x, namely 3 + 8 = 11 and 3 – 8 = –5.

9. Answer: 75 km/h. Let d the distance from Riana's home to the theatre in km. We are given $d/60 – d/100 = 1/6$ (10 minutes), which gives $d = 25$. It takes Riana 15 minutes to drive to the theatre at 100 km/h, so to arrive exactly when the movie begins she must take 20 minutes or 1/3 of an hour. The required speed s in km/h satisfies 25/s=1/3, so s = 75.

10. Answer: 6028 x 2010. Factor 2008: 2008 = 2 x 1004 = 2 x 2 x 502 = 2 x 2 x 2 x 251 = 2^3 x 251. Check that 251 is prime by verifying that no prime numbers up to 13 divide it. So, the prime factorization of 2008^{2009} is 2^{6027} x 251^{2009}. Any divisor of this number must also have only 2 and 251 in its prime factorization, with the powers at most 6027 and 2009 respectively. The number of such divisors is 6028 x 2010.

11. Answer: 1771. Count the number of numbers between 101 and 2009 that *do* include a string 13, and subtract this number from 1909, which is the total number of numbers between 101 and 2009. Systematically list the numbers containing 13. There are no 2-digit numbers since the smallest room number is 101. There are 9 3-digit numbers with 13 as the last two digits, namely 113, 213,…, 913. There are 10 3-digit numbers with 13 as the first two digits, namely 130, 131,…, 139. Note that there is no overlap between these two lists. There are 10 4-digit numbers ending in 13, namely 1013, 1113,…, 1913. There are also 10 4-digit numbers with 13 as the middle two digits, namely 1130, 1131,…, 1139, and these do not overlap with the first list of 4-digit numbers. Finally, there are 100 4-digit numbers that start with 13, namely 1300, 1301,…, 1399, but one of these, 1313, has already been counted. So, the number of numbers that include the string 13 is 9 + 10 + 10 + 10 + 99 = 138, and the number that don't is 1909 – 138 = 1771.

12. Answer: \$13.46. The perfect squares from 101 to 1001 are 11^2, 12^2, 13^2,…, 31^2. There are 21 such numbers, and those that are squares of prime numbers are 11^2, 13^2, 17^2, 19^2, 23^2, 29^2, and 31^2, which number 7. So, the maximum prize money that could be paid out is 7 x \$400 + 13 x 50 = \$3450. Since \$10001 must be raised, the total collected in ticket sales must be at least \$13451. Dividing by 1000 and rounding up to the nearest cent gives the final answer.

13. Answer: 4π. By Pythagoras, the length of the vertical altitude of the triangle is the square root of 12 – 3, i.e. 3, so the triangle has area $3 \times 2 \times 3^{1/2} / 2$. By breaking the triangle up into three congruent triangles one of whose sides is a side of the original triangle and whose third vertex is the centre of the circle and considering areas, we have that if h is the common distance from the original triangle's side to the centre of the circle, then $3 \times 2 \times 3^{1/2} / 2 = 3 \times h \times 2 \times 3^{1/2} / 2$, which gives $h = 1$. Thus the radius of the circle is 3 – 1 = 2 and the area is as given. Another solution involving the 30-60-90 right triangle is possible, if students possess this specialized knowledge.

14. Answer: $3/2 \times 10^{1/2}$. Draw a vertical cross-section, and note that within it is a right triangle whose sides have length 5 and 15 and whose hypotenuse, by Pythagoras, has length $5 \times 10^{1/2}$. The altitude of this triangle with respect to the hypotenuse is the radius r of the scoop. Computing the area of the triangle in two different ways gives $75/2 = 5r \times 10^{1/2}/2$ and $r = 3/2 \times 10^{1/2}$.

15. Answer: 280. Note first that none of the cards can have a value of 9 or larger, or else the sum would be at least 10, so we can ignore 9s, 10s, jacks, queens, and kings. If the first card picked has value 1, there are 4 choices for it, and the second card must have value between 1 and 8. Since one 1 has already been chosen, there are 3 + 4 x 7 = 31 choices for the second card. If the first card picked has value 2, the second card must have value between 1 and 7, and there are 3 + 4 x 6 = 27 choices. If the first card is a 3, there are 23 choices for the second, and if the first card is a 4 there are 19 choices. However, the pattern does not continue since if the first card is a 5, the second must have value between 1 and 4, and there are 16 choices. There are 12 choices when the first card is a 6, 8 if it is a 7, and 4 if it is an 8. Adding gives 4 x (31 + 27 + 23 + 19 + 16 + 12 + 8 + 4) = 560 *ordered* pairs, and dividing by two since the order does not matter gives the final answer. Alternate solution (sketch): There are 4 x 6 = 24 choices in which the same value appears (two aces, two 2s, two 3s, or two 4s). There are, as above, (7 + 6 + 5 + 4 + 4 + 3 + 2 + 1)/2 = 16 unordered pairs of distinct numbers whose sum is less than 10. The final count is 24 + 4 x 4 x 16 = 280.

16. Answer: Yes. For the first part: put 4 marbles on one side of the scale and the other 4 on the other side. The side with the heavier marble will be heavier in total, so we will know which set of 4 marbles contains the heavier one. Repeat, breaking the 4 marbles including the heavier one into two sets of 2 marbles to determine a pair of marbles one of which is the heavier one, and then in a final third weighing separate these 2 marbles to determine the heavier one. In fact two weighings *are* enough: Leave two marbles alone and break up the other 6 up into two sets of 3, placing each set on one side of the scale. If the heavier marble is among these 6, one of the sets of 3 will be heavier. Leave one of these heavier 3 marbles along and weigh the other 2. If these 2 marbles have the same weight, then the marble that was not weighed in the second weighing is the heavier one; if not, the heavier marble will have been determined. If the two sets of 3 marbles have the same weight, the heavier marble is one of the 2 that were not originally weighed, and the heavier one of these can be determined with a second weighing.