UBC Grade 11–12 Workshop Problems, 2009–2010

1. A sequence $\{t_n\}$ satisfies the formula $t_n = 2t_{n-1} + t_{n-2}$ for all $n > 2$. If $t_4 = 4$ and $t_6 = 20$, what is t_8?

2. A mathematician tends carefully to her prized apple tree, and it grows to a very unusual shape. The main trunk is 1 m long, and it splits into 2 branches each of which is 1/2 metre long. Each of those branches itself splits into two smaller branches, which are each 1/4 metre long, and so forth. The branches in the last layer all have length 1/1024 metre. One day the mathematician decides to chop the tree down and cut it into pieces consisting of the various individual branches. How many metres in total of tree limbs does she have?

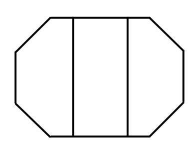
3. Cleopatra has an elaborate flower container. It consists of three nested vases, each of which has height 30 cm. The outermost vase has a square base of side length 10 cm. The middle vase is cylindrical and just fits within the outer vase. Finally, the innermost vase also has a square base and just fits within the middle vase. Cleopatra pours water only into the gap between the middle and innermost vases. What volume of water does Cleopatra use?

4. John has a jar of 57 jelly beans. There are 30 liquorice beans and 27 raspberry ones. John chooses beans from the jar randomly. If he picks a liquorice bean, he eats it, but otherwise he places it back in the jar. What is the probability that, in his first five picks, John eats the first 3 beans but not the last 2? Express your answer as a simplified fraction instead of a decimal or percentage.

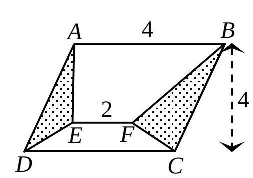
5. Alberto is creating a menu for his new pizzeria. He wants to have 15 pizzas listed on the menu, each with the same number of distinct toppings. The toppings he has available are mushrooms, peppers, pepperoni, anchovies, pineapple, and olives. What is the maximum number of toppings Alberto can put on each of the 15 pizzas?

6. Mary Ann leaves school at the same time every Monday and cycles the same path to the conservatory for her piano lesson, which starts at 5 p.m. sharp. If she cycles at 15 km/h she arrives at the conservatory at 4:30 p.m.; if she cycles at 10 km/h she arrives at 5:15 p.m. At what speed should she cycle if she wants to just make it to her lesson on time (assuming her bike has no flats)?

7. A table in the shape of a regular octagon has a side length of 50 cm. It has a rectangular table leaf that can be inserted by separating two halves, as shown in the diagram. The sides of the original, unexpanded table have the same length as the width of the insert. What is the area of the table with the insert?



8. In the figure, $ABCD$ is a parallelogram of height 4, EF is parallel to AB, AB has length 4, and EF has length 2. Determine the area of the shaded region.

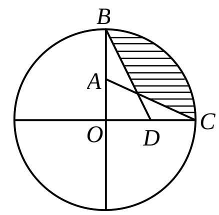


9. Sam's house is n blocks north of the school and e blocks east. Freddy's house is the same distance from the school as Sam's house, but in a 90-degree counterclockwise direction from Sam's house. Carly's house is 2 blocks north of Freddy's house and 4 blocks east, and is also 3 blocks north and 3 blocks west of Sam's house. Assuming that blocks are the same distance in the east-west direction as the north-south direction, determine n and e.

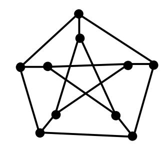
10. A circle is inscribed in an equilateral triangle. Which region has a greater area, the region inside the circle or the region inside the triangle but outside the circle?

11. In a standard 52-card deck, aces are worth 1 or 11 points, jacks, queens, and kings are worth 10, and all other cards, namely 2, 3,.., 10, are worth their face value. There are four of each of these cards. In the game of blackjack, the goal is to collect cards that total close to, but not greater than, 21. What is the probability that the first two cards dealt to a player will total *exactly* 21?

12. The circle in the diagram has area 4π and centre O. The two diameters shown meet at right angles, and the points A and D are respectively halfway between O and the points B and C. Find the area of the shaded region.



13. In the figure, there are 10 points, marked with heavy dots. Each point is coloured one of a certain number of colours. What is the minimum number of colours needed to colour all 10 points in such a way that no two points that are connected by a line have the same colour? Explain *why* your answer really is the minimum.



14. Two motorcycles are driving as fast as they can toward each other from opposite ends of a long drag strip. The first time they pass each other, they are 3 km from one end. When they reach the respective ends, they immediately turn around and drive toward the other end. At their second passing, they are 1 km from the other end. Find the length of the drag strip.

15. The number $10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 3618800$ ends in 2 0s. Determine the number of 0s at the end of 2009!

16. An empty conical cup of height 15 cm has a volume of 180π cm^3. A ping-pong ball of diameter 40 mm is dropped into the cup. How high above the bottom of the cup is the bottom of the ball? You may use the fact that the volume of a cone of radius r and height h is $\pi r^2 h/3$.

17. A taxi company replaces its cabs every N months. A new cab costs \$15,000, and the resale price of a used cab is \$(11,000-200N). Maintenance of a cab costs \$(300 + 20(k - 1)) during the kth month since the purchase (so, for example, maintenance during the month of purchase costs \$300). The average monthly cost is

$$\frac{(\text{cost of new cab}) \; + \; (\text{total maintenance over } N \text{months}) \; - \; (\text{resale price})}{N}$$

Find N so that this average monthly cost is minimal. Do *not* use calculus, but do use the fact that for any positive real numbers a and b, $(a+b)/2 \geq \sqrt{ab}$.

18. Six professors are car pooling to UBC during ride-share week. They have rented a bus that has five rows, each of which has two seats. The first row includes a driver, who is one of the professors. Dr. Chen likes company and so must share a row with someone else. Dr. Anstee dislikes Dr. Barlow and Dr. Doebeli and so cannot share a row with one of them. Dr. Barlow, for safety reasons, must ride in the third or fourth row. Dr. Ekeland is alone in a row and if there is a row in front of him then it is empty. Dr. Doebeli is riding in the second row. No information is known about where Dr. Gupta is sitting. How many different seating arrangements are there that satisfy all of these conditions? If two passengers in a given row swap seats, it is not considered a different seating arrangement.

SOLUTIONS

Note: These concise solutions are meant for workshop leaders and teachers. Presentations to Grade 11–12 students should include additional detail and motivation. The solutions outlined here are considered appropriate for school students at this level; alternate solutions are often possible.

1. Answer: 116. Using the recursion for t_6 gives t_5=8, and then t_7 = 48 and t_8=116.
2. Answer: 11 m. The *n*th layer of branches, where we consider the trunk the 1st layer, has 2^{n-1} branches each of length $1/2^{n-1}$ and hence total length 1. Since $1024 = 2^{10}$, there are 11 layers of branches.
3. Answer: 30(25π – 50) cm^3. A cross section gives a square inscribed in a circle that is inscribed in a square of side length 10. The circle has radius 5 and the small square has side length $10/\sqrt{2} = 5\sqrt{2}$. The relevant cross-sectional area is 25π – 50 and the volume is this area times the height, 30.
4. Answer: 29/836. The probability is (30/57)(29/56)(28/55)(27/54)(27/54) = 29/(4 × 11 × 19).
5. Answer: 4. There are 6 toppings and we need to be able to select at least 15 distinct *k*-subsets, so we need $\binom{6}{k}$ to be at least 15 with *k* maximal.
6. Answer: 45/4 km/h. The cycling time t in hours at 15 km/h satisfies $15t = 10(t + ¾)$, which gives $t = 3/2$ and the distance is 15(3/2) = 45/2 km. To arrive at 5:00 p.m. sharp, Mary Ann must leave her flat cycling at a speed of (45/2)/2 = 45/4 km/h.
7. Answer: $7500(1 + \sqrt{2})$ cm^2. The height of the table is $50 + 2 \times 50/\sqrt{2} = 50(1 + \sqrt{2})$ and the width is $50(2 + \sqrt{2})$. Multiply to get the area of the circumscribing rectangle and then subtract 4 × the area of the corner triangles to get an answer $2500\ (1 + \sqrt{2})\ (2 + \sqrt{2}) - 4(50/\sqrt{2})^2/2$, which simplifies to the given answer.
8. Answer: 4. The parallelogram has area 4 × 4 = 16. If *x* is the vertical separation between *AB* and *EF* then that between *EF* and *DC* is $4 - x$ and the sum of the areas of the trapezoids is $x(4 + 2)/2 + (4 - x)(2 + 4)/2 = 12$.
9. Answer: $n = 3$ and $e = 4$. Coordinatize with the school at the origin. Then $S = (e,n)$, $F = (-n,e)$, and $C = S + (-3,3) = F + (4,2)$. Solve the system of equations $e - 3 = 4 - n$, $n + 3 = e + 2$ to get the given values.
10. Answer: the region inside the circle. If the circle's radius is *r*, the triangle, by 30-60-90-triangle chasing, has side length $2\sqrt{3}r$ and area $(2\sqrt{3}r)(3r)/2 = 3\sqrt{3}r^2$. The circle has area πr^2 and now use $2\pi > 3\sqrt{3}$.
11. Answer: 32/663. There are $\binom{52}{2} = 26 \times 51$ equally likely choices for an unordered 2-card draw. We need one of these cards to be an ace and the other to have face value 10. The number of choices is 4 × 16. Cancel a 2.
12. Answer: π – 4/3. Call the intersection of *AC* and *BD E*. Coordinatize to discover, noting that the radius of the circle is 2, that $E = (2/3, 2/3)$. Now the unshaded first-quadrant area comprises a square of area $(2/3)^2$ and two triangles each of area (2/3)(4/3)/2 and the shaded area is as given, by subtracting.
13. Answer: 3. A 2-colouring is impossible since the vertices of the outer pentagon must alternate in colour and there are an odd number of them. For a 3-colouring, pick an outer vertex *A*, colour it 1 and then colour the other clockwise vertices respectively 2, 1, 2, and 3. Now colour the inner vertex closest to *A* 2 and the remaining ones, in clockwise order again, 3, 3, 1, and 2. Note the inner star is really a pentagon in disguise.
14. Answer: 8. If the required length is *x*, then equating ratios of the total distances travelled at the two meeting times gives the equation $3/(x - 3) = (x + 1)/(2x - 1)$, which yields a quadratic equation whose only positive solution is 8.
15. Answer: 500. The number of 0s is the number of factors of 5. Realizing that multiples of 5 contribute 1 factor of 5 but that multiples of 25 contribute 2 factors, etc., this number is [2009/5] + [2009/25] + [2009/125] + [2009/625], where [*x*] denotes the greatest integer less than or equal to *x*. We get 401 + 80 + 16 + 3 = 500.
16. Answer: $\sqrt{29} - 2$ cm. Draw a careful cross-section, noting the tangency of the ball to the side of the cone and hence a right angle at that point of tangency. Using the given formula, the radius of the cone is 6. If *x* is the distance form the cup's vertex to the point of tangency then similar triangles gives $x = 5$, whence the centre of the ball, by Pythagoras, is $\sqrt{29}$ high.
17. Answer: 20 months. We have to minimize the expression
$$\frac{15000 + \sum_{k=1}^{N}(300 + 20(k-1)) - (11000 - 200N)}{N}.$$
This simplifies to $4000/N + 490 + 10N$. Since the first and last terms have product $4000 = 200^2$, their sum, by the given inequality, is at least 2 × 200 = 400. The value $N = 20$ achieves this minimum.
18. Answer: 46. Call the professors A, B, C, D, E, and G. D is taken care of. Now seat E. He can't be in row 2 or 3 since D is in row 2, and also can't be in row 4 since then B would be unseatable. Suppose E is in row 5. Then the other seats in rows 4 and 5 are empty, B is in row 3, and A is in row 1. The two remaining passengers, C and G, can sit in any of the remaining seats in rows 1, 2, or 3, and there are 3 × 2 ways to assign them. Now suppose E is in row 1 by himself, and B is seated in row 3 or 4. Now seat A. He must occupy one of the two empty rows. Now seat C. If we place him in one of the 3 available occupied rows, then G can sit in any of the remaining 3 rows. If we place C in the empty row then G must sit with him. The second case gives 2 × 2 × (3 × 3 + 1) arrangements.