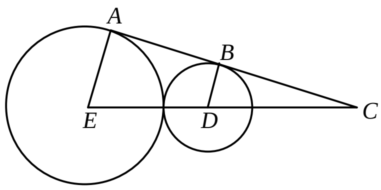
UBC Grade 11–12 Workshop Problems, 2008–2009

1. A circle has the same centre as a regular hexagon that has side length 1. The circle's diameter equals the distance from the centre to one of the vertices of the hexagon. Find the area of the circle.

2. Stephen and Jeff are trouble makers; they have decided to roll a bouncy ball down a hallway to one another. The boys are at opposite ends, in the middle of a 2-m-wide hallway. Jeff rolls the ball toward a side wall of the hallway so that the angle between the wall and the ball's path is 45°. The ball bounces away from the wall at the same angle, then hits the other side wall, bounces again, and reaches Stephen after a total of eight bounces. How long is the hallway?

3. In the diagram, the two circles are tangent, E and D are their respective centres, ABC is a straight line that is tangent to both circles at A and B respectively, and EDC is also a straight line. If the length of EC is 30 and the length of ED is 15, find the radius of the smaller circle.



4. Two walls and the ceiling of a room meet at right angles at a point P. A fly is in midair near this corner, 1 metre from one of the walls, 8 metres from the second wall, and 9 metres from P. How many metres below the ceiling is the fly?

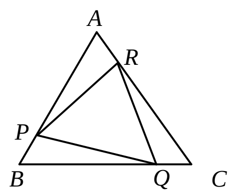
5. Define a sequence by $t_1 = 1$ and $t_n = a^n / t_{n-1}$ for n greater than or equal to 2, where a is a fixed positive real number. Express t_{1000} in terms of a.

6. Eight marbles all look alike, but one of them is actually slightly heavier than the other seven, which all have the same weight. Using a balance scale, which lets you determine which of two chosen collections of marbles, if any, has greater total weight, explain how you can find the heavier marble with just two weighings.

7. Twenty-five students are in a circle, facing the centre. Thirteen of these students are honest and always tell the truth, and 12 of them always tell a lie. Each student knows the truth-telling status of every other student. Each student makes one of the statements a) "my neighbour on the left is honest" or b) "my neighbour on the left is a liar." Exactly 1 student makes statement a), and the other 24 make statement b). Is the student who made statement a) honest or a liar?

8. An ice-cream cone has height 15 cm, and the diameter of its circular opening is 10 cm. A single scoop of ice cream, in the shape of a sphere, is placed in the cone so that the centre of the scoop coincides with the centre of the circular opening of the cone. What is the radius of the scoop?

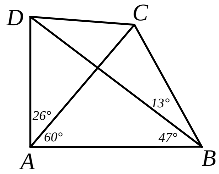
9. In the diagram, triangle ABC is equilateral with area 16. Points P, Q, and R are placed on the sides of ABC so that the distance from A to P is 3 times the distance from P to B, the distance from B to Q is 3 times the distance from Q to C, and the distance from C to R is 3 times the distance from R to A. Find the area of triangle PQR.



10. A carnival game offers you the opportunity to bet \$1 on a number from 1 through 6. Two dice are rolled. If the number you selected comes up on one die, you get \$3 back, including your \$1 bet. If it comes up on both you get \$5 back, including your \$1 bet. You lose the \$1 bet if the number does not appear on either dice. Should you play this game?

11. A stool with a circular seat is pushed into the corner of a room so that the seat touches both walls. A marked point on the edge of the seat is measured to be 5 inches from one wall and 10 inches from the other. What are the possible diameters of the stool's seat?

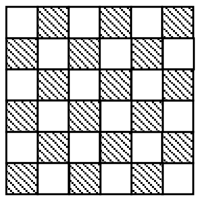
12. In the diagram, the angles *DAC, CAB, ABD,* and *DBC* are given. Find the angles *BDC* and *ACD*.



13. Jack has three daughters. He challenges his neighbour, who is good at math, to guess their ages. Jack and his neighbour are standing in the front yard of Jack's house. Jack tells his neighbour that the product of the daughters' ages is 72 and the sum of the ages is the street number of his house. The neighbour says he cannot determine the individual ages of the daughters. What is the street number of Jack's house?

14. Rufus the dog is tethered to the middle of the bottom of one side of this 1-metre-square doghouse with a 2.5-metre-long leash. If Rufus is not allowed to enter the doghouse while tethered, what is the area of the region that Rufus can cover while he is tethered?

15. Andy is playing a game on the 6 x 6 checkerboard shown in the diagram. He starts at one of the shaded squares on the top row and keeps moving diagonally downward to an adjacent shaded square until he is at the bottom row. If he has a choice of moving left or right, he picks the direction randomly. Which square should Andy start at to have the greatest chance of ending up at the middle shaded square in the last row?



16. A six-digit number has the property that if its first three digits, as a block, are interchanged with its last three digits then the number is six times as large. There is only one such number. Find this number, and explain why it is the only one.

17. Consider the sequence 8, 33, 58, 83, 108,… The first term is a perfect cube. How many more terms are perfect cubes?

SOLUTIONS

Note: These concise solutions are meant for workshop leaders and teachers. Presentations to Grade 11–12 students should include additional detail and motivation. The solutions outlined here are considered appropriate for school students at this level; alternate solutions are often possible.

1. Answer: $\pi/4$. Two adjacent vertices of the hexagon together with the centre form an equilateral triangle, so the distance from the center to a vertex is 1 and the radius of the circle is 1/2.

2. Answer: 16. Draw a picture. The first leg of the path is the hypotenuse of a 45-45-90 right triangle, and the distance from the hallway's nearest corner to the first point of impact is 1. The length of the hallway is 16 times this distance.

3. Answer: 5. The triangles *CDB* and *CEA* are similar with ratio 2, so the length of *AE* is twice that of *DB*. But, noting that if the point of tangency of the two circles is *F* then *EFD* is a straight line, these lengths are the respective radii. The sum of the radii is the length of *ED*, which is 15, so the smaller radius is 5.

4. Answer: 4. Project the fly's position vertically upward onto the point *P* on the ceiling. In the plane of the ceiling we have a right triangle with sides of length 1 m and 8 m and hypotenuse of length $65^{1/2}$ m. Now connect the corner of the room, this point P, and the fly's position to get another right triangle with hypotenuse of length 9 m, one side of length $65^{1/2}$ m, and the third side of length 4 m.

5. Answer: a^{501}. The first few terms are 1, a^2, a, a^3, a^2, a^4, a^3, and the general pattern is readily guessed and checked.

6. Leave two marbles alone and break up the other 6 up into two sets of 3, placing each set on one side of the scale. If the heavier marble is among these 6, one of the sets of 3 will be heavier. Leave one of these heavier 3 marbles along and weigh the other 2. If these 2 marbles have the same weight, then the marble that was not weighed in the second weighing is the heavier one; if not, the heavier marble will have been determined. If the two sets of 3 marbles have the same weight, the heavier marble is one of the 2 that were not originally weighed, and the heavier one of these can be determined with a second weighing.

7. Answer: honest. If a given student and the student's left neighbour are both truth tellers, the student makes statement a). If they are both liars, the student still makes statement a). If the student is a liar and the left-neighbour is honest, or vice-versa, the student makes statement b). Moving to the right through the circle, label the students S_1, S_2,…, S_{25}, where S_1 is the student who made statement a). Since each of S_2,…, S_{25} makes statement b), the truth-telling status of student S_n is different from that of student S_{n-1}, for n = 2,…, 25. So, there are 12 truth-tellers and 12 liars among students S_2,…, S_{25}, and student S_1 is a truth-teller.

8. Answer: $3/2 \times 10^{1/2}$. Draw a vertical cross-section, and note that within it is a right triangle whose sides have length 5 and 15 and whose hypotenuse, by Pythagoras, has length $5 \times 10^{1/2}$. The altitude of this triangle with respect to the hypotenuse is the radius r of the scoop. Computing the area of the triangle in two different ways gives $75/2 = 5r \times 10^{1/2}/2$ and $r = 3/2 \times 10^{1/2}$.

9. Answer: 7. The triangles *PQB*, *QRC*, and *RPA* are all congruent, as they have two common sides and an equal interior angle. Connecting *P* and *C* shows that triangle *PQB* has area (1/4) x (3/4) = 3/16 that of *ABC*, so the area of triangle *PQR* is 7/16 that of *ABC*.

10. Answer: no. There are 36 equally likely outcomes when the two dice are rolled. Of these, 5 x 5 = 25 don't contain the number you chose and 11 do. Of these 11, 10 have your number on exactly one die and 1 has it on both dice. So, the amount you expect to get back if you bet \$1 is \$3 x 10/36 + \$5 x 1/36 = \$35/36. Since you've bet a larger amount than you expect to get back, you should not play the game.

11. Answers: 10 inches and 50 inches. Coordinatize so the corner is at the origin and the stool is in the first quadrant. The base of the seat has equation $(x - r)^2 + (y - r)^2 = r^2$, where r is the radius of the base. We can set $x = 5$ and $y = 10$ by symmetry, which gives the equation $r^2 - 30r + 125 = 0$, leading to the two given solutions.

12. Answers: 30° and $y = 77°$. Angle *BDA* equals 180 – 47 – 60 – 26 = 47° so triangle *BDA* is isosceles. Also, angle *BCA* is 60° and triangle *BCA* is equilateral. Combining these observations gives that the lengths of *DA* and *CA* are equal so angles *ADC* and *ACD* are equal. Calling the angle *BDC* x and angle *ACD* y, we have $47 + x = y$ and from triangle *ACD* that $x + y + 73 = 180$. Solving gives $x = 30°$ and $y = 77°$.

13. Answer: 14. There are 12 factorizations $72 = abc$ where a, b, and c are positive integers with $a \le b \le c$; the associated triples (a,b,c) are (1,1,72), (1,2,36), (1,3,24), (1,4,18), (1,6,12), (1,8,9), (2,2,18), (2,3,12), (2,4,9), (2,6,6), (3,3,8), and (3,4,6). The respective sums are 74, 39, 28, 23, 19, 18, 22, 17, 15, 14, 14, and 13. If the sum were anything but 14 then the neighbour would infer the individual ages since only one triple has that sum. But if the sum is 14, then there are two possible triples.

14. Answer: $(127/24) \times \pi + 3^{1/2}/4$. Draw the picture to compute that the area of the region is $(1/2) \times \pi \times (5/2)^2 + 2 \times (1/4) \times \pi \times 2^2 + 2 \times (1/4) \times \pi \times 1^2$ minus the area of the double-counted region on the side of the doghouse opposite to the anchor of the tether. The area of that double-counted region is $2 \times (1/6)\ \pi \times 1^2$ minus the area of an equilateral triangle of side length 1, which is $3^{1/2}/4$. Combining gives the final answer.

15. Answer: leftmost square. If Andy starts at the leftmost shaded square in the top row, the probability that he goes through either of the two left squares in the second row is 1/2. The probability that he goes through the leftmost square in the third row is 1/2 + (1/2)(1/2) = 3/4 and that he goes through the middle square is (1/2)(1/2) = 1/4. Proceeding downward, the probability of going through a given square is the sum of the probabilities in the up-to two squares above it multiplied by a factor, where that factor equals 1 if the higher square is on a side and 1/2 if it is not. So, the probabilities for the squares in the fourth row are 3/8, 1/2, and 1/8; for the fifth row they are 5/8, 5/16, and 1/16; and for the sixth row they are 5/16, 15/32, and 7/32. If Andy starts in the middle shaded square in the top row, the respective probabilities are 1/2, 1/2; 1/4, 1/2, 1/4; 1/8, 3/8, 1/2; 5/16, 7/16, 1/4; and 5/32, 3/8, and 15/32. If he starts at the rightmost square, they are 1; 1/2, 1/2; 1/4, 3/4; 1/8, 1/2, 3/8; and 1/16, 5/16, and 5/8. The last middle probability is highest if Andy starts at the leftmost square.

16. Answer: 142,857. Write the original number as $1000a + b$, where a is a number between 100 and 999 that represents the first three digits and b is a number between 0 and 999 that represents the last three digits. We have the equation $1000b + a = 6(1000a + b)$, which becomes $994b = 5999a$. Factoring gives $2 \times 7 \times 71 \times b = 7 \times 857 \times a$. Noting that 857 is not divisible by any of the primes 2, 7, or 71, we conclude that $b = 857$ and it follows that $a = 142$.

17. Answer: infinitely many. The terms in the sequence are of the form $8 + 25n$, where n is a nonnegative integer. For this to equal a perfect cube, we need $8 + 25n = m^3$, where m is an integer ≥ 2. Rearrange this equation as $m^3 - 8 = 25n$. A given m will work provided $m^3 - 8$ is a multiple of 25. Since $m^3 - 8 = m^3 - 2^3 = (m - 2)(m^2 + 2m + 4)$, m will work if $m - 2$ is a multiple of 25. There are infinitely many such m.