

Mathematics 414, Solutions to Problem Set #7

Problem 1. Find an exact expression for the smallest positive real number x such that $\cos 3x + \sin 2x = 0$. Of course you may use any device to decide what the answer might be, but there should be a proof that your answer is correct. Formally, the question asks for the answer in radians, but you may prefer to work in degrees.

Solution. In mathematical work, $\sin u$ is the sine of the *number* u . The sine function is formally defined in any one of several ways, none of which involve “angles.” This sine function can be connected with the sine of elementary trigonometry by saying that $\sin u$ is the traditional sine of an angle of size u , where u is measured in *radians*. It will turn out that the answer to our problem (in degrees) is 54. But to be technically correct, we must convert to radians, and a 54 degree “angle” has radian measure $54\pi/180$, or more simply $3\pi/10$. Even though $3\pi/10$ is the (only) right answer, we do the analysis in “high-school” style, in degrees.

We can let microprocessors do much of the work, either by reading the answer off a graph or by using the *Solve* button on a calculator. Or else we can use even a simple scientific calculator, and a crude informal numerical procedure, or a more sophisticated numerical procedure such as *Newton’s Method*, to approximate the root.

If we work in degree notation, we fairly quickly find that the answer is either 54° or very close to that. Unfortunately, the graphing calculator, the *Solve* button, or numerical procedures can never tell us that we have found the solution *exactly*. So a numerical calculation of the type described above, followed by the statement that the answer is 54° , is not adequate.

But in this case, once we have *conjectured* that the answer is exactly 54° , *verification* of the conjecture is quick. For if $x = 54^\circ$, then $3x = 162^\circ$, and $2x = 108^\circ$.

By basic symmetry properties of the sine and cosine function, the cosine of 162 degrees is the negative of the cosine of 18 degrees. Also, the sine of 108 degrees is equal to the sine of 72 degrees, which in turn is equal to the cosine of 18 degrees. Add up: the sum is 0. So we are finished: the answer is 54° , or more properly $3\pi/10$.

Another Way. Or else we can bring out the machinery of trigonometric identities. The identity $\cos 3x = \cos 2x \cos x - \sin 2x \sin x$, together with the usual double angle identities, yields after a while $\cos 3x = 4 \cos^3 x - 3 \cos x$.

Thus we can rewrite the original equation as $4 \cos^3 x - 3 \cos x + 2 \cos x \sin x = 0$. The common factor $\cos x$ produces the obvious solution $x = \pi/2$ (90 degrees). It will turn out that this is not the *smallest* positive solution.

Now look for solutions of $4 \cos^2 x - 3 + 2 \sin x = 0$. This can be rewritten as $4 \sin^2 x - 2 \sin x - 1 = 0$. By the Quadratic Formula, the solutions are $\sin x = (1 \pm \sqrt{5})/4$. So we want the least positive solution of $\sin x = (1 + \sqrt{5})/4$.

Perhaps it is time to go to the calculator. We find that x is approximately 0.942477796 radians. Out of curiosity, we might go to degrees. To the limit of calculator accuracy, the result seems to be 54° , or close to that.

Interesting but not conclusive. We have found a numerical solution, and if we believe that answers to all problems must be ‘nice,’ we may even believe that the answer really is 54° . But we need to *prove* this. The verification that 54° is correct can be done exactly as in the first solution.

Comment. This solution was much more complicated than the first one. But as a bonus we found an exact expression for the sine of the 54° angle. And the *golden number* $(1 + \sqrt{5})/2$ showed up once again..

The identity $\cos 3x = 4 \cos^3 x - 3 \cos x$ (and the analogous $\sin 3x = 3 \sin x - 4 \sin^3 x$) have a number of uses in mathematics. The identity plays a crucial role in the standard proof that not all angles can be trisected with compass and straightedge. It also comes up in one of the classical approaches to the cubic equation.

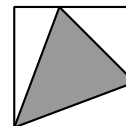
Another Way. We need to find where the two curves $y = \cos 3x$ and $y = -\sin 2x$ meet. As was remarked earlier, technically the meaning of $\sin u$ is the sine of u , where u is given in radians. But since radian measure is less familiar, we will work with degrees. A casual sketch shows that the two curves first meet at a point a little short of 60° .

Recall that $\cos(90^\circ + u) = -\sin u$. So we want the first positive solution of $\cos 3x = \cos(90^\circ + 2x)$. If we bear in mind that the answer is a little short of 60° , we can see that the angles $3x$ and $90^\circ + 2x$ should be symmetrical about 180° , and therefore

$$90^\circ + 2x - 180^\circ = 180^\circ - 3x, \quad \text{so} \quad x = 54^\circ.$$

The geometrical approach just used works for $\cos ax + \sin bx = 0$, where a and b are any real numbers, and for other closely related equations.

Problem 2. In the diagram, a square is divided into four triangles. The three outer triangles have equal area. What is the ratio of the area of the fourth (shaded) triangle to the combined area of the outer triangles?



Solution. We are only interested in ratios of areas (and we obviously cannot do better, since only scaling-independent information has been provided). So we can let the area of an outer triangle be anything we please. Of course something generic like “ a ” would be fine. But we might as well let the area be a simple specific number. So let the area of any of the outer triangles be $1/2$.

Since the two outer triangles that have a vertex at the lower left-hand corner have the same area, and one of their legs is the side of the square, the other legs are equal in length. It follows that the legs of the outer triangle at the upper right-hand corner are equal, that triangle is right-angled isosceles. The area is $1/2$, so each leg of that triangle has length 1.

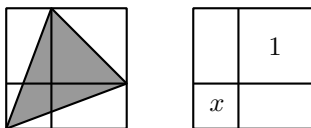
Let s be the side of the square. Then the outer triangle whose right angle is at the lower right-hand corner has legs $s - 1$ and s . The area is $1/2$, so

$$s(s - 1) = 1, \quad \text{and hence} \quad s = \frac{1 + \sqrt{5}}{2}.$$

Thus s^2 , the area of the square, is $(3 + \sqrt{5})/2$. Subtracting the combined area of the outer triangles, we find that the shaded triangle has area $\sqrt{5}/2$. The desired ratio is therefore $\sqrt{5}/3$.

Comment. The *golden ratio* $((1 + \sqrt{5})/2$ to 1) continues to show up in problems.

Another Way. Draw horizontal and vertical lines as in the picture (left) below. In the picture on the right, we have simplified things by throwing away the triangles.



Let the areas of the outer triangles be $1/2$. Then the square on the upper right has area 1. Let the area of the square on the lower left be x . The square with area x together with the rectangle on its right has area 1, so each of the remaining unlabelled rectangles has area $1 - x$.

The ratio of the areas of two rectangles of the same height is equal to the ratio of their widths. It follows that

$$\frac{x}{1-x} = \frac{1-x}{1}.$$

This equation can be rewritten as $x^2 - 3x + 1 = 0$. Thus $x = (3 - \sqrt{5})/2$ (the other root is too big). Thus the unlabelled rectangles each have area $(\sqrt{5} - 1)/2$. Adding up, we find that the area of the full square is $(3 + \sqrt{5})/2$, and therefore the inner triangle has area $\sqrt{5}/2$.

Comment. Let's generalize. Suppose that the areas of the two outer triangles that have a corner at the bottom left are a and b , and that the triangle at the upper right has area c . A calculation much like the one above shows that the area of the inner triangle is then

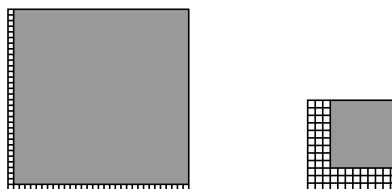
$$\sqrt{(a+b+c)^2 - 4ab}.$$

Problem 3. (a) A square is divided into 64 squares, of which at least 63 are 1×1 . What can one say about the 64-th square? (b) What can one say about the 64-th square if a *rectangle* is split into 64 squares, of which at least 63 are 1×1 ?

Solution. (a) Let the remaining square be $x \times x$. It is geometrically obvious that x is an integer. The original square has area $x^2 + 63$. This is a perfect square, say y^2 . We have reached the equation $y^2 - x^2 = 63$, or equivalently $(y-x)(y+x) = 63$. So $y-x$ must be a divisor of 63, and it is clear that since $y-x \leq y+x$, the number $y-x$ must be a "small" divisor of 63. The possibilities are $y-x = 1, 3, \text{ or } 7$, giving $y+x = 63, 21, \text{ or } 9$.

In each case, solve for x and y . We get $x = 31, y = 32, x = 9, y = 12, \text{ or } x = 1, y = 8$, a total of 3 possibilities. It is probably not absolutely clear that all these arithmetical possibilities are in fact geometrically possible. So we really need to give pictures that realize these geometrically. That is not hard. The case $x = 1, y = 8$ is geometrically obvious, the 64-th square is 1×1 , the remaining square in an 8×8 chessboard.

The remaining cases are dealt with by the pictures below (on the left, $x = 31, y = 32$, and on the right, $x = 9, y = 12$).

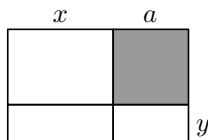


In each picture, the large shaded square is the 64-th, and it is bordered by a constellation of 63 nearly invisible 1×1 squares.

(b) Take a rectangle made up of our 64 squares. Wherever the 64-th square is, we can push it up and to the right, and rearrange the 1×1 squares so that the 64-th square is at the upper right-hand corner. (It is the shaded square in the diagram below.) Suppose that the 64-th square has side a . It is clear that a is an integer. Let the rectangle be obtained by adding to this square our 63 1×1

squares. Label the sides of the added stuff as in the picture. The area of the added stuff is 63. But it is also $xy + ax + ay$, so we obtain the equation

$$xy + ax + ay = 63, \quad \text{or equivalently} \quad (x + a)(y + a) = 63 + a^2.$$



It is clear that $a \leq 63$. We examine possible values of a , one by one. This is not particularly efficient, but with a little luck we will not miss any.

Try $a = 1$. We want $(x+1)(y+1) = 64$, obviously possible. Try $a = 2$. We want $(x+2)(y+2) = 67$, not possible since 67 cannot be expressed as a product of numbers ≥ 2 . Try $a = 3$. So we are looking at $(x+3)(y+3) = 72$, and 72 clearly can be expressed as a product of numbers ≥ 3 . Try $a = 4$. We are looking for a factorization of 79 as a product of two numbers ≥ 4 . This is not possible.

Try $a = 5$. We want to express 88 as a product of numbers ≥ 5 , which is possible. Similarly, $a = 6$ leads to a “big” factorization of 99, possible. Try $a = 7$. We want a big factorization of 112, which is possible, since $112 = 8 \cdot 14$. Try $a = 8$. So we look at factors of 127, no good. Try $a = 9$. This works nicely, we are factoring 144. Try $a = 10$. We want big factors of 163, no good. Try $a = 11$. No good, we want to factor 184, and the nearest we can get to a big factorization is 8×23 . Try $a = 12$. We want a big factorization of 207, no good.

We have examined 12 cases, only 41 to go. Let’s change tactics. We want $xy + a(x + y) = 63$, and we can assume $a \geq 13$. That forces $x + y \leq 4$. Without loss of generality we may assume $y \leq x$. Try $y = 0$. We want $ax = 63$, which gives the new possibilities $a = 21$ and $a = 63$. Try $y = 1$. If $x = 1$, we get $1 + 2a = 63$, giving $a = 31$. If $x = 2$, we get $2 + 3a = 63$, impossible. If $x = 3$, we get $3 + 4a = 63$, so $a = 15$. We cannot have $x > 3$, for that would make $x + y$ too big. Finally, try $y = 2$. Then y can only be 2, but that’s not possible, indeed we cannot ever have both x and y even.

Comment. There is I think not much prospect of a successful generalization of part (b). Suppose we have $n + 1$ squares, of which n are 1×1 . The possible sizes of the remaining squares depend on subtle arithmetical properties of n . Even part (a) is not completely easy in the general case. It is easy to connect the possible sizes to solutions of the Diophantine equation $y^2 - x^2 = n$. The solutions are easy to connect to solutions of $uv = n$. But if n is huge, factoring n is a computationally very difficult problem.