Mathematics 414, Solutions to Problem Set #6

Problem 1. Note that

$$7^2 = 49, \quad 67^2 = 4489, \quad 667^2 = 444889, \quad 6667^2 = 44448889.$$

Show that the apparent pattern continues forever.

Solution. We examine the small example 6667^2 to see what is going on. It is marginally easier to look at $6667^2 - 1$, that is, the product 6668×6666, and analyze why this is equal to 44448888. More simply, divide each term by 2, and analyze why 3334×3333 is equal to 11112222.

It takes a fair bit of effort to typeset a school multiplication, so what follows will look harder than it is. Multiply 3334 by 3. We get 10002. Multiply 33334 by 30. We get 100020. Multiply 3334 by 300. We get 1000200. Finally, multiply 3334 by 3000. We get 10002000. Add up. We get 11112222.

Precisely the same sort of thing happens when we multiply $333\ldots34$ by $333\ldots33$, where each of the two terms has n digits. Imagine doing a conventional school multiplication. We need to multiply 333…34 by 3, 33, 300, and so on, and add up. Multiply 333…34 by 3. It is easier to multiply $333\ldots33 + 1$ by 3. The result is $999\ldots99$ (n 9's) plus 3, and this sum is $1000\ldots02$ ($n-1$ 0's).

So in the school multiplication we end up finding the sum 10…02 ($n-1$ 0's) plus 10…020 plus 10…0200 and so on (the last summand has $n-1$ terminal 0's). Now add up: this is easy since every column has exactly one non-zero entry. We get (reading *from right to left* n 2's and then n 1's.

Another Way. The argument that follows is really the same as the previous one, but with a lot more "algebra" (not necessarily a good thing, unless we want to look into bases other than 10). If we give the right symbolic description of the pattern, the proof turns out to be automatic. We first "deconstruct" 3333×3334. Our pattern has to do with decimal representation, so it may be that what is significant is that

$$3333 = \frac{9999}{3} = \frac{10^4 - 1}{3} \quad \text{and} \quad 3334 = \frac{10^4 + 2}{3}.$$

Multiply:

$$3333 \times 3334 = \frac{10^8 + 10^4 - 2}{9} = \frac{10^8 - 1}{9} + \frac{10^4 - 1}{9}.$$

The first term, $(10^8 - 1)/9$, is the number a whose decimal expansion 11111111 consists of 8 1's. The next term, $(10^4 - 1)/9$ is the number b whose decimal expansion consists of 4 1's. Thus 3333×3334 is $a + b$, which with grade school addition, without even having to learn about carries, is easily seen to be 11112222.

The general argument is so close to our deconstruction of 3333×3334 that it is barely worth writing out. Let M be the number whose decimal expansion consists of n 3's, and let N be the number whose decimal expansion consists of $n-1$ 3's followed by a 4. Then

$$MN = \left(\frac{10^n - 1}{3}\right)\left(\frac{10^n + 2}{3}\right) = \frac{10^{2n} - 1}{9} + \frac{10^n - 1}{9}.$$

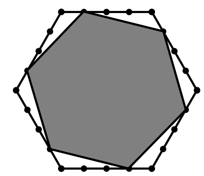
In the last expression above, the first summand is the number a whose decimal representation consists of $2n$ 1's, and the second summand is the number b whose decimal representation consists of n 1's. Add in the usual way. The number $a+b$ has decimal representation consisting of n 1's followed by n 2's.

We can also proceed more directly. Let A_n be the number whose decimal representation consists of $n-1$ 6's followed by a 7. Then $A_n = (2)(10^n+1)/3$. So

$$A_n^2 = \frac{(4)10^{2n} + (4)10^n + 1}{9} = 4\frac{10^{2n}-1}{9} + 4\frac{10^n-1}{9} + \frac{4}{9} + \frac{4}{9} + \frac{1}{9}.$$

The term $4(10^{2n}-1)/9$ is the number made up of $2n$ 4's, the term $4(10^n-1)/9$ is the number made up of n 4's, and $4/9+4/9+1/9$ is simply 1. Add up: A_n^2 has decimal expansion made up of n 4's, followed by $n-1$ 8's, followed by 9.

Problem 2. The outer hexagon is regular. Each side of this hexagon is divided into 4 equal parts. Some division points are joined as shown to form the shaded regular hexagon. What fraction of the area of the outer hexagon is shaded?

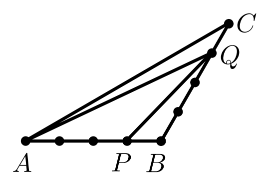


Solution. Let the sides of the outer hexagon have length $4a$. Then each little triangle has outer sides a and $3a$. Let s be the inner side of the little triangles. We compute s, or more precisely s^2. By the Cosine Law,

$$s^2 = a^2 + (3a)^2 - 2(a)(3a)\cos(120^\circ) = 13a^2.$$

Now we know the square of the side of the shaded hexagon. The ratio of the area of this to the area of the whole hexagon is $13a^2/16a^2$, that is, $13/16$. By choosing our unit of length appropriately, we could have let the side of the outer hexagon be 4, and not have to bother carrying "a" around.

Another Way. Let A, B, and C be three consecutive vertices of the outer hexagon, and let PBQ be one of our small triangles.



The area of $\triangle PBQ$ is $1/4$ of the area of $\triangle ABQ$, since its base PB is $1/4$ the base AB of $\triangle ABQ$, and the heights are the same. But the area of $\triangle ABQ$ is $3/4$ of the area of $\triangle ABC$ (same base AB, but $3/4$ of the height). It follows that the area of $\triangle PBQ$ is $3/16$ of the area of $\triangle ABC$.

By joining A and C to the centre of the hexagon, it is easy to show that the area of $\triangle ABC$ is one-sixth of the area of the hexagon. It follows that the area of $\triangle PBQ$ is $(1/6)(3/16)$ times the area of the hexagon. So the combined area of the 6 triangles is $3/16$ times the area of the outer hexagon. Thus the area of the shaded hexagon is $(1-3/16)$ times the area of the outer hexagon: the required ratio is $13/16$.

There is a more unpleasant way of calculating the area of a small triangle. Let b be the side of the outer hexagon. Then the outer sides of a little triangle have length $b/4$ and $3b/4$. Thus the area of a little triangle is $(1/2)(b/4)(3b/4)\sin(120^\circ)$, that is, $3\sqrt{3}b^2/64$ By a standard and easily verified fact, the area of any one of the 6 equilateral triangles that make up the hexagon is $\sqrt{3}b^2/4$, so the ratio of the sum of the areas of the little triangles to the area of the outer hexagon is, after some simplification, $3/16$.

Problem 3. For what values of p are there 3 real numbers in geometric progression whose sum is p and the sum of whose squares is 1?

Solution. It is reasonable to let the 3 numbers in geometric progression be a, ar, and ar^2. (Actually, it would be better to let them be a/r, a, and ar, things would be prettier, more symmetrical. But perhaps in his case familiarity is more valuable than symmetry.) With this choice our conditions become

$$(1) \quad a + ar + ar^2 = p \quad \text{and} \quad (2) \quad a^2 + a^2r^2 + a^2r^4 = 1.$$

Square each side of Equation 1. We obtain

$$a^2 + a^2r^2 + a^2r^4 + 2(a^2r + a^2r^2 + a^2r^3) = p^2.$$

Using Equation 2, we conclude that

$$a^2r + a^2r^2 + a^2r^3 = \frac{p^2 - 1}{2}.$$

From this, by dividing both sides by the corresponding sides of Equation 1, we obtain

$$ar = \frac{p^2 - 1}{2p}.$$

But we need to be cautious: the division makes no sense if $p = 0$. Could $p = 0$? Then $a(1+r+r^2) = 0$. It is easy to see that $1 + r + r^2 = 0$ has no real roots, so $p = 0$ forces $a = 0$, which produces a fairly boring geometric progression—there might be some argument about whether 0, 0, 0 is a geometric progression at all. However, that doesn't matter here, for if $a = 0$ then Equation 2 cannot hold.

Note that we have obtained the "middle term" of the geometric progression, probably another indication that it would have been better to let the terms be a/r, a, and ar. Using Equation 1 again, and dividing by ar, we obtain

$$\frac{1}{r} + 1 + r = \frac{2p^2}{p^2 - 1}.$$

Again we need to step back and ask whether division by $p^2 - 1$ is legitimate. There is a problem only when $ar = 0$. We have already dealt with $a = 0$. The other possibility is $r = 0$. One could argue about whether this is possible: it would give the 3-term progression a, 0, 0. Is this a geometric progression? That depends on how one defines geometric progression, and in fact opinions about a, 0, 0, ... differ. Let's allow it: then $p = \pm 1$ is allowed. Having dealt with that case, we can from now on assume that $p \neq \pm 1$.

A little manipulation now gives the quadratic equation

$$(p^2 - 1)r^2 - (p^2 + 1)r + (p^2 - 1) = 0.$$

This has a real solution if and only if the discriminant $(p^2 + 1)^2 - 4(p^2 - 1)^2$ is non-negative. The discriminant factors as

$$\left((p^2 + 1) - 2(p^2 - 1)\right)\left((p^2 + 1) + 2(p^2 - 1)\right),$$

that is, as $(3 - p^2)(3p^2 - 1)$.

The discriminant is non-negative precisely when $1/\sqrt{3} \leq |p| \leq \sqrt{3}$, so these are the values of p for which there is a geometric progression that satisfies Equations 1 and 2.

Another Way. We start again from

$$(1) \quad a + ar + ar^2 = p \quad \text{and} \quad (2) \quad a^2 + a^2r^2 + a^2r^4 = 1.$$

We will use the attractive and occasionally useful factorization

$$1 + u^2 + u^4 = (1 + u + u^2)(1 - u + u^2).$$

The identity above, like most identities, is easy to verify: just multiply out the right-hand side (there are more conceptual ways of getting to the identity). Square both sides of Equation 1, and use the identity on Equation 2. We get

$$a^2(1 + r + r^2)^2 = p^2 \quad \text{and} \quad [a(1 + r + r^2)][a(1 - r + r^2)] = 1.$$

From these equations we obtain

$$\frac{1 + r + r^2}{1 - r + r^2} = p^2.$$

We could now multiply both sides by $1 - r + r^2$ and obtain the quadratic equation obtained in the first solution. But for fun we finish the argument in a different (and probably more complicated!) way. Let $f(r) = 1 + r + r^2/(1 - r + r^2)$. Then

$$f(r) = \frac{1/r + 1 + r}{1/r - 1 + r} = 1 + \frac{2}{1/r - 1 + 1/r}.$$

Using the Arithmetic Mean – Geometric Mean Inequality, or some other way, it is not hard to see that if r is positive then $1/r + r \geq 2$, with equality when $r = 1$. Thus, as r increases from 1, or decreases towards 0, $f(r)$ (for $r \geq 0$) goes through all values from 1 to 3.

To deal with negative r, use the fact that $f(-r) = 1/f(r)$. So $f(r)$ attains a minimum of $1/3$ at $r = -1$, and as r travels through the reals ≤ 0, $f(r)$ takes on all values from $1/3$ to 1. Thus the range of values for p^2 is $1/3 \leq p^2 \leq 3$.