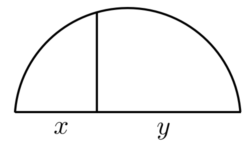
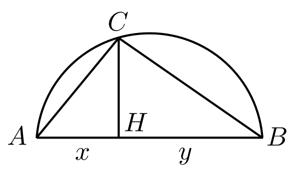
Mathematics 414, Solutions to Problem Set #4

Problem 1. Use the semicircle below to give a geometric argument for the two-variable Arithmetic Mean — Geometric Mean Inequality (if x and y are positive, then $(x + y)/2 \geq \sqrt{xy}$, with equality precisely if $x = y$).



Solution. It is reasonable to complete the picture as follows. By a basic property of circles, $\angle C$ is a right angle. That fact is almost certainly too familiar to bother proving: we can't prove everything. But it might be worthwhile to recall a proof in case it is needed. Let O be the centre of the circle, and draw the line segment OC. Triangles AOC and COB are isosceles. Let their base angles be, respectively, α and β. Then the sum of the angles of $\triangle ABC$ is $2\alpha + 2\beta$. This sum is $180°$, so $\alpha + \beta$, the measure of $\angle C$, is $90°$.



The vertical line of the original picture, now called CH, is meant to be perpendicular to the diameter of the semicircle. That makes all three triangles in the above picture similar. The fact that they are similar need not be "proved," it is essentially obvious, but if proof is wanted simple angle-chasing does the job.

By the similarity of $\triangle AHC$ and $\triangle CHB$, we have $HC/x = y/HC$, and hence $HC = \sqrt{xy}$. The rest is easy. The radius of the semicircle is $(x + y)/2$. Draw the line OC. . Then $OC = (x + y)/2$, and therefore $OC \geq HC$, with equality if and only if $H = O$. Thus $(x + y)/2 \geq \sqrt{xy}$ if and only if $x = y$.

Comment. The fact that the radius $OC \geq HC$ is geometrically clear. It can be proved easily by using the Pythagorean Theorem, but it would be overkill to do so. Or is it? Note that $OC = |x - y|/2$, so the Pythagorean Theorem in this case is a geometric version of the algebraic fact that

$$((x + y)/2)^2 = (|x - y|/2)^2 + xy,$$

which is essentially the same identity as the one used in the standard algebraic proof of the two variable AM – GM Inequality.

We could also get the same result by drawing a vertical line upwards from O, meeting the circle at say P. It is obvious from our familiar mental picture of the circle that $OP \geq HC$. Proof is unnecessary, but easy if demanded.

Note that we have been systematically sloppy: the notation UV was used to denote a line segment, and also the length of that line segment. Once upon a time, people did not worry much about this (deliberate) ambiguity, since whether UV refers to a line segment or to its length is usually clear from the context. Nowadays, many of the school textbooks are much stricter, they use for example $|UV|$ or mUV to refer to the length of the line segment UV, and insist on calling that length the *measure* of UV. Or else the line segment is $\overline{UV}$ and its "measure" is one of several possibilities. Similar issues come up in distinguishing between an angle and the several ways of specifying the size of that angle. Amusingly, the now common insistence on notational purity has been accompanied by a large decrease in the amount of geometry actually done.

Problem 2. (a) Define the sequence A_0, A_1, A_2, and so on by $A_0 = A_1 = 1$, and $A_n = 2A_{n-1}+A_{n-2}$ for $n \geq 2$. Let $x = 1/3$. Calculate

$$A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \cdots + A_n x^n + \cdots .$$

Manipulate infinite sums freely, ignoring issues of convergence.

(b) For $x = 1/3$, find $\sum_{n=1}^{\infty} nA_n x^n$.

Solution. (a) The calculation imitates the corresponding calculation with the Fibonacci numbers, and the familiar method for summing an infinite geometric series. Let our sum be $S(x)$. Multiply $S(x)$ by $2x$, and subtract from the expression for $S(x)$. Gathering like powers of x together, we obtain

$$S(x) - 2xS(x) = A_0 + (A_1 - 2A_0)x + (A_2 - 2A_1)x^2 + (A_3 - 2A_2)x^3 + (A_4 - 2A_3)x^4 + \cdots .$$

If $n \geq 2$, then $A_n - 2A_{n-1} = A_{n-2}$. Using this, we find that

$$S(x) - 2xS(x) = 1 - x + A_0 x^2 + A_1 x^3 + A_2 x^4 + \cdots = 1 - x + x^2 S(x).$$

A little manipulation now gives $S(x) = (1-x)/(1-2x-x^2)$. When $x = 1/3$, this is equal to 3.

Comment. Things are somewhat more messy looking if from the beginning we work with 1/3 rather than x. This increases the probability of error, and more importantly makes it more likely that a nice structural pattern will be missed. Quite often in problems, even when specific numbers are mentioned, it can be useful to replace them by letters. Any "algebra" will look much neater, and one may get a general result. Working with specific numbers from the beginning may be necessary, but it should be postponed if possible.

As instructed, we operated "formally" on the series, ignoring issues of convergence. It turns out that our series converges if $|x| < \sqrt{2} - 1$, which (no accident!) is one the roots of the equation $1 - 2x - x^2 = 0$. That root is roughly 0.4142, and 1/3 is safely smaller. A proof that there is convergence at $x = 1/3$ is not hard. It is enough to show (say by induction) that $A_n < 0.35^n$ if n is large enough.

(b) We use more or less the same ideas, but the details are somewhat messy. There are general procedures for solving this kind of problem, but we will temporarily try to stay away from them. Define $T(x)$ by

$$T(x) = A_1 x + 2A_2 x^2 + 3A_3 x^3 + 4A_4 x^4 + \cdots .$$

Imitating part (a), we may want to look at $T(x) - 2xTx)$, or more boldly, look at $T(x) - 2xT(x) - x^2 T(x)$, since that turned out to be very nice with $S(x)$. Exploit the fact that $A_n = 2A_{n-1} + A_{n-2}$. After a bit (or a lot) of playing around, we find that

$$(1 - 2x - x^2)T(x) + 2xS(x) - 2S(x) = -2A_0 + (2A_0 - A_1)x = x - 2.$$

(On the left hand side there are infinite series, but almost everything cancels.) Since we know an expression for $S(x)$, we can now find one for $T(x)$. For numerical computation with $x = 1/3$, we don't need to bother with the algebra, since we already know that $S(1/3) = 3$. Calculate. It turns out that $T(1/3) = 21/2$.

Comment. Using some algebra on the previous expression for $T(x)$, we can after some work get

$$T(x) = \frac{2x - x^2}{(1 - 2x - x^2)^2}.$$

There is a much simpler way to get this result by using calculus. Since $S(x) = A_0 + A_1x + a_2x^2 + \cdots$, the derivative $S'(x)$ of $S(x)$ is $A_1 + 2A_2x + 3A_3x^2 + \cdots$. It follows immediately that $T(x) = xS'(x)$. Since we know $S(x)$ from part (a), we now know $T(x)$. The device just mentioned here is useful and general. Note that even if we only care about the numerical value of $T(x)$ for the single value $x = 1/3$, it can be valuable to have a general procedure that works for all x. For more information, look for the key expression "generating functions."

Problem 3. Solve *exactly one* of the following two problems:
(i) Find the slope of the tangent line to the curve $xy = 4$ at the point $(1, 4)$ in two ways, neither of which involves "calculus." One way I can think of uses routine algebra, another uses a transformation.

(ii) Beth has a biased loonie that lands heads with probability $p \neq 0$, and tails with probability $1 - p$. Alicia tosses the coin repeatedly, and keeps a running count of heads and tails. If the number of heads is *ever* greater than the number of tails, Alicia wins the game (and the coin). What is the probability that Alicia wins the game?

Solution. (i) Draw a picture of the curve (not done). It is a hyperbola, symmetrical about the line with equation $y = x$. The full geometry will be clearer if we also draw the third quadrant part of the curve.

The tangent line is not vertical, so it has equation of the shape $y - 4 = m(x - 1)$, where m is the slope. To find the x-coordinates of the points where this line meets the hyperbola, substitute $4 + m(x - 1)$ for y in the equation $xy = 4$, and simplify. We arrive at the equation

$$mx^2 - (m - 4)x - 4 = 0.$$

It is geometrically clear that the slope is negative, and in particular non-zero. One of the roots of the equation is given by $x = 1$, and the sum of the roots is $(m - 4)/m$, that is, $1 - 4/m$, so the other root is $-4/m$. Alternately and more simply, the product of the roots is $-4/m$, and since one of the roots is 1, the other is $-4/m$.

Since m is negative, there are two distinct positive roots, unless, of course, 1 and $-4/m$ coincide, in which case there is only one root. But the picture shows that in the case of tangency, there is only one root. Thus $-4/m = 1$, and $m = -4$.

A slightly different way of doing the same thing is to refer to the Quadratic Formula. The "$b^2 - 4ac$" part of that formula is called the *discriminant.* The discriminant is $(m - 4)^2 + 16m$, which simplifies to $(m + 4)^2$. This discriminant is always non-negative, and it is positive unless $m = -4$. So there are two distinct real roots unless $m = -4$. A scan of $xy = 4$ shows that unequal roots means non-tangency, so we have tangency precisely if $m = -4$. The discriminant approach is somewhat more complicated than the earlier approach through the sum or product of the roots. but the discriminant *is* important, so maybe it is a good idea to use it even though it complicates things. But then again, the sum and product stuff about the roots is arguably even more important than the discriminant. Our fixation with formulas makes us focus unnecessarily on the discriminant.

Essentially the same argument can be used to compute the slope of the tangent line to $xy = k$ at the point $(a, k/a)$.

Another Way. Let $t = 4x$. Then equation $xy = 4$ becomes $ty = 16$, and the point $x = 1$, $y = 4$ becomes $t = 4$, $y = 4$. Algebraically more attractive, but let's think about the geometry. Graph, *separately* the curves $xy = 4$ and $ty = 16$, letting y in each case refer to the vertical direction. The graph of $ty = 16$ is just the graph of $ty = 4$, scaled in the horizontal direction by a stretching factor of 4. Alternately, on the same graph, draw $xy = 4$ and $xy = 16$. The second curve is the just the first, stretched by a factor of 4 in the x-direction.

The tangent line to $xy = 4$ at $(1, 4)$ is transformed by the stretching into the tangent line to $ty = 16$ (or $xy = 16$) at the point $(4, 4)$. Now we take advantage of symmetry, which was the whole point of the game. The tangent line to $xy = 16$ at $(4, 4)$ obviously has slope -1. Now *Transform Back*, by scaling in the x-direction by the factor $1/4$. The tangent line to $xy = 16$ at $(4, 4)$ is transformed back into the tangent line to $xy = 4$ at $(1, 4)$. The scaling by $1/4$ in the x-direction multiplies slopes by 4. This is easy to verify algebraically, and is geometrically obvious. So our slope is -4.

Comment. We have used a particular example of a general technique often called "Transform, Solve, Transform Back." Many techniques, both elementary and not so elementary, fall under this rubric. As a familiar example, suppose that we are interested in the curve $y = x^2 + 4x - 17$. Rewrite as $y = (x + 2)^2 - 21$. Let $t = x + 2$. We are looking at $y = t^2 - 21$, which has pleasant symmetry about $t = 0$. Equivalently, move the curve to the left by the amount 2. We arrive at $y = x^2 - 21$. It is obvious where this curve crosses the x-axis. Transform back to solve $x^2 + 4x - 17 = 0$. Techniques of linear algebra such as diagonalization are important because the transformed problem is often easy to solve.

(ii) Let x be the probability that Alicia wins the game. Winning for Alicia can happen in two ways: (1) the first toss is a head or (2) the first toss is a tail. The first toss is a head with probability p. If the first toss is a head, the game is over, Alicia has won.

If the first toss is a tail (probability $1 - p$) then in order to win, Alicia has to at some time draw even, and then at some later time has to get ahead. The probability that Alicia draws even at some time in the race, and then wins. The probability of some time drawing even is x, for drawing even when she is "1 down" is the same problem as getting 1 head ahead when you are tied, and that probability is x. And *given* that Alicia has drawn even, the probability Alicia ultimately wins is x. We have obtained the equation

$$x = p + (1 - p)(x)(x).$$

If $p = 1$, there is no issue, Alicia wins with probability 1. If $p \neq 1$, Look at the quadratic equation above. In standard form, it is $(1 - p)x^2 - x + p = 0$. Solve. The simplest way is to observe that 1 is a root. But the product of the roots is $p/(1 - p)$, so the other root is $p/(1 - p)$.

Our equation has two roots. Which one is the answer? If $p > 1/2$, the root $p/(1 - p)$ is greater than 1, so cannot be a probability. Thus $x = 1$. If $p = 1/2$, the two roots are equal, so again $x = 1$.

The case $p < 1/2$ needs some argument. Then $p/(1 - p)$ is between 0 and 1. So is it the answer? We must eliminate the other root $x = 1$ as a possibility. If $p < 1/2$, then it is intuitively reasonable (and true) that after n tosses, where n is large, the number of heads obtained, divided by n, is close to p, so for large n, with positive probability we never return to a situation where the number of heads is equal to the number of tails. That means that there is a positive probability that after the first toss, we never have equality of heads and tails. That eliminates the root $x = 1$.

One might note that $p/(1 - p) = 1/2$ if and only if $p = 1$. So the game is a fair one if the loonie lands heads with probability $1/3$.

Comment. The probability that A and B are tied after $2n$ coin flips is $\binom{2n}{n}p^n(1-p)^n$. We estimate $\binom{2n}{n}$. It is not hard to verify that

$$\binom{2n}{n} = \frac{(2n)(2n-1)}{n^2}\binom{2n-2}{n-1} < 4\binom{2n-2}{n-1}.$$

From this we can conclude that $\binom{2n}{n} < 4^n$ if $n > 0$.

Suppose that $p \neq 1/2$, and assume that $p \neq 0$, $p \neq 1$. By completing the square, we can show that $p(1-p) < 1/4$. The probability that there is a tie at the $2m$-th toss *or beyond* is less than

$$\sum_{k=m}^{\infty} \frac{2k}{k} p^k (1-p)^k.$$

Let $t = 4p(1-p)$. If $p \neq 1/2$, then $t < 1$, so the above sum is less than

$$\sum_{k=m}^{\infty} t^k.$$

The above sum can be made < 1 (indeed arbitrarily close to 0) by taking m large enough. From this we can show that if $p \neq 1/2$, the probability that the contestants will never be tied is greater than 0. We argued earlier that this was intuitively very reasonable, but wanted to show that with some effort a proof can be given.

There is a huge literature on questions related to our "Alicia" problem. One term to search for is "random walks." The subject, beside being mathematically beautiful, has applications in many branches of science.