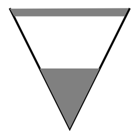
Mathematics 414, Solutions to Problem Set #3

Problem 1. A conical cup was filled to the rim with brandy. Xavier had some. When he was finished, the brandy level was down one-quarter of an inch. Yolande then drank, leaving the bottom two inches for Zoë. It turns out that Xavier and Zoë had exactly the same amount of brandy. How tall is the cup? (You may use grade 11/12 techniques, but preferably should avoid the standard volume of cone formula.)

Solution. Call a conical cup *similar* to ours, but of height 1 inch, a *baby-cone.* Let the volume of a baby cone be k. (We could let our unit of volume be the volume of a baby-cone, in which case $k = 1$. That choice of unit of volume would mildly simplify what follows.)



Let h be the height of our cone. Our cone is the baby-cone with all dimensions scaled by the factor h. Scaling by the factor h scales volume by the factor h^3, So our full cone has volume kh^3.

After X drank, the height of the cone of brandy is $h - 1/4$, so the volume of brandy left is $k(h - 1/4)^3$. It follows that the amount that X drank is $kh^3 - k(h - 1/4)^3$.

Since Z drank a cone similar to our baby-cone, but of height 2, she drank the amount $k(2^3)$. It follows that

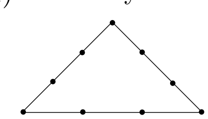
$$h^3 - (h - 1/4)^3 = 2^3.$$

Expand $(h - 1/4)^3$, or use the identity $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$ to simplify. We arrive at the equation

$$(3/4)h^2 - (3/16)h + 1/64 - 8 = 0.$$

Note that the second and third term will be smallish compared to the first and fourth. Thus $(3/4)h^2 \approx 8$, Numerically, that gives the estimate $h \approx 3.266$. And then we could refine our estimate in various ways. Of course there is no need of this, we can obtain an explicit expression for h by solving the full quadratic equation. That gives $h \approx 3.39$. It is a bit silly to give the height of the cup to the nearest hundredth of an inch, 3.4 would be good enough for all practical purposes, since presumably our drinking friends are not measuring with great accuracy, the cone is unlikely to be perfect, there are meniscus effects, and so on. (The drinkers should be told that it is barbaric to drink brandy from a paper cup, and also barbaric to fill a brandy glass more than halfway.)

Problem 2. How many triangles have their vertices at 3 of the 9 points in the figure below? A triangle has been drawn to make the location of the points clearer—and it is one of the triangles we need to count. The solution *cannot* (explicitly) use the binomial coefficients $\binom{n}{r}$. Use only tools available at the grade 6–7 level.



Solution. There are many ways to count. We first do a count which is sort of organized but suboptimal. Life would be easier if we did not have the corner points. So let's take care of them first. There is 1 triangle all of whose vertices are corner points. How many triangles have 2 corner points as vertices? There are 3 ways to choose the 2 corner points that are vertices. For [every] such choice, there are 4 ways to choose the vertex which is not a corner point. That gives a total of 12 triangles with 2 corner point vertices.

How many triangles have exactly 1 corner point vertex? The corner point can be chosen in 3 ways. Now we count the number of triangles which have this point as a vertex, but no other corner point. Look at the side opposite the corner point. Maybe our triangle has as vertices *both* non-corner points on that side (1 triangle). Maybe our triangle has exactly one of these non-corner points as vertex (2 choices). And for every such choice, there are 4 ways to choose the remaining vertex, for a total of (2)(4) triangles. Finally, the triangle may have no vertices on the side opposite the corner point. So we must choose 1 point on each of the two sides that meet at the corner point. There are (2)(2) ways to choose these 2 points. So there is a total of $(3)(1 + 8 + 4)$, that is, 39 triangles with exactly 1 corner point as a vertex.

There are other ways to count the triangles with exactly 1 corner point vertex. Once we have chosen a corner point, we must select 2 non-corner points to go with it. There are 6 points to choose from. By basically listing cases, we can see that there are 15 ways to choose 2 points from these. But if we choose points collinear with our corner point, that will be bad. It is easy to see there are only 2 bad choices, leaving 13. Now multiply by 3 to get 39.

Finally, we count the triangles with no corner point as vertex. Maybe the triangle has exactly 1 vertex on each side of the triangle in the picture. It is not hard to see that there are (2)(2)(2) ways to choose these vertices. Or maybe the triangle has 2 vertices on one side, plus another one. The *side* that has 2 vertices can be chosen in 3 ways. Once we have chosen the side, the actual vertices are determined. And then the third vertex can be chosen in 4 ways, for a total of (3)(4). So there are 20 triangles with no corner point as vertex.

Add up. There are $1 + 12 + 39 + 20$, that is, 72 triangles.

Comment (A confession). During the counting, I made a couple of errors. Since I knew, using more machinery, that the answer is 72, and the errors led to an answer bigger than 72, I knew there were mistakes somewhere. The errors were then easy to track down. What lesson should one draw from this? If possible, do things in more than one way. But more importantly, one should understand that counting problems, even when they are "elementary," can be very tricky.

Another Way. (Ciorán Leavitt) Label the points in some reasonably systematic way. For instance, the labels could be 1 to 9, with 1 at the lower right-hand corner, and then $2, 3, \ldots, 9$ going clockwise around the big triangle. Now list the triangles, by giving their vertices in *increasing label order.*

List first the triangles that begin with 1. First list the triangles that begin with 12. They are 125, 126, 127, 128, 129. Then list the triangles that begin with 13. They are 135, 136, 137, 138, 139. Now deal with the ones that begin with 14, and so on. It would be easier to just *count* (for example) the triangles that begin with 12, instead of listing, but listing is probably safer. After a while one should end up with 22 triangles that begin with 1.

Now list the triangles that begin with 2. First list the ones that begin with 23. They are 235, 236, 237, 238, 239. Continue, dealing then with triangles that begin with 24, and so on.

The work is not too hard, and with some care one should not miss any triangles. Then count. We get 72 of them.

Another Way. Suppose that the numbers of points on the sides of the triangle are, respectively, a, b, and c, and let n be the total number of points. (The number n is not necessarily equal to $a + b + c$, since $a + b + c$ counts any vertex among our points twice.) Then there are $\binom{n}{3}$ ways to choose 3 points from our n points. If the 3 points are on a line, they do not form a triangle, so we must throw out $\binom{a}{3}+\binom{b}{3}+\binom{c}{3}$ of the choices, and hence the number of triangles is

$$\binom{n}{3}-\binom{a}{3}-\binom{b}{3}-\binom{c}{3}.$$

Now we will try, in our special case, to imitate this argument without *explicitly* using the binomial coefficients. Whether we have succeeded depends on the interpretation of "explicitly."

We need to choose 3 points as vertices. It is clear that some choices are forbidden, namely those in which the 3 chosen points are all on a side of the big triangle. How many forbidden configurations are there? Pick a specific side of the big triangle. There are exactly 4 ways to pick 3 points on that side (just leave out any one of the 4 points). Since there are 4 forbidden choices on each side, the number of forbidden choices is (4)(3).

Now we need to count the number of ways of choosing 3 points from 9, with no restrictions. Call our points P_1, P_2, …, P_9, or more simply 1, 2, 3, …, 9. We want to choose 3 numbers from these 9.

Maybe the biggest of our chosen numbers is 9. Then we need to choose two numbers from 1, 2, …, 8. Let's tackle that simpler problem, patiently. So the biggest is 9, and will remain 9 throughout this paragraph. Maybe the second smallest is 8. Then there are 7 ways to choose the smallest. Maybe the second smallest is 7. Then there are 6 ways to choose the smallest. Maybe the second smallest is 6. Then there are 5 ways to choose the smallest. Continue. Finally, maybe the second smallest is 2, giving 1 way to choose the smallest. So the total number of choices with the biggest number equal to 9 is $7+6+5+\cdots+1$. By the "Gauss" trick, or more simply by adding, we find that the number of choices with biggest number equal to 9 is 28.

Maybe the biggest of our chosen numbers is 8. By almost exactly the same argument as the one above, the remaining two numbers can be chosen in $6+5+4+\cdots+1$ ways, so in 21 ways.

Similarly, the number of choices with 7 the biggest number is 15, the number of choices with 6 the biggest number is 10, the number of choices with 5 the biggest number is 6, the number of choices with 4 the biggest is 3, and there is only one choice with 3 the biggest. (Possibly the analysis should have been done the other way, from simplest case (3 is the biggest number chosen) to most complicated (9 is the biggest).

We have made a complete list. There are $28+21+15+10+6+3+1$ ways to choose 3 objects from 9, a total of 84. Now subtract 12 for the forbidden choices, giving 72.

Comment. (Youngshin Oh) There is a more sophisticated but simpler way to count the number of ways to choose 3 objects from our 9. How many 3-digit *numbers* are there, with all digits different, and with the digits chosen from 1, 2, …, 9? The first digit can be chosen in 9 ways. For each such choice, there are 8 ways to choose the second digit, so there are (9)(8) ways to choose the first two digits. And for each such choice, there are 7 ways to choose the third digit, for a total of (9)(8)(7). However, this grossly over counts the number of ways to choose a *bag* of 3 numbers. Any bag of 3 numbers, like $\{2,5,7\}$ gives rise to six 3-digit numbers, in our case 257, 275, 527, 572, 725, and 752. So (9)(8)(7) over counts our desired number by a factor of 6. Thus there are (9)(8)(7)/6 ways to choose 3 objects from 9.

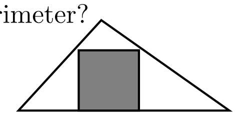
Another Way. (Yimo Ni) This is my favourite, and needs a picture. Colour a vertex of the big triangle, and the next two points (say counterclockwise) red. Then colour the next 3 points green, and the last 3 blue. There are 2 types of triangle: (i) the ones with vertices of 3 different colours and (ii) the ones with vertices of 2 colours.

For (i), pick the red vertex (3 ways). For each choice, there are 3 ways of choosing the green vertex, and once that has been done, 3 choices for the blue, a total of 27.

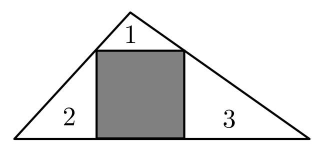
For (ii), pick the colour we have 2 of (3 ways). For each such choice, pick the actual 2 vertices of that colour (3 choices). For every way of doing these two things, there are 5 ways of picking the remaining vertex, since only 1 of the unused points is forbidden. So there are 45 type (ii) triangles.

Add up: we get 72. This idea works just as nicely with any number k of "inside" points on each edge. (In our problem we had $k = 2$.) The idea generalizes readily to a, b, and c inside points on the 3 edges.

Problem 3. A square has area 1. (a) What is the smallest possible area among all triangles such that (i) one side of the square lies on a side of the triangle and (ii) the triangle contains the square? (b) Among all such triangles of smallest area, what is the smallest possible perimeter?



Solution. (a) It was intended that the two top corners of the square not quite reach the two top sides of the triangle, but the scale is too small to make that clear. But in any case it is obvious that to minimize the area of the triangle, the top corners of the square must reach the two top sides of the triangle, else we can easily produce a triangle of smaller area that meets our specifications. So we need only study situations in which the top two corners of the square *do* reach the triangle. We redraw the picture in order to make that clear, make things a bit larger, and introduce some labels.



Note that Triangle 1 is similar to the full triangle. Imagine removing the square, and sliding Triangle 2 to the right until it meets Triangle 3. The result is a triangle (which we call Triangle 4, not drawn). Triangle 4 is also similar to the full triangle, and hence toTriangle 1.

To minimize the area of the full triangle, we need to minimize the sum of the areas of Triangles 1 and 4. Let x be the height of Triangle 1, and let y be the base of Triangle 4. Since the base of Triangle 1 is 1, and the height of Triangle 4 is 1, the sum of their areas is $(1/2)(x + y)$. And since the triangles are similar, we have

$$\frac{x}{1} = \frac{1}{y},$$

or equivalently $xy = 1$. So we want to minimize $x + y$ subject to the condition $xy = 1$. This is a problem that has shown up before, and can be settled by the ever useful Arithmetic Mean—Geometric Mean Inequality, or more explicitly by using the fact that

$$(x + y)^2 = (x - y)^2 + 4xy = (x - y)^2 + 4.$$

To minimize $(x + y)^2$ (and hence $x + y$) we should make $(x - y)^2$ as small as possible. The best we can do is to make $x = y$. That forces $x = y = 1$.

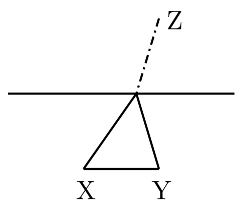
Can we actually *achieve* this geometrically? Yes, and in many ways. Pick any line segment of length 2 that contains the bottom edge of the square. Draw the line that goes through the left end of this line segment and the top left corner of the square. Do the same with the right end of the line segment and the top right corner of the square. The triangle thus produced has the desired properties. Any such triangle has area 2.

(b) In part (a), we saw that the "winning" triangles are precisely the triangles for which the triangle we called Triangle 1 has height equal to 1. And since Triangle 1 and the full triangle are similar, to minimize the perimeter of the full triangle it is enough to minimize the perimeter of Triangle 1.

It seems very plausible, some might even say obvious, that to minimize this perimeter we should make the triangle isosceles. If that is true, then the calculation of the smallest possible perimeter is easy. For then the winning full triangle is isosceles with base 2 and height 2. The two non-horizontal sides of the triangle each have length $\sqrt{1^2+2^2}$, so the triangle has perimeter $2+2\sqrt{5}$. By the way, $1+\sqrt{5}$ is intimately connected with the Fibonacci numbers. A coincidence? Probably.

If we set up the problem algebraically, the minimization problem is very difficult to do without calculus, and would be considered kind of hard in a first year calculus exam. It is easier to look at the closely related problem of maximizing the area given the base and the perimeter. You might want to try. Heron's formula for the area of a triangle turns out to be the key.

But here is a little story that settles things . Friends X and Y live 1 mile apart at the ends of the top edge of the square. A river runs 1 mile to the north, and parallel to this edge. X wants to go to the river, get some water, and take it to Y. What is the shortest possible path? Draw a possible path, like the one in the picture.



Now it so happens that X has another friend, Z, who lives exactly opposite Y, just as far to the north of the river as Y is to the south. The river is narrow and shallow, easy to cross, so X thinks about visiting Z instead. If so, the second part of his trip would be the dashed line in the picture—the reflection of the second part of the trip to Y. If the path south of the river was the minimal path to Y, then the part to the river, followed by the dashed line, is the minimal path to Z. The minimal path to Z, however, must be a straight line. Draw it. The line obviously meets the river due north of the halfway point between X's house and Y's house. Reflect the second part of this straight line path back so that it goes to Y (X decided not to jilt Y). The two parts of the trip have the same length, the triangle is isosceles.

Comment. Look at the isosceles triangle $\mathcal{T}$ with (horizontal) base 1 and height 1. Call the endpoints of the base by the strange names F_1 and F_2. The sum of the distances from the top vertex to F_1 and F_2 is a certain number, which happens to be $\sqrt{5}$.

Once upon a time, most high school students knew that the collection of points P such that $PF_1 + PF_2 = k$ is a nice curve called an ellipse with foci F_1 and F_2. (Nowadays, in BC, they associate the ellipse only with the equation $x^2/a^2 + b^2/y^2 = 1$. Soon they will not meet the ellipse in mathematics classes.)

The points with sum of distances from F_1 and F_2 equal to $\sqrt{5}$ trace out an ellipse. Points inside the ellipse have sum of distances from the foci less than $\sqrt{5}$. Points outside the ellipse have sum of distances from the foci greater than $\sqrt{5}$.

Now look at the top vertex of $\mathcal{T}$, and imagine moving it, parallel to the base, either to the left or to the right. Then this top vertex moves *outside* the ellipse (we need a Java applet here!), so the sum of distances becomes greater than $\sqrt{5}$.