

Mathematics 414, Solutions to Problem Set #2

Problem 1. Yolande has \$77 more than Xavier, and Zoë has \$77 more than Yolande. Between them, they have \$777. How much money does each of them have? Solve the problem (preferably in more than one interesting way) without using formal algebra.

Solution. The names—any names—are a nuisance, they take up too much space, and one ends up attempting to balance sexes and ethnic groups. And sadistic problem setters call their heroes Jill, Jodie, and Julia, in an all too often successful attempt to confuse. Abbreviate our rich kids' names as X, Y, and Z. Why can't everybody in "word problems" have sensible names like that, or like A, B, and C? (Note that abbreviating names is *not* algebra.)

Guess how much X has. Any guess will do. Guess, for example, that X has \$23. Then Y has \$100, and Z has \$177, for a total of \$300. But we are aiming for \$777, so we are \$477 short, the guess for Xavier has to be adjusted upwards. For each additional dollar we assign to X, we must assign one to Y and one to Z, so our sum increases by \$3. The sum needs to increase by \$477, so we must assign an additional $477/3$ dollars to X. It follows that X has $23 + 159$ dollars, that is \$182. Hence Y has \$259 and Z has \$336.

Another Way. Concentrate instead on Y. Since X has \$77 less, and Z has \$77 more, between the two of them they have twice as much as Y. Thus Y has $777/3$ dollars, and the rest is trivial. Or else Z gives \$77 to X. Then everyone has the same amount, namely $777/3$ dollars. Since Y's fortune did not change, we know she started with \$259, and we are essentially finished.

Another Way. Y and Z have pangs of guilt over their wealth, so Y gives \$77 to a semi-destitute mathematician so he can buy more presentable clothing, and Z gives $77 + 77$ to the mathematician to help support his tobacco habit, for a total of \$231. That leaves \$546, equally distributed between the three friends. Since X's fortune has not changed, he started with $546/3$ dollars.

Comment. Any of the three arguments, particularly the last two, could and perhaps should be accompanied by a picture.



Problem 2. Some positive integers n have the property that $1/n$ has a terminating decimal expansion. For example, $1/40$ has the terminating decimal expansion 0.025. (a) How many such integers are there that are less than or equal to 10^6 ? (b) What is the sum of the reciprocals of *all* the positive integers n such that $1/n$ has a terminating decimal expansion? So we want

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{8} + \frac{1}{10} + \frac{1}{16} + \frac{1}{20} + \frac{1}{25} + \frac{1}{32} + \frac{1}{40} + \dots$$

Solution. (a) It is almost obvious that $1/n$ has a terminating decimal expansion if and only if $n = 2^a 5^b$ for some non-negative integers a and b . For completeness we give a proof. If $1/n$ has a terminating decimal expansion, then $1/n = k/10^c$, where k is a positive integer and c is a non-negative integer. Thus $nk = 10^c$. It follows that any prime divisor of n divides 10^c , so must be 2 or

5. Conversely, suppose that $n = 2^a 5^b$. Let c be the larger of a and b . Then

$$n = \frac{2^c 5^c}{2^{c-a} 5^{c-b}} = \frac{10^c}{2^{c-a} 5^{c-b}}$$

and hence $1/n$ has a terminating decimal expansion.

Now list and then count. How shall we list? Perhaps first list all our numbers that are not divisible by 5. These are $2^0, 2^1, 2^2, 2^3$, and so on. Note that 2^{10} is roughly 1000, so 2^{20} is a bit bigger than 10^6 . Our “last” number is therefore 10^{19} , so this portion of the list has 20 numbers.

Now list the numbers that are divisible by 5 but not by 5^2 . These are $5 \times 2^0, 5 \times 2^1, 5 \times 2^2$, and so on. Our last number is 5×2^{17} , so this part of the list has 18 numbers.

Now list the numbers that are divisible by 5^2 but not by 5^3 . These are $5^2 \times 2^0, 5^2 \times 2^1, 5^2 \times 2^2$, and so on. Our last number is $5^2 \times 2^{15}$, so this part of the list has 16 numbers.

Now list the numbers that are divisible by 5^3 but not by 5^4 . These are $5^3 \times 2^0, 5^3 \times 2^1, 5^3 \times 2^2$, and so on. Our last number is $5^3 \times 2^{12}$, so this part of the list has 13 numbers.

Continue. For “ 5^4 ” we get 11 numbers, for 5^5 we get 9, for 5^6 we get 7, for 5^7 we get 4, and for 5^8 we get 2. The total is 100. Nice and round! When we get such a pretty number, we should ask ourselves whether we are missing something interesting. In this case, probably not.

A student familiar with logarithms might calculate in general as follows. We want to count the numbers $n \leq N$ such that $1/n$ has a terminating decimal expansion. Imagine listing these numbers using the strategy we used for $N = 10^6$. Then the first list has about $\log_2 N$ members, the second list has about $\log_2(N/5)$ members, the third list has about $\log_2(N/25)$, and so on. Add up. We can further observe that $\log_2(N/5^k) = \log_2 N - k \log_2 5$, and continue. But already we are getting beyond (BC) high school material.

Another Way. We could list in other ways. For example, we might first list the numbers not divisible by 10, then the numbers divisible by 10 but not by 10^2 , then the numbers divisible by 10^2 but not by 10^3 , and so on.

The numbers not divisible by 10 are just the powers of 2 or 5 that are $\leq 10^6$. There are 20 such powers of 2 and 9 such powers of 5, but they have the number 1 in common, giving a total of 28.

Now look at the numbers divisible by 10 but not by 10^2 . These are the numbers of the form $10m$, where m is any number $\leq 10^5$ which is a power of 2 or of 5. There are $17 + 8 - 1$ of these. Now look at the numbers divisible by 10^2 but not by 10^3 . These are the numbers of the form $10^2 m$, where m is any number $\leq 10^4$ which is a power 2 or of 5. There are $14 + 6 - 1$ of these. Continue. We get the counts $10 + 5 - 1, 7 + 3 - 1, 4 + 2 - 1$, and 1. Add up. We get 100. The counts agree, I did not make a mistake! But maybe I made more than one. (If we feel like it, we can express everything in terms of logarithms. That would be very good for giving a simple *approximate* value of the count.)

(b) First add up the fractions whose denominator is a power of 2. So we want the sum of the geometric series $1 + 1/2 + 1/4 + \dots$. This sum is 2.

Next add up the fractions whose denominator is divisible by 5 but by no higher power of 5. That sum is $(1/5)(1 + 1/2 + 1/4 + \dots)$, that is, $2/5$.

Then add up the fractions whose denominator is divisible by 25 but by no higher power of 5. This sum is $2/25$. Continue in this way forever.

Finally, calculate the sum of the infinite geometric series $2 + 2/5 + 2/25 + \dots$. This sum is $5/2$.

Another Way. Imagine expanding out the product

$$\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\right) \left(1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \dots\right).$$

We obtain precisely the sum of the fractions $1/n$ where n has no prime divisor other than 2 or 5. The two infinite geometric series have sum 2 and $5/4$ respectively. Multiply.

Comment. The second solution is a bit shorter, and generalizes nicely. Let p_1, p_2, \dots, p_k be distinct primes, and let S be the sum of all numbers $1/n$ where any prime factor of the positive integer n lies among the p_i . Then

$$S = \left(1 + \frac{1}{p_1} + \frac{1}{p_1^2} + \dots\right) \left(1 + \frac{1}{p_2} + \frac{1}{p_2^2} + \dots\right) \cdots \left(1 + \frac{1}{p_k} + \frac{1}{p_k^2} + \dots\right).$$

Now sum the k infinite geometric series above. We get

$$\frac{1}{S} = \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right).$$

The above calculations can be used to show that there are infinitely many primes. For suppose to the contrary that there are exactly k primes, p_1 to p_k . Then S is “finite.” But if there were only these k primes, then *all* positive integers would be divisible by at most the primes p_1 to p_k , which means that our sum would be the sum of the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \dots.$$

However, it is well-known that the sum of the harmonic series is not finite, the harmonic series diverges.

The above proof that there are infinitely many primes is due to Euler, the greatest mathematician of the eighteenth century. He used more or less the same idea to show that in fact the sum of the reciprocals of the primes is “infinite,” which informally means that actually the primes are fairly thick on the ground.

Elaborate variants of the idea have been used to study the distribution of primes ever since Euler. Ultimately this line of thought leads to the Riemann zeta-function, and to the most famous unsolved problem in mathematics, the *Riemann Hypothesis*.

Comment. We operated with infinite sums as if they were long finite sums, and in particular rearranged the order of summation freely. That can be done without risk for series that only have positive terms. But this kind of rearrangement may be illegitimate in some cases. For example, it turns out that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4},$$

but the series can be rearranged so as to have sum 1024, or any other number we like!

Comment. We took as “known” the usual formula for the sum of the geometric progression $1 + a + a^2 + \dots$. The standard approach to the formula involves finding an explicit expression for the finite sum $1 + a + a^2 + \dots + a_n$ and studying what happens to this sum as n gets large.

But there is a clever slightly dishonest approach. Let $S = 1 + a + a^2 + \dots$. Then

$$S = 1 + a(1 + a + a^2 + \dots) = 1 + aS.$$

Now solve for S . We get $S = 1/(1 - a)$.

This pretty argument assumes that infinite sums can be treated as “long” finite sums, and can be subjected to analogous algebraic manipulations. To show that one has to be cautious, let’s apply the method to $1 + 2 + 4 + 8 + \dots$. Let the sum be S . Then

$$S = 1 + 2(1 + 2 + 4 + \dots) = 1 + 2S.$$

Solve for S . We get $S = -1$!

Problem 3. Find all ordered pairs (x, y) of integers (where x and y need not be positive) such that $xy - 2x + 3y = 2010$.

Solution. We use an analogue of “completing the square.” Note that

$$xy - 2x + 3y = (x + 3)(y - 2) + 6,$$

so our equation can be rewritten as

$$(x + 3)(y - 2) = 2004.$$

The algebra could be given geometric content by drawing a suitable picture involving rectangles. A picture *is* important in the “completing the square” situation, for historical and other reasons. But I don’t think there is a strong argument for the necessity of a picture in this problem. (And rectangles with negative sides are not much fun to draw.)

We now look for integer solutions of the equation $uv = 2004$. The prime factorization of 2004 is given by

$$2004 = 2^2 \cdot 3 \cdot 167.$$

Recall that if p_1, p_2, \dots, p_k are distinct primes, then $p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$ has $(e_1 + 1)(e_2 + 1) \dots (e_k + 1)$ positive factors. If you do not recall this, let’s count for example the number of positive factors of N , where $N = 2^3 \cdot 3 \cdot 7^2$. To make a positive factor m of N , let’s first decide how many 2’s to give to m . There can be 0, or 1, or 2, or 3, 4 choices in all. For emphevery one of the decisions about how many 2’s, there are 2 ways to decide how many 3’s to give to m (0 or 1), so there are $4 \cdot 2$ ways of deciding how many 2’s and how many 3’s the factor m will get. And for every one of the decisions about how many 2’s and 3’s, there are 3 ways to decide how many 7’s m will get, for a total of $4 \cdot 2 \cdot 3$.

Thus 2004 has 12 positive factors, and 12 negative factors. This means that the equation $uv = 2004$ will have 24 ordered pairs (u, v) of integer solutions. It is not really worthwhile to list them all, since we have already for all practical purposes given a *method* for listing them all. But for the same of concreteness, we generate a few of the solutions.

We could have $u = 1, v = 2004$. That gives $x + 3 = 1, y - 2 = 2004$, that is, $x = -2, y = 2006$. Or else $u = 2004, v = 1$, giving $x = 2001, y = 3$. Or else $u = 2, v = 1002$, giving $x = -1, y = 1004$. Continue. Don’t forget about $u = -1, v = -2004$, giving $x = 4, y = -2002$, and so on, and so on.

Comment. The number 2004 is pretty easy to factor. But a huge number N can be extremely painful to factor, even with computer assistance. So finding the solutions of $xy - 2x + 3y = N + 6$ could be very difficult if N is huge.