Mathematics 414, Solutions to Problem Set #1

Problem 1. A rectangle has area 110 cm^2 and perimeter 44 cm. If each side of the rectangle is expanded by 2 cm, what is the area of the expanded rectangle?

Solution. We first sketch an inelegant solution. Let the sides of the rectangle be x and y. Then $xy = 110$ and $x + y = 22$. Since $y = 22 - x$, we can rewrite $xy = 110$ as $x(22 - x) = 110$. After minor manipulation, we obtain $x^2 - 22x + 110 = 0$. This has the solutions

$$x = 11 \pm \sqrt{11}.$$

If x is one of these roots, then $22 - x$ is the other, so we have found the sides of our rectangle. The expanded rectangle has sides $13 \pm \sqrt{11}$, and therefore has area 158.

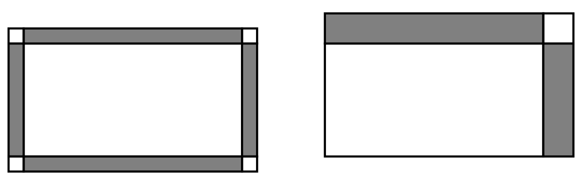
Comment. All too many high school students (but of course, not you) would do the following awful thing. They would somehow use a calculator to find a good numerical approximation to the sides, perhaps by evaluating $11 \pm \sqrt{11}$, or, with a more sophisticated calculator, by using the "*Solve*" button. Then they would add 2 to each of these, and multiply. The answer obtained is then numerically right, or almost right, but the underlying structure is totally missed.

Another Way. There is a much simpler way to solve the problem. Let x and y be the sides of the original rectangle. Then the sides of the expanded rectangle are $x + 2$ and $y + 2$. So the area is $(x + 2)(y + 2)$. But note that

$$(x + 2)(y + 2) = xy + 2(x + y) + 4.$$

Since $xy = 110$ and $2(x + y) = 44$, the expanded rectangle has area 158.

Another Way. The previous solution is certainly simple enough. But there is an at least aesthetic flaw. The problem is about areas and perimeters of rectangles, but all that I see in the solution is a bunch of x and y. Where's the rectangle? To *see* what's happening, look at the left-hand picture below.



The white rectangle in the middle represents our original rectangle. The outer rectangle represents the expanded rectangle, that is, the rectangle with each dimension increased by 2. The increasing has been done by adding a border of width 1, like a frame around a picture. This frame is made up of the shaded rectangles, plus four 1×1 squares at the corners.

The shaded rectangles, if cut out and lined up, would make a rectangle of height 1 and base the perimeter of the original rectangle. So the shaded stuff has total area 1×44. The corner squares have combined area 4, and the inner white rectangle has area 110, for a total of 158.

We can draw the diagram a little differently. Expand the original rectangle as in the right-hand picture, by adding the two shaded "decks" of width 2, together with the 2×2 square at the upper right corner. Again, a quick calculation shows that the area of the expanded rectangle is 158.

The pictures have nothing much to do with the actual values 110, 44, and 2 in the problem: the idea is obviously general.

Comment. We solve an equation like $x^2 + 36x = 700$ by "completing the square." Can one draw the geometric figure that is "incomplete," and how are we completing it to a real actual geometric square?

Problem 2. What is the smallest value taken on by $2x + 3/x$ as x ranges over the positive reals? No calculus, please. Hint: think maybe about minimizing $u + v$, subject to the condition $uv = 6$.

Solution. The hint is relevant. Once we have found positive u and v that satisfy $uv = 6$ and minimize $u + v$, we can find the value(s) of x that minimize $2x + 3/x$ by setting $x = u/2$.

One good thing about the altered version of the problem is the nice symmetry between u and v. Symmetry is very useful, useful enough to make it worthwhile to have two variables instead of one.

We will use the very important (and easily verified) identity

$$(u + v)^2 = (u - v)^2 + 4uv.$$

Given that $uv = 6$, we conclude that $(u + v)^2 = (u - v)^2 + 24$. Since $(u - v)^2$ is always ≥ 0, we conclude that

$$(u + v)^2 \geq 24,$$

with equality if and only if $u = v$. Since u and v are are positive, it follows that $u + v \geq \sqrt{24} = 2\sqrt{6}$, with equality if and only if $u = v$.

If $u = v$ and $u + v = \sqrt{24}$, we obtain $u = v = \sqrt{6}$. Thus $2x + 3/x$ is minimized over positive x if $x = \sqrt{6}/2$. The minimum *value* of $2x + 3/x$ over positive x is $2\sqrt{6}$.

Comment. We return to the inequality $(u + v)^2 \geq 4uv$. This can be rewritten as

$$\frac{(u + v)^2}{4} \geq uv.$$

If u and v are non-negative, we can take the square root of both sides and conclude that if u and v are non-negative, then

$$\frac{u + v}{2} \geq \sqrt{uv}$$

with equality if and only if $u = v$. This is the famous (two variable) Arithmetic Mean – Geometric Mean (AM – GM) Inequality. (Note that $(u + v)/2$ is the arithmetic mean of u and v, and $\sqrt{uv}$ is the geometric mean of u and v.)

This two variable inequality generalizes nicely. Let u_1, u_2, …, u_n be non-negative. It turns out that

$$\frac{u_1 + u_2 + \cdots + u_n}{n} \geq \sqrt[n]{u_1 u_2 \cdots u_n},$$

with equality if and only if all the u_i are equal. There are many proofs, none really simple. The above result is called the AM – GM Inequality. It is more or less the simplest inequality that one needs to know and use in Olympiad level contests. It is also highly useful outside contests!

Another Way. We next give a very *geometric* approach. There really should be a picture, but inserting the output of drawing software into a document is a tedious business, so let us *imagine* the picture. We are working in the u–v plane. First imagine drawing the part of the curve $uv = 6$, where u (and hence v) are positive. We get the first quadrant half of a hyperbola, one which is beautifully symmetric about the line $u = v$.

Now look at the lines $u + v = k$, for various positive values of the parameter k. We get a family of lines, all of them with slope -1. As k decreases, the associated line moves southwestwards. We want to make $u+v$ as small as possible consistent with $uv=6$. So we want to choose m so that the line $u+v=m$ meets the positive part of the hyperbola $uv=6$, but so that if $k<m$, the line $u+v=k$ does not meet our half-hyperbola. It is clear from the geometry that for this "last" (and least) m, the line meets the hyperbola at a point (u,v) with $u=v$. Since $uv=6$, it follows that $u=v=\sqrt{6}$, and therefore $m=2\sqrt{6}$.

Another Way. There is no compulsion to use the hint. Let $f(x)=2x+3/x$. We can graph the curve $y=f(x)$ using a graphing calculator. Or else we can use graphing software, which is often far more powerful than the graphing calculator, and doesn't make you squint at a pathetically little screen. (There are a number of good *free* graphing programs.)

By zooming appropriately on the graph of $y=f(x)$, we can find a good *estimate* of the minimum value of $f(x)$ over positive x. This does not fully answer the question. Ideally, we would like if possible to find an explicit expression for the minimum value.

Here is one way to do it. Let m be the minimum value taken on by $f(x)$. Then (if we believe the picture), the number m is the only positive number for which the horizontal line $y=m$ meets the curve $y=f(x)$ in exactly one point. So we want to determine the positive number m such that the equation

$$m=2x+\frac{3}{x}$$

has exactly one solution.

The above equation can be rewritten as

$$2x^2-mx+3=0.$$

Now we can proceed slowly, completing the square, or quote the usual formula for the roots of a quadratic equation, or quote some half-remembered result about something called the *discriminant.* I will proceed slowly. Multiplying through by 8 ("$4a$"), we obtain the equivalent equation

$$16x^2-8mx+24=0,$$

which can be rewritten as

$$(4x-m)^2=24-m^2.$$

For any given m, there is a unique solution x if and only if $24-m^2=0$, so the required smallest positive value is given by $m=\sqrt{24}$.

Another Way. We sketch a strange solution that introduces a couple of new ideas. Our function is not quite symmetric enough, $x+1/x$ would look nicer. Let $x=ku$. Then we are looking at $2ku+3/(ku)$. Choose k so that $2k=3/k$, so let $k=\sqrt{3/2}$. We end up wanting to miniimize $\sqrt{6}(u+1/u)$. Now deal with $u+1/u$.

Let $u=\tan(\theta)$. A bit of manipulation gets us to $1/(\sin(\theta)(\cos\theta))$. The denominator is $(1/2)\sin(2\theta)$. Make that as big as possible. Clearly this biggest possible value is 1.

Problem 3. Let $x=2000-\sqrt{999999}-\sqrt{1000001}$. Evaluate x (in "scientific" notation), correct to (a) 3 significant figures and (b) 15 significant figures. Do the evaluation without a calculator, or with at most a simple scientific calculator.

Solution. (a) We might as well cheat a little. There is a high precision calculator bundled with Microsoft Windows. Or else we could use the very interesting (and free) Wolfram Alpha. There are many other free programs that do high precision calculations. And if one insists on paying for what is available for free, there is Mathematica, or (O Canada) Maple. The Microsoft Windows calculator gives something like

$$2.5000000000000781249999999734 \times 10^{-10}.$$

Not entirely reliable of course, but interesting looking. Maybe our analysis will end up with something similar.

Now let us attack the calculation with a simple scientific calculator, whatever that means. Mine, which are all simple, or at least cheap, all claim that $x = 0$. The number x is indeed *close* to 0. But x is not equal to 0, so none of my calculators gives the result correct to even 1 significant figure! The problem is that 2000 and $\sqrt{999999} + \sqrt{10000001}$ are large and nearly equal, and a simple calculator does not carry enough digits to correctly find the difference. The difficulty we are in may seem artificial. But there are many real scientific computations in which the quantity we are interested in is the difference between nearly equal large numbers, so analogous issues come up surprisingly often in the real world.

Comment. Should we give up on the calculator? It may be a good idea to work with smaller numbers that the calculator really can handle. For example, one of my calculators gives

$$20 - \sqrt{99} - \sqrt{101} \approx 2.500079 \times 10^{-4}, \quad \text{and}$$

$$200 - \sqrt{9999} - \sqrt{10001} \approx 2.51 \times 10^{-7}$$

(here the third significant digit is wrong). We could calculate related things, like $50 - \sqrt{24} - \sqrt{26}$. After some calculator experimentation, we could form a plausible conjecture about what happens in general. We will not pursue this idea any further, but this kind of investigation can be very valuable.

Enough preliminary comments! Let's tackle the problem directly, by finding excellent approximations to $\sqrt{999999}$ and $\sqrt{1000001}$. Any simple scientific calculator, however primitive it may be, will show that $\sqrt{999999}$ is approximately 999.9995, and that $\sqrt{1000001}$ is approximately 1000.0005. Unfortunately, when we use these estimates to calculate x, we get 0. So we will find better approximations.

Look first at $\sqrt{1000001}$. We want to solve the equation $u^2 = 1000001$. It is clear that u is close to 1000. So let $u = 1000 + v$. Substitute $1000 + v$ for u in the equation $u^2 = 1000001$. Expand and simplify. We get $v^2 + 2000v = 1$. Since v is close to 0, the term v^2 is negligible compared to the other two terms. So $2000v \approx 1$. Thus $v \approx 1/2000$. Let $v = 1/2000 + w$. Substitute in the equation $v^2 + 2000v = 1$ and simplify a bit. We get

$$w^2 + (2000 + 1/1000)w + 1/2000^2 = 0.$$

The term w^2 is negligibly small in comparison with the other terms. And the coefficient of w is nearly 2000. So w is approximately equal to $-1/2000^3$. We conclude that

$$\sqrt{1000001} \approx 1000 + \frac{1}{2000} - \frac{1}{2000^3}.$$

Use the same technique to approximate $\sqrt{999999}$. We get

$$\sqrt{999999} \approx 1000 - \frac{1}{2000} - \frac{1}{2000^3}.$$

Add the two estimates, subtract the result from 2000. We get

$$x \approx 22000^3 = 2.50 \times 10^{-10}.$$

Our approximation is in fact excellent, x agrees with $2/2000^3$ to many decimal places. That will be shown later. Note that we did not bother to write down explicitly the decimal expansions for our square roots, but instead expressed each square root as a sum.

Comment. The approximation technique that we used turns out to be the same (for polynomials) as Newton's Method. In fact, the above kind of calculation is exactly what Newton did. The familiar formulation in terms of derivatives came quite a bit later, and is not due to Newton.

Another Way. It is convenient, but not necessary, to notice that $\sqrt{1000001} = 1000\sqrt{1.000001}$. To save typing, let $t = 0.000001 = 10^{-6}$. We want to find an excellent estimate for $\sqrt{1+t}$.

Note that $(1+t/2)^2 = 1 + t + t^2/4$. So the square of $(1+t/2)$ is a tiny bit bigger than $1+t$, meaning that the square root of $1+t$ is a tiny bit smaller than $1+t/2$. How can we adjust $1+t/2$ so that on squaring the $t^2/4$ term disappears? Look at $1+t/2-t^2/8$. When we square this, we get $1+t-t^3/8+t^4/64$. That's *awfully close* (technical term) to $1+t$, a very tiny bit less. So our square root is a very tiny bit bigger than $1+t/2-t^2/8$.

More or less the same reasoning works for $\sqrt{999999}$, which is equal to $1000\sqrt{1-t}$. The square of $1-t/2-t^2/8$ is $1-t+t^3/8+t^4/64$, a very tiny bit too big. So $\sqrt{1-t}$ is a very tiny bit smaller than $1-t/2-t^2/8$. Note that the errors in these new estimates for $\sqrt{1+t}$ and $\sqrt{1-t}$ are in opposite directions, and will at least partially cancel. So we have good reason to think that

$$x \approx 1000(2 - (1 - t/2 - t^2/8) - (1 + t/2 - t^2/8)) = 1000t^2/4.$$

But $t = 10^{-6}$, and our estimate for x is $10^{-9}/4$, or, in scientific notation, to 3 significant figures, 2.50×10^{-10}.

We have *almost* solved part (a) (in two ways). Our estimates for the square roots are fantastically good, but a formal proof has not been given that they are indeed fantastically good. Maybe one should relax, this is an applied numerical problem, maybe being morally sure is good enough. But we can easily check that, for example with $t = 10^{-6}$, the number $1+t/2-t^2/8$ is close enough to $\sqrt{1+t}$. We had already noted it is a tiny bit too small. Look at the slightly bigger number $1+t/2-t^2/9$. Its square is $1+t+t^2/36-t^3/9+t^4/81$, which is bigger than $1+t$. So our estimate for $\sqrt{1+t}$ has error less than $t^2/8-t^2/9$, which is $t^2/72$, about 1.4×10^{-14}, good enough, by a lot.

Comment. The approximation $\sqrt{1+t} \approx 1+t/2-t^2/8$ will be familiar to students of calculus, since $1+x-x^2/8$ is the degree 2 Taylor polynomial for $(1+x)^{1/2}$. But the method by which the approximation was derived makes no explicit use of the calculus.

There are other ways to show that $1+t/2-t^2/8$ is a good enough approximation to $\sqrt{1+t}$. We want to estimate the absolute value of

$$\sqrt{1+t} - (1+t/2-t^2/8).$$

Multiply "top" and "bottom" by $\sqrt{1+t} + (1+t/2-t^2/8)$. When the smoke clears, we get

$$\frac{t^3/8 - t^4/64}{\sqrt{1+t} + (1+t/2-t^2/8)}.$$

The denominator is clearly greater than 2, and the numerator is less than $t^3/8$, so the error is less than $t^3/16$, thus less than 6.25×10^{-20}. We get a similar estimate for the size of the error in our estimate of $\sqrt{1-t}$, and since the errors are in opposite directions, our ultimate error is less than 6.25×10^{-17} (remember the 1000 in front of our expression for x). We found that x is roughly 2.5×10^{-10}. Our error estimate now shows that this is correct to at least 6 significant figures. Unfortunately, that is not good enough to settle part (b).

Another Way. It is handy, and fairly natural, to note that

$$x = \left(1000 - \sqrt{999999}\right) - \left(\sqrt{1000001} - 1000\right).$$

Look first at $1000 - \sqrt{999999}$. We have

$$1000 - \sqrt{999999} = \frac{(1000 - \sqrt{999999})(1000 + \sqrt{999999})}{1000 + \sqrt{999999}} = \frac{1}{1000 + \sqrt{999999}}.$$

A similar calculation with the rest of the expression shows that

$$x = \frac{1}{1000 + \sqrt{999999}} - \frac{1}{\sqrt{1000001} + 1000} = \frac{\sqrt{1000001} - \sqrt{999999}}{(1000 + \sqrt{999999})(\sqrt{1000001} + 1000)}.$$

We still have a difference of nearly equal quantities. But now multiply "top" and "bottom" by $\sqrt{1000001} - \sqrt{999999}$. We get

$$x = \frac{2}{(1000 + \sqrt{999999})(\sqrt{1000001} + 1000)(\sqrt{1000001} + \sqrt{999999})}.$$

Finally, something that gives no roundoff issues! The denominator is a product of 3 terms, all of them nearly equal to 2000. Use a calculator, or more sensibly decide that for our purposes the denominator is almost exactly 2000^3. We obtain the estimate $x \approx 2.5 \times 10^{-10}$.

Another Way. Actually, we will write the preceding solution over again, the way it should have been done. The previous way involved an awful lot of typing: 999999 and 1000001 are not pleasant to write down over and over again. So let $a = 1000$ and let $e = 1$. Then we are trying to estimate

$$2a - \sqrt{a^2 - e} - \sqrt{a^2 + e}.$$

But

$$\sqrt{a^2 - e} - \sqrt{a^2 + e} = a\left(\sqrt{1 - \epsilon} + \sqrt{1 + \epsilon}\right),$$

where $\epsilon = e/a^2$. So we are estimating

$$a\left(2 - \sqrt{1 - \epsilon} - \sqrt{1 + \epsilon}\right).$$

(In our case, $\epsilon = 10^{-6}$.)

Now do *exactly* the same simplification as in the previous solution. After not very long, we arrive at

$$\frac{2a\epsilon^2}{(1 + \sqrt{1 - \epsilon})(1 + \sqrt{1 + \epsilon})(\sqrt{1 - \epsilon} + \sqrt{1 + \epsilon})}.$$

Since ϵ is close to 0, the denominator is nearly equal to 8. So our expression is nearly equal to $a\epsilon^2/4$, or, since $\epsilon = e/a^2$, our expression is nearly equal to $e^2/(4a^3)$. (By *nearly equal* we mean that the *ratio* of our expression to $e^2/(4a^3)$ is nearly equal to 1. Note that although 0.0002 and 0.0003 are not far from each other, they are not "nearly equal" in this sense.)

There is nothing special about $a = 1000$ and $e = 1$. The only thing that matters is that e is much smaller than a. So the approximation we have obtained is in fact *general.* If a is positive and $|d/a|$ is close to 0, then $2a - \sqrt{a^2 - e} - \sqrt{a^2 + e}$ is nearly equal to $e^2/(4a^3)$.

(b) We had observed in the solution to (a) that

$$x = 1000(2 - \sqrt{1-t} - \sqrt{1+t}),$$

where $t = 10^{-6}$. Now we try to get even better approximations to $\sqrt{1-t}$ and $\sqrt{1+t}$ than we found in two of the solutions to part (a). After a while we find that

$$\sqrt{1+t} \approx 1 + \frac{t}{2} - \frac{t^2}{8} + \frac{3t^3}{16} - \frac{5t^4}{128} \qquad \text{and}$$

$$\sqrt{1-t} \approx 1 - \frac{t}{2} - \frac{t^2}{8} - \frac{3t^3}{16} - \frac{5t^4}{128}.$$

That gives the following estimate for x:

$$x = 10^3\left(\frac{t^2}{4} + \frac{5t^4}{64}\right).$$

But $1000t^2/4 = 2.5 \times 10^{-10}$ and $5t^4/64 = 7.8125 \times 10^{-23}$. We conclude that x is probably awfully close to

$$2.50000000000078125 \times 10^{-10}.$$

(For 15 significant figures, we could cut off the "125" part, but it seems a shame to do that.) We have not shown that our estimate is indeed correct to 15 significant figures. An approach much like the one used for part (a) will work.

Comment. This problem has roots in the calculus. If $f(x)$ is a nice function, and h is small, then the second-order difference

$$\frac{(f(a+h) - f(a)) - (f(a) - f(a-h)}{h^2}$$

is a good approximation to $f''(a)$. Or, a little more sloppily,

$$f(a+h) - 2f(a) + f(a-h) \approx h^2 f''(a).$$

Let $f(x) = -\sqrt{1+x}$, and let $a = 0$. Then $f''(a) = 1/4$. It follows that $-\sqrt{1+h}+2-\sqrt{1-h} \approx h^2/4$, which is an informal version of what we showed in part (a).

When we try to estimate $f'(a)$ by calculating $(f(a+h) - f(a))/h$, where h is very small, we end up taking the difference of "large" nearly equal numbers. It is very easy, unless one is careful, to end up with seriously flawed results because of roundoff error. The situation is ordinarily much worse if we attempt the numerical evaluation of a second derivative. Our problem is a good illustration of this.