CLP 4

VECTOR CALCULUS

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CLP-4 Vector Calculus

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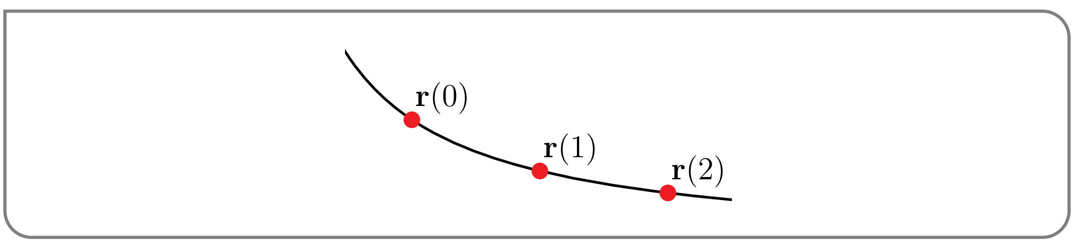
Chapter 1

CURVES

We are now going to study vector-valued functions of one real variable. That is, we are going to study functions that assign to each real number t (typically in some interval) a vector[1] $\mathbf{r}(t)$. For example

$$\mathbf{r}(t) = \big(x(t), y(t), z(t)\big)$$

might be the position of a particle at time t. As t varies, $\mathbf{r}(t)$ sweeps out a curve.

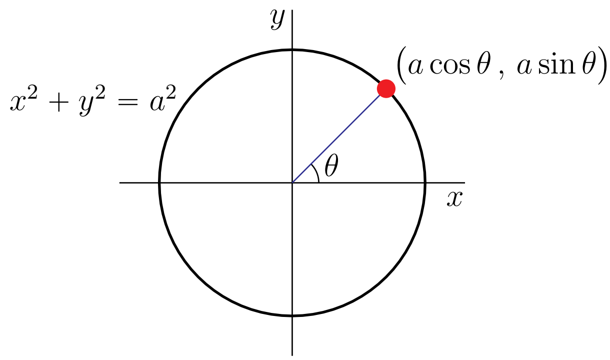


While in some applications t will indeed be "time", it does not have to be. It can be simply a parameter that is used to label the different points on the curve that $\mathbf{r}(t)$ sweeps out. We then say that $\mathbf{r}(t)$ provides a parameterization of the curve.

Example 1.0.1 $\big($Parametrization of $x^2 + y^2 = a^2\big)$

While we will often use t as the parameter in a parametrized curve $\mathbf{r}(t)$, there is no need to call it t. Sometimes it is natural to use a different name for the parameter. For example, consider the circle $x^2 + y^2 = a^2$. It is natural to use the angle θ in the sketch below to label the point $\big(a\cos\theta\,,\ a\sin\theta\big)$ on the circle.

1 We are going to use boldface letters, like $\mathbf{r}$, to designate vectors. When writing by hand, it is clearer to use arrows, like $\vec{r}$, instead.



That is,

$$\mathbf{r}(\theta) = \big(a\cos\theta\,,\, a\sin\theta\big) \qquad 0 \leqslant \theta < 2\pi$$

is a parametrization of the circle $x^2 + y^2 = a^2$. Just looking at the figure above, it is clear that, as θ runs from 0 to 2π, $\mathbf{r}(\theta)$ traces out the full circle.

However beware that just knowing that $\mathbf{r}(t)$ lies on a specified curve does not guarantee that, as t varies, $\mathbf{r}(t)$ covers the entire curve. For example, as t runs over the whole real line, $\frac{2}{\pi}\arctan(t)$ runs over the interval $(-1, 1)$. For all t,

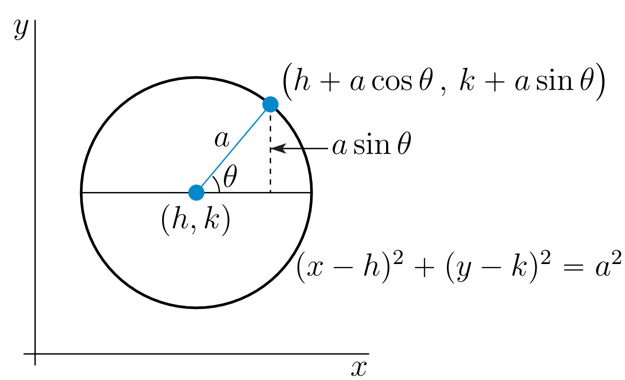
$$\mathbf{r}(t) = \big(x(t), y(t)\big) = a\left(\frac{2}{\pi}\arctan(t)\,,\, \sqrt{1 - \frac{4}{\pi^2}\arctan^2(t)}\right)$$

is well-defined and obeys $x(t)^2 + y(t)^2 = a^2$. But this $\mathbf{r}(t)$ does not cover the entire circle because $y(t)$ is always positive.

Example 1.0.1

Example 1.0.2 (Parametrization of $(x-h)^2 + (y-k)^2 = a^2$)

We can tweak the parametrization of Example 1.0.1 to get a parametrization of the circle of radius a that is centred on (h, k). One way to do so is to redraw the sketch of Example 1.0.1 with the circle translated so that its centre is at (h, k).



We see from the sketch that

$$\mathbf{r}(\theta) = \big(h + a\cos\theta\,,\, k + a\sin\theta\big) \qquad 0 \leqslant \theta < 2\pi$$

is a parametrization of the circle $(x-h)^2 + (y-k)^2 = a^2$.

A second way to come up with this parametrization is to observe that we can turn the trig identity $\cos^2 t + \sin^2 t = 1$ into the equation $(x-h)^2 + (y-k)^2 = a^2$ of the circle by

- multiplying the trig identity by a^2 to get $(a\cos t)^2 + (a\sin t)^2 = a^2$ and then
- setting $a\cos t = x - h$ and $a\sin t = y - k$, which turns $(a\cos t)^2 + (a\sin t)^2 = a^2$ into $(x-h)^2 + (y-k)^2 = a^2$.

Example 1.0.2

Example 1.0.3 $\left(\text{Parametrization of } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ and of } x^{2/3} + y^{2/3} = a^{2/3}\right)$

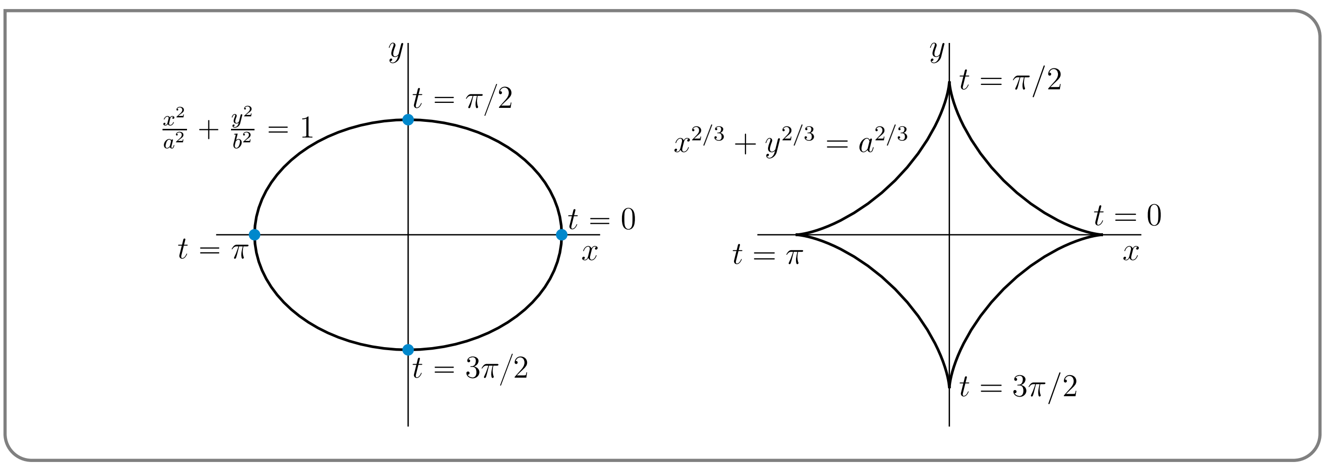
We can build parametrizations of the curves $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $x^{2/3} + y^{2/3} = a^{2/3}$ from the trig identity $\cos^2 t + \sin^2 t = 1$, like we did in the second part of the last example.

- Setting $\cos t = \frac{x}{a}$ and $\sin t = \frac{y}{b}$ turns $\cos^2 t + \sin^2 t = 1$ into $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
- Setting $\cos t = \left(\frac{x}{a}\right)^{\frac{1}{3}}$ and $\sin t = \left(\frac{y}{a}\right)^{\frac{1}{3}}$ turns $\cos^2 t + \sin^2 t = 1$ into $\frac{x^{2/3}}{a^{2/3}} + \frac{y^{2/3}}{a^{2/3}} = 1$.

So

$$\begin{aligned} \mathbf{r}(t) &= \big(a\cos t\,,\, b\sin t\big) & 0 \leqslant t < 2\pi \\ \mathbf{r}(t) &= \big(a\cos^3 t\,,\, a\sin^3 t\big) & 0 \leqslant t < 2\pi \end{aligned}$$

give parametrizations of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $x^{2/3} + y^{2/3} = a^{2/3}$, respectively. To see that running t from 0 to 2π runs $\mathbf{r}(t)$ once around the curve, look at the figures below.



The curve $x^{2/3} + y^{2/3} = a^{2/3}$ is called an astroid. From its equation, we would expect its sketch to look like a deformed circle. But it is probably not so obvious that it would have the pointy bits of the right hand figure. We will not explain here why they arise. The astroid is studied in some detail in Example 1.1.9. In particular, the above sketch is carefully developed there.

Example 1.0.3

Example 1.0.4 (Parametrization of $e^y = 1 + x^2$)

A very easy method that can often create parametrizations for a curve is to use x or y as a parameter. Because we can solve $e^y = 1 + x^2$ for y as a function of x, namely $y = \ln(1 + x^2)$, we can use x as the parameter simply by setting $t = x$. This gives the parametrization

$$\mathbf{r}(t) = \big(t\,,\,\ln(1+t^2)\big) \qquad -\infty < t < \infty$$

Example 1.0.4

Example 1.0.5 (Parametrization of $x^2 + y^2 = a^2$, again)

It is also quite common that one can use either x or y to parametrize part of, but not all of, a curve. A simple example is the circle $x^2 + y^2 = a^2$. For each $-a < x < a$, there are two points on the circle with that value of x. So one cannot use x to parametrize the whole circle. Similarly, for each $-a < y < a$, there are two points on the circle with that value of y. So one cannot use y to parametrize the whole circle. On the other hand

$$\mathbf{r}(t) = \big(t\,,\,\sqrt{a^2 - t^2}\big) \qquad -a < t < a$$
$$\mathbf{r}(t) = \big(t\,,\,-\sqrt{a^2 - t^2}\big) \qquad -a < t < a$$

provide parametrizations of the top half and bottom half, respectively, of the circle using x as the parameter, and

$$\mathbf{r}(t) = \big(\sqrt{a^2 - t^2}\,,\,t\big) \qquad -a < t < a$$
$$\mathbf{r}(t) = \big(-\sqrt{a^2 - t^2}\,,\,t\big) \qquad -a < t < a$$

provide parametrizations of the right half and left half, respectively, of the circle using y as the parameter.

Example 1.0.5

Example 1.0.6 (Unparametrization of $\mathbf{r}(t) = (\cos t, 7 - t)$)

In this example, we will undo the parametrization $\mathbf{r}(t) = (\cos t, 7 - t)$ and find the Cartesian equation of the curve in question. We may rewrite the parametrization as

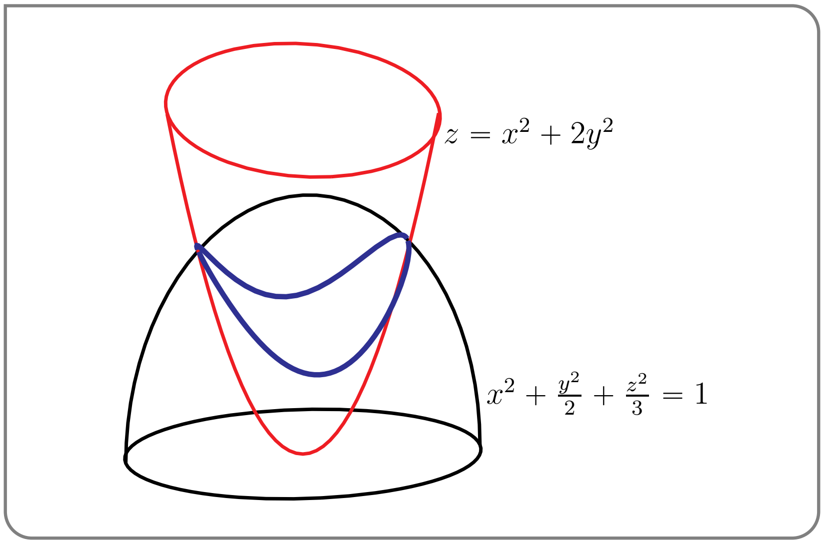
$$x = \cos t$$
$$y = 7 - t$$

Note that we can eliminate the parameter t simply by using the second equation to solve for t as a function of y. Namely $t = 7 - y$. Substituting this into the first equation gives us the Cartesian equation

$$x = \cos(7 - y)$$

Example 1.0.6

Curves often arise as the intersection of two surfaces. For example, the intersection of the ellipsoid $x^2 + \frac{y^2}{2} + \frac{z^2}{3} = 1$ with the paraboloid $z = x^2 + 2y^2$ is the blue curve in the figure below. One way to parametrize such curves is to choose one of the three



coordinates x, y, z as the parameter, and solve the two given equations for the remaining two coordinates, as functions of the parameter. Here are two examples.

Example 1.0.7

The set of all (x, y, z) obeying

$$\begin{aligned} x^3 - e^{3y} &= 0 \\ x^2 - e^y + z &= 0 \end{aligned}$$

is a curve. We can choose to use y as the parameter and think of

$$\begin{aligned} x^3 &= e^{3y} \\ x^2 + z &= e^y \end{aligned}$$

as a system of two equations for the two unknowns x and z, with y being treated as a given constant, rather than as an unknown. We can now solve the first equation for x, substitute the result into the second equation, and finally solve for z.

$$\begin{aligned} x^3 &= e^{3y} \implies x = e^y \\ x^2 + z &= e^y \qquad\qquad \implies e^{2y} + z = e^y \implies z = e^y - e^{2y} \end{aligned}$$

So

$$\mathbf{r}(y) = \left(e^y\,,\,y\,,\,e^y - e^{2y}\right)$$

is a parametrization for the given curve.

Example 1.0.7

Example 1.0.8

The previous example was rigged so that it was easy to solve for x and z as functions of y. In practice it is not always easy, or even possible, to do so. A more realistic example is the set of all (x, y, z) obeying

$$\begin{aligned} x^2 + \frac{y^2}{2} + \frac{z^2}{3} &= 1 \\ x^2 + 2y^2 &= z \end{aligned}$$

which is the blue curve in the figure above. Substituting $x^2 = z - 2y^2$ (from the second equation) into the first equation gives

$$-\frac{3}{2}y^2 + z + \frac{z^2}{3} = 1$$

or, completing the square,

$$-\frac{3}{2}y^2 + \frac{1}{3}\left(z + \frac{3}{2}\right)^2 = \frac{7}{4}$$

If, for example, we are interested in points (x, y, z) on the curve with $y \geqslant 0$, this can be solved to give y as a function of z.

$$y = \sqrt{\frac{2}{9}\left(z + \frac{3}{2}\right)^2 - \frac{14}{12}}$$

Then $x^2 = z - 2y^2$ also gives x as a function of z. If $x \geqslant 0$,

$$\begin{aligned} x &= \sqrt{z - \frac{4}{9}\left(z + \frac{3}{2}\right)^2 + \frac{14}{6}} \\ &= \sqrt{\frac{4}{3} - \frac{4}{9}z^2 - \frac{1}{3}z} \end{aligned}$$

The other signs of x and y can be gotten by using the appropriate square roots. In this example, (x, y, z) is on the curve, i.e. satisfies the two original equations, if and only if all of $(\pm x, \pm y, z)$ are also on the curve.

Example 1.0.8

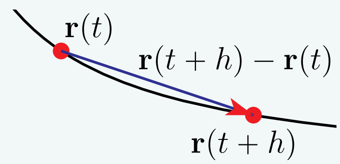
1.1▲ Derivatives, Velocity, Etc.

This being a Calculus text, one of our main operations is differentiation. We are now interested in parametrizations $\mathbf{r}(t)$. It is very easy and natural to extend our definition of derivative to $\mathbf{r}(t)$ as follows.

Definition 1.1.1.

The derivative of the vector valued function $\mathbf{r}(t)$ is defined to be

$$\mathbf{r}'(t) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$



when the limit exists. In particular, if $\mathbf{r}(t) = \big(x(t)\,,\, y(t)\,,\, z(t)\big)$, then

$$\mathbf{r}'(t) = \big(x'(t)\,,\, y'(t)\,,\, z'(t)\big)$$

That is, to differentiate a vector valued function of t, just differentiate each of its components.

And of course differentiation interacts with arithmetic operations, like addition, in the obvious way. Only a little more thought is required to see that differentiation interacts quite nicely with dot and cross products too. Here are some examples.

Example 1.1.2

Let

$$\begin{aligned} \mathbf{a}(t) &= t^2\,\hat{\boldsymbol{\imath}} + t^4\,\hat{\boldsymbol{\jmath}} + t^6\,\hat{\mathbf{k}} \\ \mathbf{b}(t) &= e^{-t}\,\hat{\boldsymbol{\imath}} + e^{-3t}\,\hat{\boldsymbol{\jmath}} + e^{-5t}\,\hat{\mathbf{k}} \\ \gamma(t) &= t^2 \\ s(t) &= \sin t \end{aligned}$$

We are about to compute some derivatives. To make it easier to follow what is going on, we'll use some colour. When we apply the product rule

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[f(t)\,g(t)\big] = f'(t)\,g(t) + f(t)\,g'(t)$$

we'll use blue to highlight the factors $f'(t)$ and $g'(t)$. Here we go.

$$\begin{aligned} \gamma(t)\,\mathbf{b}(t) &= t^2 e^{-t}\,\hat{\boldsymbol{\imath}} + t^2 e^{-3t}\,\hat{\boldsymbol{\jmath}} + t^2 e^{-5t}\,\hat{\mathbf{k}} \\ \implies \frac{\mathrm{d}}{\mathrm{d}t}\big[\gamma(t)\mathbf{b}(t)\big] &= \big[2te^{-t} - t^2 e^{-t}\big]\hat{\boldsymbol{\imath}} + \big[2te^{-3t} - 3t^2 e^{-3t}\big]\hat{\boldsymbol{\jmath}} + \big[2te^{-5t} - 5t^2 e^{-5t}\big]\hat{\mathbf{k}} \\ &= 2t\big\{e^{-t}\,\hat{\boldsymbol{\imath}} + e^{-3t}\,\hat{\boldsymbol{\jmath}} + e^{-5t}\,\hat{\mathbf{k}}\big\} + t^2\big\{-e^{-t}\,\hat{\boldsymbol{\imath}} - 3e^{-3t}\,\hat{\boldsymbol{\jmath}} - 5e^{-5t}\,\hat{\mathbf{k}}\big\} \\ &= \gamma'(t)\mathbf{b}(t) + \gamma(t)\mathbf{b}'(t) \end{aligned}$$

and

$$\begin{aligned}
\mathbf{a}(t)\cdot\mathbf{b}(t) &= t^2e^{-t} + t^4e^{-3t} + t^6e^{-5t} \\
\implies \frac{\mathrm{d}}{\mathrm{d}t}\big[\mathbf{a}(t)\cdot\mathbf{b}(t)\big] &= \big[2te^{-t} - t^2e^{-t}\big] + \big[4t^3e^{-3t} - 3t^4e^{-3t}\big] + \big[6t^5e^{-5t} - 5t^6e^{-5t}\big] \\
&= \big[2te^{-t} + 4t^3e^{-3t} + 6t^5e^{-5t}\big] + \big[-t^2e^{-t} - 3t^4e^{-3t} - 5t^6e^{-5t}\big] \\
&= \big\{2t\,\hat{\pmb{\imath}} + 4t^3\,\hat{\pmb{\jmath}} + 6t^5\,\hat{\mathbf{k}}\big\}\cdot\big\{e^{-t}\,\hat{\pmb{\imath}} + e^{-3t}\,\hat{\pmb{\jmath}} + e^{-5t}\,\hat{\mathbf{k}}\big\} \\
&\qquad + \big\{t^2\,\hat{\pmb{\imath}} + t^4\,\hat{\pmb{\jmath}} + t^6\,\hat{\mathbf{k}}\big\}\cdot\big\{-e^{-t}\,\hat{\pmb{\imath}} - 3e^{-3t}\,\hat{\pmb{\jmath}} - 5e^{-5t}\,\hat{\mathbf{k}}\big\} \\
&= \mathbf{a}'(t)\cdot\mathbf{b}(t) + \mathbf{a}(t)\cdot\mathbf{b}'(t)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{a}(t)\times\mathbf{b}(t) &= \det\begin{bmatrix} \hat{\pmb{\imath}} & \hat{\pmb{\jmath}} & \hat{\mathbf{k}} \\ t^2 & t^4 & t^6 \\ e^{-t} & e^{-3t} & e^{-5t} \end{bmatrix} \\
&= \hat{\pmb{\imath}}\big(t^4e^{-5t} - t^6e^{-3t}\big) - \hat{\pmb{\jmath}}\big(t^2e^{-5t} - t^6e^{-t}\big) + \hat{\mathbf{k}}\big(t^2e^{-3t} - t^4e^{-t}\big) \\
\implies \frac{\mathrm{d}}{\mathrm{d}t}\big[\mathbf{a}(t)\times\mathbf{b}(t)\big] &= \hat{\pmb{\imath}}\big(\,4t^3e^{-5t} - 6t^5e^{-3t}\big) - \hat{\pmb{\jmath}}\big(\,2te^{-5t} - 6t^5e^{-t}\big) + \hat{\mathbf{k}}\big(\,2te^{-3t} - 4t^3e^{-t}\big) \\
&\quad + \hat{\pmb{\imath}}\big(-5t^4e^{-5t} + 3t^6e^{-3t}\big) - \hat{\pmb{\jmath}}\big(-5t^2e^{-5t} + t^6e^{-t}\big) + \hat{\mathbf{k}}\big(-3t^2e^{-3t} + t^4e^{-t}\big) \\
&= \big\{2t\,\hat{\pmb{\imath}} + 4t^3\,\hat{\pmb{\jmath}} + 6t^5\,\hat{\mathbf{k}}\big\}\times\big\{e^{-t}\,\hat{\pmb{\imath}} + e^{-3t}\,\hat{\pmb{\jmath}} + e^{-5t}\,\hat{\mathbf{k}}\big\} \\
&\quad + \big\{t^2\,\hat{\pmb{\imath}} + t^4\,\hat{\pmb{\jmath}} + t^6\,\hat{\mathbf{k}}\big\}\times\big\{-e^{-t}\,\hat{\pmb{\imath}} - 3e^{-3t}\,\hat{\pmb{\jmath}} - 5e^{-5t}\,\hat{\mathbf{k}}\big\} \\
&= \mathbf{a}'(t)\times\mathbf{b}(t) + \mathbf{a}(t)\times\mathbf{b}'(t)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{a}\big(s(t)\big) &= (\sin t)^2\,\hat{\pmb{\imath}} + (\sin t)^4\,\hat{\pmb{\jmath}} + (\sin t)^6\,\hat{\mathbf{k}} \\
\implies \frac{\mathrm{d}}{\mathrm{d}t}\big[\mathbf{a}\big(s(t)\big)\big] &= 2(\sin t)\cos t\,\hat{\pmb{\imath}} + 4(\sin t)^3\cos t\,\hat{\pmb{\jmath}} + 6(\sin t)^5\cos t\,\hat{\mathbf{k}} \\
&= \big\{2(\sin t)\,\hat{\pmb{\imath}} + 4(\sin t)^3\hat{\pmb{\jmath}} + 6(\sin t)^5\hat{\mathbf{k}}\big\}\cos t \\
&= \mathbf{a}'\big(s(t)\big)\,s'(t)
\end{aligned}$$

Example 1.1.2

Of course these examples extend to general (differentiable) $\mathbf{a}(t)$, $\mathbf{b}(t)$, $\gamma(t)$ and $s(t)$ and give us (most of) the following theorem.

Theorem 1.1.3 (Arithmetic of differentiation)**.**

Let

- $\mathbf{a}(t), \mathbf{b}(t)$ be vector valued differentiable functions of $t \in \mathbb{R}$ that take values in $\mathbb{R}^n$ and
- $\alpha, \beta \in \mathbb{R}$ be constants and
- $\gamma(t)$ and $s(t)$ be real valued differentiable functions of $t \in \mathbb{R}$

Then

(a) $\frac{\mathrm{d}}{\mathrm{d}t}\big[\alpha\,\mathbf{a}(t) + \beta\,\mathbf{b}(t)\big] = \alpha\,\mathbf{a}'(t) + \beta\,\mathbf{b}'(t)$ (linear combination)

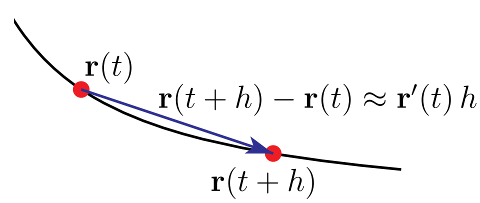
(b) $\frac{\mathrm{d}}{\mathrm{d}t}\big[\gamma(t)\mathbf{b}(t)\big] = \gamma'(t)\mathbf{b}(t) + \gamma(t)\mathbf{b}'(t)$ (multiplication by scalar function)

(c) $\frac{\mathrm{d}}{\mathrm{d}t}\big[\mathbf{a}(t) \cdot \mathbf{b}(t)\big] = \mathbf{a}'(t) \cdot \mathbf{b}(t) + \mathbf{a}(t) \cdot \mathbf{b}'(t)$ (dot product)

(d) $\frac{\mathrm{d}}{\mathrm{d}t}\big[\mathbf{a}(t) \times \mathbf{b}(t)\big] = \mathbf{a}'(t) \times \mathbf{b}(t) + \mathbf{a}(t) \times \mathbf{b}'(t)$ (cross product)

(e) $\frac{\mathrm{d}}{\mathrm{d}t}\big[\mathbf{a}(s(t))\big] = \mathbf{a}'(s(t))\,s'(t)$ (composition)

Let's think about the geometric significance of $\mathbf{r}'(t)$. In particular, let's think about the relationship between $\mathbf{r}'(t)$ and distances along the curve. The derivative $\mathbf{r}'(t)$ is the limit of $\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$ as $h \to 0$. The numerator, $\mathbf{r}(t+h) - \mathbf{r}(t)$, is the vector with head at $\mathbf{r}(t+h)$ and tail at $\mathbf{r}(t)$.



When h is very small this vector

- has the essentially the same direction as the tangent vector to the curve at $\mathbf{r}(t)$ and
- has length being essentially the length of the part of the curve between $\mathbf{r}(t)$ and $\mathbf{r}(t+h)$.

Taking the limit as $h \to 0$ yields that

- $\mathbf{r}'(t)$ is a tangent vector to the curve at $\mathbf{r}(t)$ that points in the direction of increasing t and
- if $s(t)$ is the length of the part of the curve between $\mathbf{r}(0)$ and $\mathbf{r}(t)$, then $\frac{\mathrm{d}s}{\mathrm{d}t}(t) = \big|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\big|$.

This is worth stating formally.

Lemma 1.1.4.

Let $\mathbf{r}(t)$ be a parametrized curve.

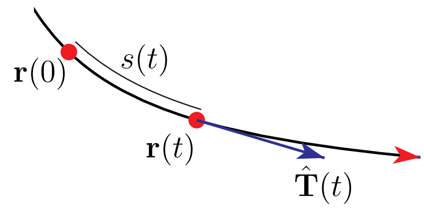
(a) Denote by $\hat{\mathbf{T}}(t)$ the unit tangent vector to the curve at $\mathbf{r}(t)$ pointing in the direction of increasing t. If $\mathbf{r}'(t) \neq 0$ then

$$\hat{\mathbf{T}}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

(b) Denote by $s(t)$ the length of the part of the curve between $\mathbf{r}(0)$ and $\mathbf{r}(t)$. Then

$$\frac{\mathrm{d}s}{\mathrm{d}t}(t) = \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\right|$$

$$s(T) - s(T_0) = \int_{T_0}^{T} \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\right| \mathrm{d}t$$



(c) In particular, if the parameter happens to be arc length, i.e. if $t = s$, so that $\frac{\mathrm{d}s}{\mathrm{d}s} = 1$, then

$$\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}(s)\right| = 1 \qquad \hat{\mathbf{T}}(s) = \mathbf{r}'(s)$$

As an application, we have the

Lemma 1.1.5.

If $\mathbf{r}(t) = \big(x(t)\,,\, y(t)\,,\, z(t)\big)$ is the position of a particle at time t, then

$$\begin{aligned}
\text{position at time } t &= \mathbf{r}(t) = \big(x(t)\,,\, y(t)\,,\, z(t)\big)\\
\text{velocity at time } t &= \mathbf{v}(t) = \mathbf{r}'(t) = \big(x'(t)\,,\, y'(t)\,,\, z'(t)\big) = \frac{\mathrm{d}s}{\mathrm{d}t}(t)\,\hat{\mathbf{T}}(t)\\
\text{speed at time } t &= \frac{\mathrm{d}s}{\mathrm{d}t}(t) = |\mathbf{v}(t)| = |\mathbf{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\\
\text{acceleration at time } t &= \mathbf{a}(t) = \mathbf{r}''(t) = \mathbf{v}'(t) = \big(x''(t)\,,\, y''(t)\,,\, z''(t)\big)
\end{aligned}$$

and the distance travelled between times T_0 and T is

$$s(T) - s(T_0) = \int_{T_0}^{T} \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\right| \mathrm{d}t = \int_{T_0}^{T} \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, \mathrm{d}t$$

Note that the velocity $\mathbf{v}(t) = \mathbf{r}'(t)$ is a vector quantity while the speed $\frac{\mathrm{d}s}{\mathrm{d}t}(t) = |\mathbf{r}'(t)|$ is a scalar quantity.

Example 1.1.6 (Circumference of a circle)

In general it can be quite difficult to compute arc lengths. So, as an easy warmup example, we will compute the circumference of the circle $x^2 + y^2 = a^2$. We'll also find a unit tangent to the circle at any point on the circle. We'll use the parametrization

$$\mathbf{r}(\theta) = \big(a\cos\theta\,,\ a\sin\theta\big) \qquad 0 \leqslant \theta \leqslant 2\pi$$

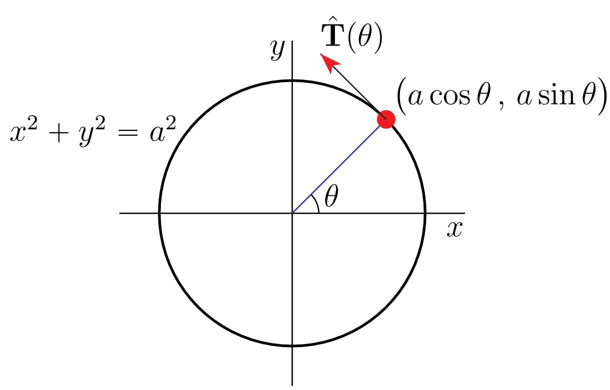
of Example 1.0.1. Using Lemma 1.1.4, but with the parameter t renamed to θ

$$\begin{aligned}
\mathbf{r}'(\theta) &= a\big(-\sin\theta\,,\cos\theta\big)\\
\hat{\mathbf{T}}(\theta) &= \frac{\mathbf{r}'(\theta)}{|\mathbf{r}'(\theta)|} = \big(-\sin\theta\,,\cos\theta\big)\\
\frac{\mathrm{d}s}{\mathrm{d}\theta}(\theta) &= \big|\mathbf{r}'(\theta)\big| = a\\
s(\Theta) - s(0) &= \int_0^\Theta \big|\mathbf{r}'(\theta)\big|\,\mathrm{d}\theta = a\Theta
\end{aligned}$$

As[2] $s(\Theta)$ is the arc length of the part of the circle with $0 \leqslant \theta \leqslant \Theta$, the circumference of the whole circle is

$$s(2\pi) = 2\pi a$$

which is reassuring, since this formula has been known[3] for thousands of years. The



formula $s(\Theta) - s(0) = a\Theta$ also makes sense — the part of the circle with $0 \leqslant \theta \leqslant \Theta$ is the fraction $\frac{\Theta}{2\pi}$ of the whole circle, and so should have length $\frac{\Theta}{2\pi} \times 2\pi a$. Also note that

$$\mathbf{r}(\theta)\cdot\hat{\mathbf{T}}(\theta) = \big(a\cos\theta\,,\ a\sin\theta\big)\cdot\big(-\sin\theta\,,\cos\theta\big) = 0$$

so that the tangent to the circle at any point is perpendicular to the radius vector of the circle at that point. This is another geometric fact that has been known[4] for thousands of years.

2 You might guess that Θ is a capital Greek theta. You'd be right.
3 The earliest known written approximations of π, in Egypt and Babylon, date from 1900–1600BC. The first recorded algorithm for rigorously evaluating π was developed by Archimedes around 250 BC. The first use of the symbol π, for the ratio between the circumference of a circle and its diameter, in print was in 1706 by William Jones.
4 It is Proposition 18 in Book 3 of Euclid's Elements. It was published around 300BC.

Example 1.1.6

Example 1.1.7 (Arc length of a helix)

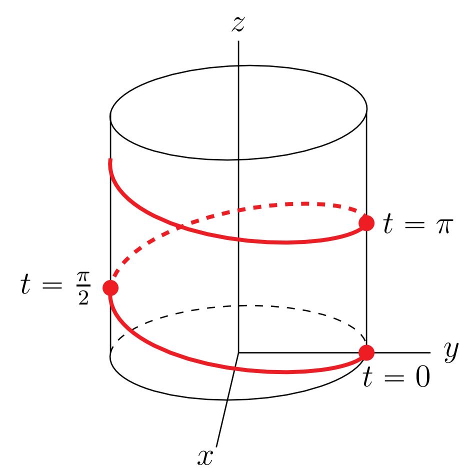
Consider the curve

$$\mathbf{r}(t) = 6\sin(2t)\hat{\imath} + 6\cos(2t)\hat{\jmath} + 5t\hat{\mathbf{k}}$$

where the standard basis vectors $\hat{\imath} = (1,0,0)$, $\hat{\jmath} = (0,1,0)$ and $\hat{\mathbf{k}} = (0,0,1)$. We'll first sketch it, by observing that

- $x(t) = 6\sin(2t)$ and $y(t) = 6\cos(2t)$ obey $x(t)^2 + y(t)^2 = 36\sin^2(2t) + 36\cos^2(2t) = 36$. So all points of the curve lie on the cylinder $x^2 + y^2 = 36$ and
- as t increases, $\big(x(t), y(t)\big)$ runs clockwise around the circle $x^2 + y^2 = 36$ and at the same time $z(t) = 5t$ just increases linearly.

Our curve is the helix



We have marked three points of the curve on the above sketch. The first has $t = 0$ and is $0\hat{\imath} + 6\hat{\jmath} + 0\hat{\mathbf{k}}$. The second has $t = \frac{\pi}{2}$ and is $0\hat{\imath} - 6\hat{\jmath} + \frac{5\pi}{2}\hat{\mathbf{k}}$, and the third has $t = \pi$ and is $0\hat{\imath} + 6\hat{\jmath} + 5\pi\hat{\mathbf{k}}$. We'll now use Lemma 1.1.4 to find a unit tangent $\hat{\mathbf{T}}(t)$ to the curve at $\mathbf{r}(t)$ and also the arclength of the part of curve between $t = 0$ and $t = \pi$.

$$\begin{aligned}
\mathbf{r}(t) &= 6\sin(2t)\hat{\imath} + 6\cos(2t)\hat{\jmath} + 5t\hat{\mathbf{k}} \\
\mathbf{r}'(t) &= 12\cos(2t)\hat{\imath} - 12\sin(2t)\hat{\jmath} + 5\hat{\mathbf{k}} \\
\frac{\mathrm{d}s}{\mathrm{d}t}(t) &= \big|\mathbf{r}'(t)\big| = \sqrt{12^2\cos^2(2t) + 12^2\sin^2(2t) + 5^2} = \sqrt{12^2 + 5^2} \\
&= 13 \\
\hat{\mathbf{T}}(t) &= \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t))|} = \frac{12}{13}\cos(2t)\hat{\imath} - \frac{12}{13}\sin(2t)\hat{\jmath} + \frac{5}{13}\hat{\mathbf{k}} \\
s(\pi) - s(0) &= \int_0^\pi \big|\mathbf{r}'(t)\big|\,\mathrm{d}t = 13\pi
\end{aligned}$$

Example 1.1.7

Example 1.1.8 (Velocity and acceleration)

Imagine that, at time t, a particle is at

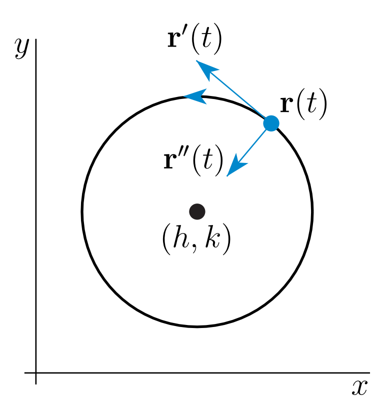
$$\mathbf{r}(t) = \left[h + a\cos\left(2\pi\frac{t}{T}\right)\right]\hat{\boldsymbol{\imath}} + \left[k + a\sin\left(2\pi\frac{t}{T}\right)\right]\hat{\boldsymbol{\jmath}}$$

As $|\mathbf{r}(t) - h\hat{\boldsymbol{\imath}} - k\hat{\boldsymbol{\jmath}}| = a$, the particle is running around the circle of radius a centred on (h, k). When t increases by T, the argument, $2\pi\frac{t}{T}$, of $\cos\left(2\pi\frac{t}{T}\right)$ and $\sin\left(2\pi\frac{t}{T}\right)$ increases by exactly 2π and the particle runs exactly once around the circle. In particular, it travels a distance $2\pi a$. So it is moving at speed $\frac{2\pi a}{T}$. According to Lemma 1.1.5, it has

$$\begin{aligned}
\text{velocity } &= \mathbf{r}'(t) = -\frac{2\pi a}{T}\sin\left(2\pi\frac{t}{T}\right)\hat{\boldsymbol{\imath}} + \frac{2\pi a}{T}\cos\left(2\pi\frac{t}{T}\right)\hat{\boldsymbol{\jmath}} \\
\text{speed} &= \frac{\mathrm{d}s}{\mathrm{d}t}(t) = |\mathbf{r}'(t)| = \frac{2\pi a}{T} \\
\text{acceleration} &= \mathbf{r}''(t) = -\frac{4\pi^2 a}{T^2}\cos\left(2\pi\frac{t}{T}\right)\hat{\boldsymbol{\imath}} - \frac{4\pi^2 a}{T^2}\sin\left(2\pi\frac{t}{T}\right)\hat{\boldsymbol{\jmath}} = -\frac{4\pi^2}{T^2}\left[\mathbf{r}(t) - h\hat{\boldsymbol{\imath}} - k\hat{\boldsymbol{\jmath}}\right]
\end{aligned}$$

Here are some observations.

- The velocity $\mathbf{r}'(t)$ has dot product zero with $\mathbf{r}(t) - h\hat{\boldsymbol{\imath}} - k\hat{\boldsymbol{\jmath}}$, which is the radius vector from the centre of the circle to the particle. So the velocity is perpendicular to the radius vector, and hence parallel to the tangent vector of the circle at $\mathbf{r}(t)$.
- The speed given by Lemma 1.1.5 is exactly the speed we found above, just before we started applying Lemma 1.1.5.
- The acceleration $\mathbf{r}''(t)$ points in the direction opposite to the radius vector.



Example 1.1.8

Example 1.1.9 (Perimeter of the astroid)

In this example, we find the perimeter of the astroid[5]

$$x^{2/3} + y^{2/3} = a^{2/3}$$

A geometric construction of this curve, as well as a derivation of its equation is given in the optional section 1.11 later. We'll start by finding a convenient parametrization.

- To do so, notice that $x^{2/3} + y^{2/3} = a^{2/3}$ looks somewhat like the equation of the circle $x^2 + y^2 = a^2$.
- The standard parametrization of the circle, namely $x = a\cos t$, $y = a\sin t$ works because of the elementary trig identity $\cos^2 t + \sin^2 t = 1$.
- If we can arrange that $x(t)^{2/3} = a^{2/3}\cos^2 t$ and $y(t)^{2/3} = a^{2/3}\sin^2 t$, then the same elementary trig identity will give $x(t)^{2/3} + y(t)^{2/3} = a^{2/3}$, as desired.
- But of course its easy to arrange that: just solve $x(t)^{2/3} = a^{2/3}\cos^2 t$ for $x(t)$, namely $x(t) = a\cos^3 t$, and solve $y(t)^{2/3} = a^{2/3}\sin^2 t$ for $y(t)$, namely $y(t) = a\sin^3 t$.

Our parametrization is

$$\mathbf{r}(t) = a\cos^3 t\,\hat{\boldsymbol{\imath}} + a\sin^3 t\,\hat{\boldsymbol{\jmath}}$$

By Lemma 1.1.4

$$\begin{aligned}
\mathbf{r}(t) &= a\cos^3 t\,\hat{\boldsymbol{\imath}} + a\sin^3 t\,\hat{\boldsymbol{\jmath}} \\
\mathbf{r}'(t) &= -3a\sin t\cos^2 t\,\hat{\boldsymbol{\imath}} + 3a\sin^2 t\cos t\,\hat{\boldsymbol{\jmath}} \\
\frac{\mathrm{d}s}{\mathrm{d}t}(t) &= \big|\mathbf{r}'(t)\big| = \sqrt{9a^2\sin^2 t\cos^4 t + 9a^2\sin^4 t\cos^2 t} \\
&= 3a\sqrt{\sin^2 t\cos^2 t(\cos^2 t + \sin^2 t)} \\
&= 3a\big|\sin t\cos t\big| \\
\hat{\mathbf{T}}(t) &= \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t))|} = \frac{\sin t\cos t}{|\sin t\cos t|}\,\big(-\cos t\,\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}}\big) \\
&= \mathrm{sgn}\big(\sin t\cos t\big)\,\big(-\cos t\,\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}}\big)
\end{aligned}$$

Here $\mathrm{sgn}\big(\sin t\cos t\big)$ means "the sign of $\sin t\cos t$", i.e $+1$ when $\sin t\cos t > 0$ and -1 when $\sin t\cos t < 0$. So

$$\begin{aligned}
\hat{\mathbf{T}}(t) &= \begin{Bmatrix} 1 & \text{if } \sin t > 0,\ \cos t > 0 & \text{or} & \sin t < 0,\ \cos t < 0 \\ -1 & \text{if } \sin t > 0,\ \cos t < 0 & \text{or} & \sin t < 0,\ \cos t > 0 \end{Bmatrix}\big(-\cos t\,\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}}\big) \\
&= \begin{Bmatrix} 1 & \text{if } 0 < t < \frac{\pi}{2} & \text{or} & \pi < t < \frac{3\pi}{2} \\ -1 & \text{if } \frac{\pi}{2} < t < \pi & \text{or} & \frac{3\pi}{2} < t < 2\pi \end{Bmatrix}\big(-\cos t\,\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}}\big)
\end{aligned}$$

Before we go on to sketch the astroid and compute its perimeter, we can make a few observations that will simplify our lives.

5 Astroid should not be confused with asteroid, though both words derive from the Greek word for star.

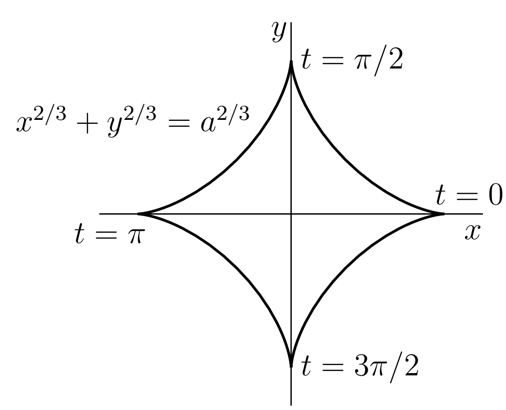
- The signs of both components of $\mathbf{r}(t)$ are the same as the signs of the components of $\cos t\,\hat{\imath} + \sin t\,\hat{\jmath}$; and the signs of both components of $\mathbf{r}'(t)$ are the same as the signs of the components of $-\sin t\,\hat{\imath} + \cos t\,\hat{\jmath}$. Consequently the astroid looks somewhat like a circle in that
  - when $0 \leqslant t \leqslant \frac{\pi}{2}$, $\mathbf{r}(t)$ lies in the first quadrant and moves upward and to the left as t increases and
  - when $\frac{\pi}{2} \leqslant t \leqslant \pi$, $\mathbf{r}(t)$ lies in the second quadrant and moves downward and to the left as t increases and
  - when $\pi \leqslant t \leqslant \frac{3\pi}{2}$, $\mathbf{r}(t)$ lies in the third quadrant and moves downward and to the right as t increases and
  - when $\frac{3\pi}{2} \leqslant t \leqslant 2\pi$, $\mathbf{r}(t)$ lies in the fourth quadrant and moves upward and to the right as t increases and
  - $\mathbf{r}(2\pi) = \mathbf{r}(0)$ so that the astroid is a closed curve that circumnavigates the origin exactly once as t runs from 0 to 2π.
- Something weird happens at those values of t where $\sin t \cos t$ changes sign[6], i.e. at $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, etc. Namely $\hat{T}(t)$ flips. To be precise

$$\lim_{t\to 0-} \hat{T}(t) = \lim_{t\to 0-} \operatorname{sgn}\big(\sin t \cos t\big) \lim_{t\to 0-} \big(-\cos t\,\hat{\imath} + \sin t\,\hat{\jmath}\big) = \hat{\imath}$$
$$\lim_{t\to 0+} \hat{T}(t) = \lim_{t\to 0+} \operatorname{sgn}\big(\sin t \cos t\big) \lim_{t\to 0+} \big(-\cos t\,\hat{\imath} + \sin t\,\hat{\jmath}\big) = -\hat{\imath}$$

and

$$\lim_{t\to \pi/2-} \hat{T}(t) = \lim_{t\to \pi/2-} \operatorname{sgn}\big(\sin t \cos t\big) \lim_{t\to \pi/2-} \big(-\cos t\,\hat{\imath} + \sin t\,\hat{\jmath}\big) = \hat{\jmath}$$
$$\lim_{t\to \pi/2+} \hat{T}(t) = \lim_{t\to \pi/2+} \operatorname{sgn}\big(\sin t \cos t\big) \lim_{t\to \pi/2+} \big(-\cos t\,\hat{\imath} + \sin t\,\hat{\jmath}\big) = -\hat{\jmath}$$

and so on. This signals cusps in the curve at $t = 0$, i.e. at $\mathbf{r}(0) = a\hat{\imath}$, and at $t = \frac{\pi}{2}$, i.e. at $\mathbf{r}(\frac{\pi}{2}) = a\hat{\jmath}$, and so on. So while the astroid looks somewhat like a circle, it has cusps at $\pm a\hat{\imath}$ and $\pm a\hat{\jmath}$. Here is the sketch.



6 Like a cross-walk sign.

- The astroid is invariant under reflections in the x-axis and in the y-axis. That is, $x^{2/3} + y^{2/3} = a^{2/3}$ is invariant under $x \to -x$ and also under $y \to -y$. So to find the whole perimeter, it suffices to find the arc length of the part of the astroid in the first quadrant, and then multiply by 4.

$$\begin{aligned} \text{perimeter} &= 4\int_0^{\pi/2} \frac{\mathrm{d}s}{\mathrm{d}t}\,\mathrm{d}t = 4\int_0^{\pi/2} 3a\sin t\cos t\,\mathrm{d}t = 6a\int_0^{\pi/2} \sin(2t)\,\mathrm{d}t \\ &= 6a\Big[-\frac{\cos(2t)}{2}\Big]_0^{\pi/2} = 6a \end{aligned}$$

Example 1.1.9

Example 1.1.10 ($\mathbf{r}'(t) = \mathbf{0}$)

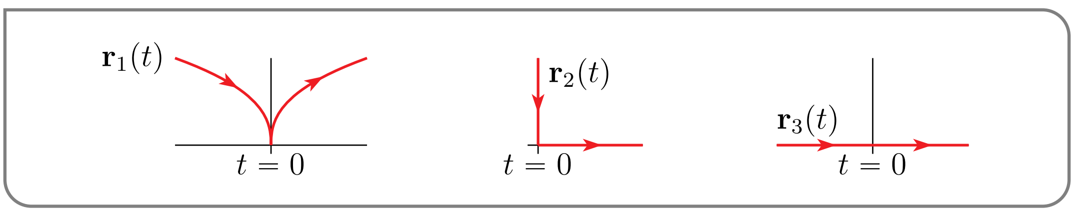
In the last example, we found that the astroid had cusps at those points $\mathbf{r}(t)$ where the velocity $\mathbf{r}'(t)$ vanished. In this example, we will explore a little further what can happen when $\mathbf{r}'(t) = \mathbf{0}$.

Suppose that you are out for a walk and that your position at time t is $\mathbf{r}(t)$. If at some time you have nonzero velocity, it is very hard for you to change your direction of motion discontinuously[7]. On the other hand, when $\mathbf{r}'(t) = 0$, you are not moving at all and it is easy for you to turn and leave in any direction you choose. You could reverse direction completely, or make a sharp left turn, or not change direction at all. Here are examples of all of these. They all have $\mathbf{r}'(t) = 0$. They are sketched below.

$$\mathbf{r}_1(t) = (t^5, t^2) \qquad \mathbf{r}_1'(t) = (5t^4, 2t)$$

$$\mathbf{r}_2(t) = \begin{Bmatrix} (t^2, 0) & \text{if } t \geqslant 0 \\ (0, t^2) & \text{if } t \leqslant 0 \end{Bmatrix} \qquad \mathbf{r}_2'(t) = \begin{Bmatrix} (2t, 0) & \text{if } t \geqslant 0 \\ (0, 2t) & \text{if } t \leqslant 0 \end{Bmatrix}$$

$$\mathbf{r}_3(t) = (t^3, 0) \qquad \mathbf{r}_3'(t) = (3t^2, 0)$$



Example 1.1.10

Example 1.1.11 (Corkscrew)

We'll find the arc length of

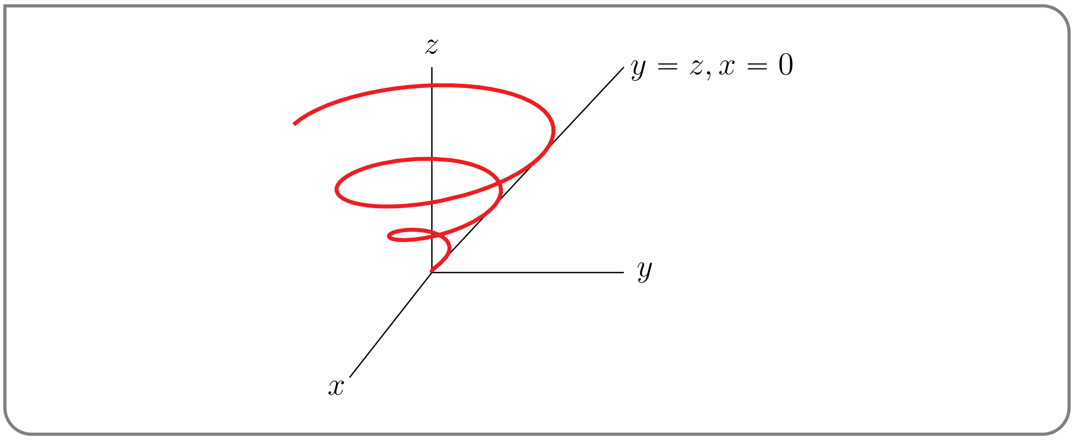
$$\mathbf{r}(t) = t\cos t\,\hat{\boldsymbol{\imath}} + t\sin t\,\hat{\boldsymbol{\jmath}} + t\,\hat{\mathbf{k}} \qquad 0 \leqslant t \leqslant \sqrt{2}$$

7 For your velocity to jump discontinuously, your acceleration has to be infinite, which requires an infinite force. You might not look so healthy afterwards

We'll first sketch it, by observing that

- $x(t) = t\cos t$, $y(t) = t\sin t$ and $z(t) = t$ obey $x(t)^2 + y(t)^2 = t^2 = z(t)^2$. So all points of the curve lie on the cone $x^2 + y^2 = z^2$ and
- as t increases, $\big(x(t), y(t)\big)$ runs counterclockwise around a "circle whose radius increases linearly with t and at the same time $z(t)$ also increases linearly.

Our curve is the "corkscrew"



By Lemma 1.1.4

$$\begin{aligned}
\mathbf{r}(t) &= t\cos t\,\hat{\boldsymbol{\imath}} + t\sin t\,\hat{\boldsymbol{\jmath}} + t\,\hat{\mathbf{k}} \\
\mathbf{r}'(t) &= [\cos t - t\sin t]\hat{\boldsymbol{\imath}} + [\sin t + t\cos t]\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}} \\
\frac{ds}{dt}(t) &= |\mathbf{r}'(t)| \\
&= \sqrt{\big(\cos^2 t - 2t\sin t\cos t + t^2\sin^2 t\big) + \big(\sin^2 t + 2t\sin t\cos t + t^2\cos^2 t\big) + 1} \\
&= \sqrt{2 + t^2}
\end{aligned}$$

Our goal, stated at the beginning of this example, was to compute

$$s(\sqrt{2}) - s(0) = \int_0^{\sqrt{2}} |\mathbf{r}'(t)|\,dt = \int_0^{\sqrt{2}} \sqrt{2 + t^2}\,dt$$

To evaluate the integral, we'll use three techniques that you learned in your first integral calculus course. First, motivated by the $\sqrt{2+t^2}$, we'll use the trigonometric substitution

$$t = \sqrt{2}\tan u \qquad dt = \sqrt{2}\sec^2 u\,du \qquad 2 + t^2 = 2\big[1 + \tan^2 u\big] = 2\sec^2 u$$

When $t = 0$, $u = 0$ and when $t = \sqrt{2}$, $\tan u = 1$ so that $u = \frac{\pi}{4}$ and

$$s(\sqrt{2}) - s(0) = \int_0^{\pi/4} \sqrt{2\sec^2 u}\,\sqrt{2}\sec^2 u\,du = 2\int_0^{\pi/4} \sec^3 u\,du$$

You may have evaluated this integral in first year. There are several ways of doing so. Perhaps the most straight forward, but also most tedious, method is to rewrite the integral as

$$s(\sqrt{2}) - s(0) = 2\int_0^{\pi/4} \frac{\cos u}{\cos^4 u}\,du$$

We recognize that this is a trigonometric integral that contains an odd power of $\cos u$, so we substitute $w = \sin u$, $\mathrm{d}w = \cos u\,\mathrm{d}u$, $\cos^2 u = 1 - w^2$. When $u = 0$, $w = 0$ and when $u = \frac{\pi}{4}$, $w = \frac{1}{\sqrt{2}}$ so that

$$s(\sqrt{2}) - s(0) = 2\int_0^{1/\sqrt{2}} \frac{\mathrm{d}w}{(1-w^2)^2}$$

The integrand is now a rational function, i.e. a ratio of polynomials. So we apply partial fractions.

$$\begin{aligned} s(\sqrt{2}) - s(0) &= 2\int_0^{1/\sqrt{2}} \frac{\mathrm{d}w}{[(1-w)(1+w)]^2} \\ &= \frac{1}{2}\int_0^{1/\sqrt{2}} \Big[\frac{1}{1-w} + \frac{1}{1+w}\Big]^2 \mathrm{d}w \\ &= \frac{1}{2}\int_0^{1/\sqrt{2}} \Big[\frac{1}{(1-w)^2} + \frac{2}{(1-w)(1+w)} + \frac{1}{(1+w)^2}\Big]\mathrm{d}w \\ &= \frac{1}{2}\int_0^{1/\sqrt{2}} \Big[\frac{1}{(1-w)^2} + \frac{1}{1-w} + \frac{1}{1+w} + \frac{1}{(1+w)^2}\Big]\mathrm{d}w \\ &= \frac{1}{2}\Big[\frac{1}{1-w} - \ln|1-w| + \ln|1+w| - \frac{1}{1+w}\Big]_0^{1/\sqrt{2}} \\ &= \frac{1}{2}\Big[\frac{2w}{1-w^2} + \ln\frac{1+w}{1-w}\Big]_0^{1/\sqrt{2}} = \frac{1}{2}\Big[2\sqrt{2} + \ln\frac{\sqrt{2}+1}{\sqrt{2}-1}\Big] \approx 2.2956 \end{aligned}$$

Ooof!

Example 1.1.11

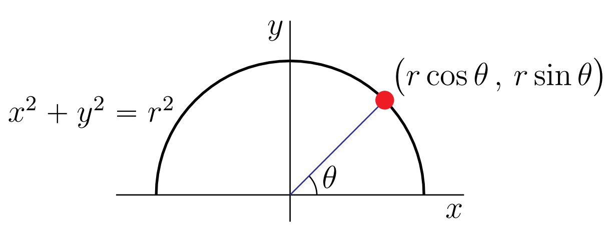
1.2 Reparametrization

There are invariably many ways to parametrize a given curve. Kind of trivially, one can always replace t by, for example, $3u$. But there are also more substantial ways to reparametrize curves. It often pays to tailor the parametrization used to the application of interest. For example, we shall see in the next couple of sections that many curve formulae simplify a lot when arc length is used as the parameter.

Example 1.2.1

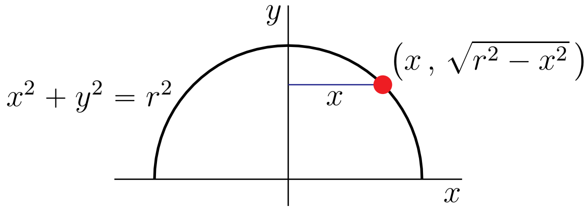
Here are three different parametrizations of the semi-circle $x^2 + y^2 = r^2$, $y \geqslant 0$.

- The first uses the polar angle θ as the parameter. We have already seen, in Example 1.0.1, the parametrization



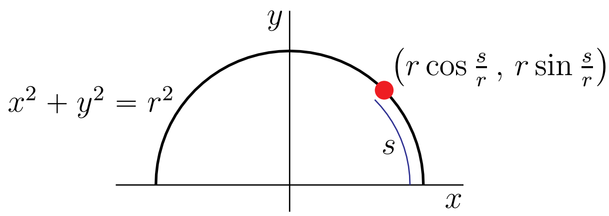
$$\mathbf{r}_1(\theta) = \big(r\cos\theta\,,\ r\sin\theta\big) \qquad 0 \leqslant \theta \leqslant \pi$$

- The second uses x as the parameter. Just solving $x^2 + y^2 = r^2$, $y \geqslant 0$ for y as a function of x, gives $y(x) = \sqrt{r^2 - x^2}$ and so gives the parametrization



$$\mathbf{r}_2(x) = \left(x\,,\ \sqrt{r^2 - x^2}\,\right) \qquad -r \leqslant x \leqslant r$$

- The third uses arc length from $(r, 0)$ as the parameter. We have seen, in Example 1.1.6, that the arc length from $(r, 0)$ to $\mathbf{r}_1(\theta)$ is just $s = r\theta$. So the point on the semi-circle that is arc length s away from $(r, 0)$ is



$$\mathbf{r}_3(s) = \mathbf{r}_1\Big(\frac{s}{r}\Big) = \Big(r\cos\frac{s}{r}\,,\ r\sin\frac{s}{r}\Big)$$

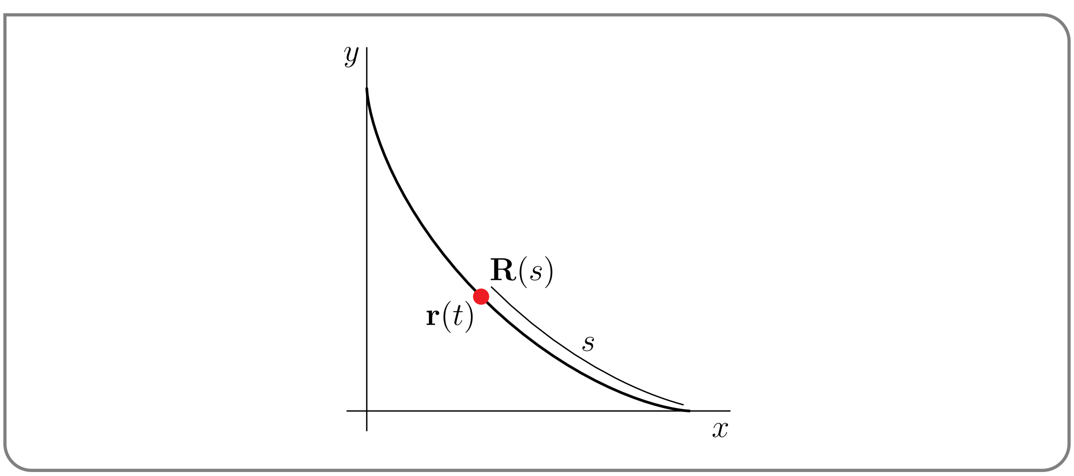
with $0 \leqslant s \leqslant \pi r$.

Example 1.2.1

We shall see that, for some purposes, it is convenient to use parametrization by arc length. Here is a messier example in which we reparametrize a curve so as to use the arc length as the parameter.

Example 1.2.2

We saw in Example 1.1.9, that, as t runs from 0 to $\frac{\pi}{2}$, $\mathbf{r}(t) = a\cos^3 t\,\hat{\boldsymbol{\imath}} + a\sin^3 t\,\hat{\boldsymbol{\jmath}}$ runs from $(a, 0)$ to $(0, a)$ along the astroid $x^{2/3} + y^{2/3} = a^{2/3}$. Suppose that we want a new parametrization $\mathbf{R}(s)$ chosen so that, as s runs from 0 to some appropriate value, $\mathbf{R}(s)$ runs from $(a, 0)$ to $(0, a)$ along $x^{2/3} + y^{2/3} = a^{2/3}$, with s being the arc length from $(a, 0)$ to $\mathbf{R}(s)$ along $x^{2/3} + y^{2/3} = a^{2/3}$.



We saw, in Example 1.1.9, that, for $0 \leqslant t \leqslant \frac{\pi}{2}$, $\frac{ds}{dt} = \frac{3a}{2}\sin(2t)$ so that the arclength from $(a,0) = \mathbf{r}(0)$ to $\mathbf{r}(t)$ is

$$s(t) = \int_0^t \frac{3a}{2}\sin(2t')\,dt' = \frac{3a}{4}\big[1-\cos(2t)\big]$$

which runs from 0, at $t = 0$, to $\frac{3a}{2}$, at $t = \frac{\pi}{2}$. This is relatively clean and we can invert $s(t)$ to find t as a function of s. The value, $T(s)$, of t that corresponds to any given $0 \leqslant s \leqslant \frac{3a}{2}$ is determined by

$$s = \frac{3a}{4}\big[1-\cos\big(2T(s)\big)\big] \qquad \Longleftrightarrow \qquad T(s) = \frac{1}{2}\arccos\Big(1-\frac{4s}{3a}\Big)$$

and

$$\mathbf{R}(s) = \mathbf{r}\big(T(s)\big) = a\cos^3\big(T(s)\big)\hat{\imath} + a\sin^3\big(T(s)\big)\hat{\jmath}$$

We can simplify $\cos^3\big(T(s)\big)$ and $\sin^3\big(T(s)\big)$ by just using trig identities to convert the $\cos\big(2T(s)\big)$ in $s = \frac{3a}{4}\big[1-\cos\big(2T(s)\big)\big]$ into $\cos\big(T(s)\big)$'s and $\sin\big(T(s)\big)$'s.

$$s = \frac{3a}{4}\big[1-\cos\big(2T(s)\big)\big] = \frac{3a}{4}\big[1-\big\{2\cos^2\big(T(s)\big)-1\big\}\big] \iff \cos^2\big(T(s)\big) = 1-\frac{2s}{3a}$$
$$s = \frac{3a}{4}\big[1-\cos\big(2T(s)\big)\big] = \frac{3a}{4}\big[1-\big\{1-2\sin^2\big(T(s)\big)\big\}\big] \iff \sin^2\big(T(s)\big) = \frac{2s}{3a}$$

Consequently the desired parametrization is

$$\mathbf{R}(s) = a\left[1-\frac{2s}{3a}\right]^{3/2}\hat{\imath} + a\left[\frac{2s}{3a}\right]^{3/2}\hat{\jmath} \qquad 0 \leqslant s \leqslant \frac{3a}{2}$$

which is remarkably simple.

Example 1.2.2

1.3▲ Curvature

So far, when we have wanted to approximate a complicated curve by a simple curve near some point, we drew the tangent line to the curve at the point. That's pretty crude. In particular tangent lines are straight — they don't curve. We will get a much better idea of what the complicated curve looks like if we approximate it, locally, by a very simple "curvy curve" rather than by a straight line. Probably the simplest "curvy curve" is a circle[8] and that's what we'll use.

8 Circles are good for studying "curvature", because, unlike parabolas for example, the rate at which a circle curves is uniform over the entire circle.

Definition 1.3.1.

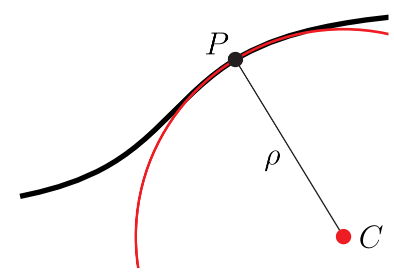
(a) The circle which best approximates a given curve near a given point is called the *circle of curvature* or the *osculating circle*[9] at the point.

(b) The radius of the circle of curvature is called the *radius of curvature* at the point and is normally denoted ρ.

(c) The *curvature* at the point is $\kappa = {}^1/_\rho$.

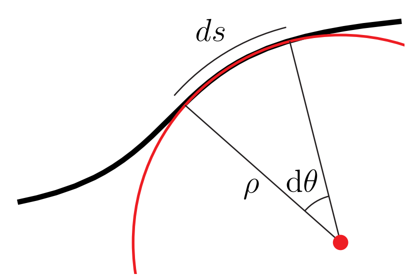
(d) The centre of the circle of curvature is called *centre of curvature* at the point.

These definitions are illustrated in the figure below. It shows (part of) the osculating circle at the point P. The point C is the centre of curvature.



Note that when the curvature κ is large, the radius of curvature ρ is small and we have a very curvy curve. On the other hand when the curvature κ is small, the radius of curvature ρ is large and our curve is almost straight. In particular, straight lines have curvature exactly zero.

We are now going to determine how to find the circle of curvature, starting by figuring out what its radius should be. We'll first look at curves[10] that lie in the xy-plane and then move on to curves in 3d. Consider the black curve in the figure below.



That figure also contains a (portion of a) red circle that fits the curve really well between the two radial lines that are (a very small) angle $d\theta$ apart. So the arclength ds of the part

9 "Osculare" is the Latin verb "to kiss". The German mathematician Gottfried Wilhelm (von) Leibniz (1646–1716) named the circle the "circulus osculans".

10 We'll also assume that the curves of interest are smooth, with no cusps for example, and not straight, so that the radius of curvature $0 < \rho < \infty$.

of the black curve between the two radial lines, should be (essentially) the same as the arc length of the circle between the two radial lines, which is $\rho\,|\mathrm{d}\theta|$, where ρ is the radius of the circle. (We put in absolute values to take into account the possibility that $\mathrm{d}\theta$ could be negative.) Thus $\mathrm{d}s = \rho\,|\mathrm{d}\theta|$. When $\mathrm{d}\theta$ is a macroscopic angle, this is of course an approximation. But in the limit as $\mathrm{d}\theta \to 0$, we should end up with

$$\rho = \left|\frac{\mathrm{d}s}{\mathrm{d}\theta}\right|$$

We now have a formula for the radius of curvature, but not in a very convenient form, because to evaluate it we would need to know the arc length along the curve as a function of the angle θ in the rightmost figure below. We'll now spend some time developing more convenient formulae for ρ. First consider the three figures below. They all show the same curve as in the last figure. The leftmost figure just shows

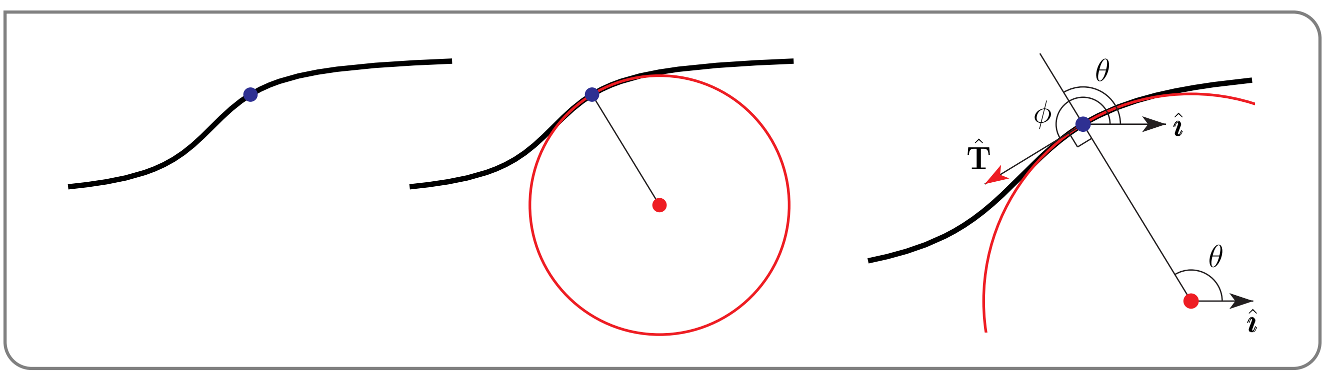
- the curve of interest, which is the black curve, and
- the (blue) point of interest on the black curve. We want to find the curvature at that point.

The middle figure shows the same curve and point of interest and also shows

- the red circle of curvature (i.e. best fitting circle) for the black curve at the blue dot.
- The red dot is the centre of curvature.

The rightmost figure shows the same black curve, blue point of interest and red circle of curvature (at least part of it) somewhat enlarged.

- The angle θ is the angle between $\hat{\boldsymbol{\imath}}$ and the radius vector from the red dot (the centre of curvature) to the blue dot (the point of interest).
- $\hat{\mathbf{T}}$ is the tangent vector to the black curve at the blue dot.
- The angle ϕ is the angle between $\hat{\boldsymbol{\imath}}$ and $\hat{\mathbf{T}}$. The vector $\hat{\mathbf{T}}$ is also tangent to the red circle. As the tangent and radius vectors for circles are perpendicular to each other[11], we have that $\phi = \theta + \frac{\pi}{2}$ and hence $\rho = \left|\frac{\mathrm{d}s}{\mathrm{d}\phi}\right|$ too.



We are now in a position to develop a bunch of formulae for the radius of curvature ρ and the curvature $\kappa = \frac{1}{\rho}$, that are more convenient than $\kappa = \left|\frac{\mathrm{d}s}{\mathrm{d}\phi}\right|^{-1}$. These formulae will use the

11 We saw that in Example 1.1.6.

Notation 1.3.2.

If $\mathbf{r}(t)$ is a parametrized curve, then

- $\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}(t)$ is the velocity vector at $\mathbf{r}(t)$
- $\mathbf{a}(t) = \frac{d^2\mathbf{r}}{dt^2}(t)$ is the acceleration vector at $\mathbf{r}(t)$
- $\hat{\mathbf{T}}(t)$ is the unit tangent vector to the curve at $\mathbf{r}(t)$ that points in the direction of increasing t.
- $\hat{\mathbf{N}}(t)$ is the unit normal vector to the curve at $\mathbf{r}(t)$ that points toward the centre of curvature.
- $\kappa(t)$ is the curvature at $\mathbf{r}(t)$
- $\rho(t)$ is the radius of curvature at $\mathbf{r}(t)$

Theorem 1.3.3.

(a) Given[12] $s(\phi)$, i.e. if we know the arc length along the curve as a function of the angle[13] $\phi = \measuredangle(\hat{\imath}, \hat{\mathbf{T}})$, then

$$\rho = \left|\frac{ds}{d\phi}\right| \qquad \kappa = \left|\frac{ds}{d\phi}\right|^{-1} \qquad \kappa = \left|\frac{d\phi}{ds}\right|$$

(b) Given $\mathbf{r}(s)$, i.e. if we have a parametrization of the curve in terms of arc length, then

$$\frac{d\hat{\mathbf{T}}}{ds}(s) = \kappa(s)\,\hat{\mathbf{N}}(s)$$

where $\hat{\mathbf{N}}(s)$ is the unit normal vector to the curve at $\mathbf{r}(s)$ that points toward the centre of curvature.

(c) Given $\mathbf{r}(t)$, i.e. if we have a general parametrized curve, then

$$\frac{d\hat{\mathbf{T}}}{dt} = \kappa\frac{ds}{dt}\hat{\mathbf{N}} \qquad \mathbf{v}(t) = \frac{ds}{dt}(t)\,\hat{\mathbf{T}}(t) \qquad \mathbf{a}(t) = \frac{d^2s}{dt^2}\hat{\mathbf{T}} + \kappa\left(\frac{ds}{dt}\right)^2\hat{\mathbf{N}}$$

(d) Given $\big(x(t)\,,\,y(t)\big)$, (for curves in the xy-plane)

$$\kappa = \left|\frac{\mathbf{v}(t)\times\mathbf{a}(t)}{\left(\frac{ds}{dt}\right)^3}\right| = \frac{\left|\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}\right|}{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{3/2}}$$

12 The equation $s = s(\phi)$ is called the "intrinsic equation of the curve".

Theorem 1.3.3 (continued).

(e) Given $y(x)$, (again for curves in the xy-plane)

$$\kappa = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}$$

Proof. (a) Given $s(\phi)$, then

$$\rho = \left|\frac{ds}{d\phi}\right| \qquad \kappa = \left|\frac{ds}{d\phi}\right|^{-1}$$

As we are assuming that $0 < \rho = \left|\frac{ds}{d\phi}\right| < \infty$, the inverse function theorem says that we can invert the function $s(\phi)$ (at least locally) to get ϕ as a function of s, and that

$$\kappa = \left|\frac{d\phi}{ds}\right|$$

(b) Given $\mathbf{r}(s)$, then, by Lemma 1.1.4.c, $\hat{\mathbf{T}}(s) = \mathbf{r}'(s)$ is a unit tangent to the curve at $\mathbf{r}(s)$ and

$$\frac{d\hat{\mathbf{T}}}{ds} = \frac{d\hat{\mathbf{T}}}{d\phi}\frac{d\phi}{ds} \tag{$*$}$$

Now up to a sign $\frac{d\phi}{ds}$ is κ, and just because $\phi = \measuredangle(\hat{\imath}, \hat{\mathbf{T}})$, with $\hat{\mathbf{T}}$ a unit vector,

$$\begin{aligned} \hat{\mathbf{T}} &= \cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath} \\ \implies \frac{d\hat{\mathbf{T}}}{d\phi} &= -\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath} \end{aligned} \tag{$**$}$$

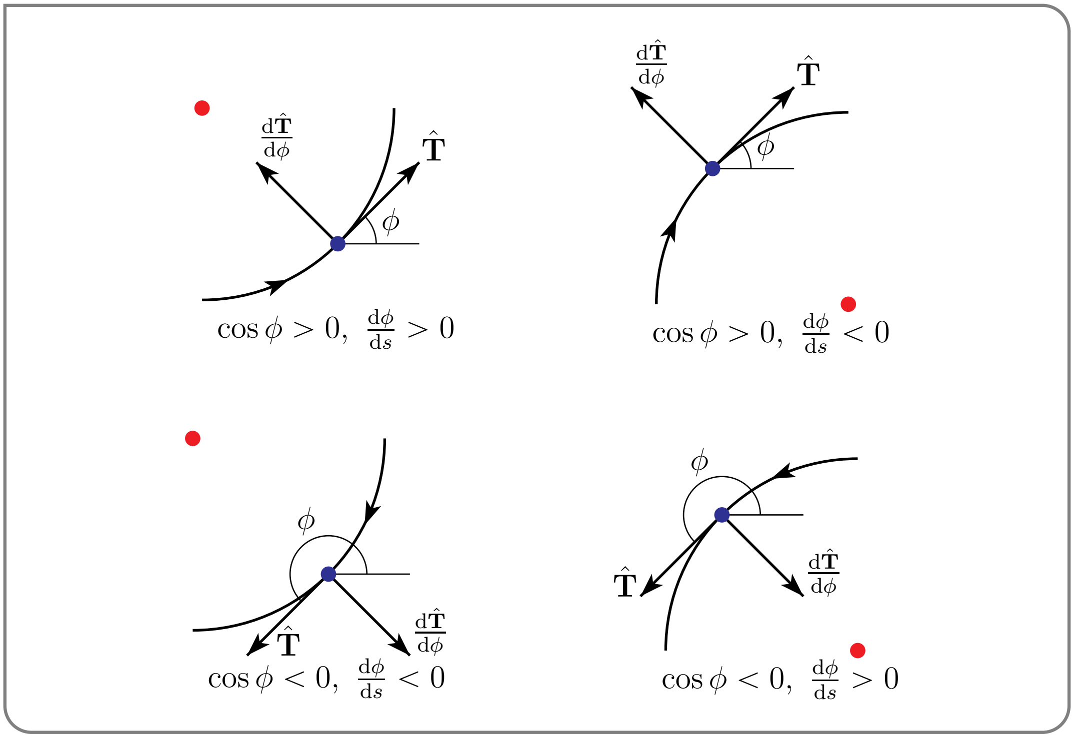
So $\frac{d\hat{\mathbf{T}}}{d\phi}$ is a unit vector that is perpendicular[14] to $\hat{\mathbf{T}}$, and hence to the curve at $\mathbf{r}(s)$, and

$$\frac{d\hat{\mathbf{T}}}{ds}(s) = \kappa(s)\,\hat{\mathbf{N}}(s) \tag{$\dagger$}$$

with $\hat{\mathbf{N}}(s)$ a unit normal vector to the curve at $\mathbf{r}(s)$. In fact, $\hat{\mathbf{N}}(s)$ is the unit normal vector to the curve at $\mathbf{r}(s)$ that points toward the centre of curvature.

To see that, look at the figures below[15], and note that substituting the sign information from each figure into ($*$) gives ($\dagger$). For example, consider the figure on the lower left. In

13 The notation $\measuredangle(\hat{\imath}, \hat{\mathbf{T}})$ means "the angle between $\hat{\imath}$ and $\hat{\mathbf{T}}$".
14 Think about why this should be the case. In particular, sketch $\hat{\mathbf{T}}$ and ϕ and think about what the sketch says about $\frac{d\hat{\mathbf{T}}}{d\phi}$.
15 In each of the four figures, the arrow on the curve specifies the direction of increasing arc length s and the red dot is the centre of curvature for the curve at the blue dot.



that figure,

- the x component of $\hat{\mathbf{T}}$ is negative ($\hat{\mathbf{T}}$ is leftward pointing in the figure),
  - which makes $\cos\phi$ negative (see ($**$)),
  - which makes the y component of $\frac{d\hat{\mathbf{T}}}{d\phi}$ negative (see ($**$) again),
  - so $\frac{d\hat{\mathbf{T}}}{d\phi}$ is downward pointing,

  so $\frac{d\hat{\mathbf{T}}}{d\phi} = -\hat{\mathbf{N}}$ (the centre of curvature is the red dot above the curve) and
- as s increases (i.e. as you move in the direction of the arrow on the curve), ϕ decreases (on the far right hand part of the curve $\phi \approx \frac{3\pi}{2}$, while on the far left hand part of the curve $\phi \approx \pi$), so $\frac{d\phi}{ds} < 0$ and $\kappa = \left|\frac{d\phi}{ds}\right| = -\frac{d\phi}{ds}$.
- So by ($*$), $\frac{d\hat{\mathbf{T}}}{ds} = \frac{d\hat{\mathbf{T}}}{d\phi}\frac{d\phi}{ds} = \big(-\hat{\mathbf{N}}\big)(-\kappa) = \kappa\hat{\mathbf{N}}$.

In each of the three other figures we also end up with $\frac{d\hat{\mathbf{T}}}{ds} = \kappa(s)\hat{\mathbf{N}}(s)$.
Note that if $\kappa(s) = 0$, then $\hat{\mathbf{N}}(s)$ is not defined. This makes sense: if the curve is (locally) a straight line, there is no "best fitting circle".

(c) Given $\mathbf{r}(t)$, i.e. if we have a general parametrized curve, we can determine a unit tangent vector by using Lemma 1.1.4:

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}(t) = \frac{ds}{dt}(t)\,\hat{\mathbf{T}}(t) \qquad \Longrightarrow \qquad \hat{\mathbf{T}}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

Then we can determine κ and $\hat{\mathbf{N}}$ by differentiating $\hat{\mathbf{T}}(t)$ and using the chain rule:

$$\frac{d\hat{\mathbf{T}}}{dt} = \frac{d\hat{\mathbf{T}}}{ds}\frac{ds}{dt} = \kappa\frac{ds}{dt}\hat{\mathbf{N}} \qquad \Longrightarrow \qquad \kappa(t) = \frac{|\hat{\mathbf{T}}'(t)|}{|\mathbf{r}'(t)|}$$

Also, if we differentiate $\mathbf{v}(t) = \frac{\mathrm{d}s}{\mathrm{d}t}\hat{\mathbf{T}}(t)$, we get that the acceleration

$$\mathbf{a}(t) = \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\hat{\mathbf{T}} + \frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}t} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\hat{\mathbf{T}} + \kappa\Big(\frac{\mathrm{d}s}{\mathrm{d}t}\Big)^2\hat{\mathbf{N}}$$

(d) Given $\big(x(t)\,,\,y(t)\big)$, (for curves in the xy-plane), we can read off the curvature from

$$\begin{aligned}\mathbf{v}(t)\times\mathbf{a}(t) &= \Big(\frac{\mathrm{d}s}{\mathrm{d}t}(t)\,\hat{\mathbf{T}}(t)\Big)\times\Big(\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\hat{\mathbf{T}} + \kappa\Big(\frac{\mathrm{d}s}{\mathrm{d}t}\Big)^2\hat{\mathbf{N}}\Big)\\ &= \kappa\Big(\frac{\mathrm{d}s}{\mathrm{d}t}\Big)^3\hat{\mathbf{T}}\times\hat{\mathbf{N}} \qquad (\text{since } \hat{\mathbf{T}}\times\hat{\mathbf{T}} = \mathbf{0})\end{aligned}$$

Think of $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$ as 3d vectors that whose z-components happen to be zero. As $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$ are mutually perpendicular unit vectors in the xy-plane, the cross-product $\hat{\mathbf{T}}\times\hat{\mathbf{N}}$ will be either $+\hat{\mathbf{k}}$ or $-\hat{\mathbf{k}}$. In both cases, $|\mathbf{v}(t)\times\mathbf{a}(t)| = \kappa\big|\frac{\mathrm{d}s}{\mathrm{d}t}\big|^3$. So

$$\begin{aligned}\kappa &= \left|\frac{\mathbf{v}(t)\times\mathbf{a}(t)}{\big(\frac{\mathrm{d}s}{\mathrm{d}t}\big)^3}\right| = \left|\frac{\big[\frac{\mathrm{d}x}{\mathrm{d}t}\hat{\imath} + \frac{\mathrm{d}y}{\mathrm{d}t}\hat{\jmath}\big]\times\big[\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\hat{\imath} + \frac{\mathrm{d}^2y}{\mathrm{d}t^2}\hat{\jmath}\big]}{\big(\frac{\mathrm{d}s}{\mathrm{d}t}\big)^3}\right| = \left|\frac{\big[\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\big]\hat{\mathbf{k}}}{\big(\frac{\mathrm{d}s}{\mathrm{d}t}\big)^3}\right|\\ &= \frac{\big|\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\big|}{\big[\big(\frac{\mathrm{d}x}{\mathrm{d}t}\big)^2 + \big(\frac{\mathrm{d}y}{\mathrm{d}t}\big)^2\big]^{3/2}}\end{aligned}$$

(e) Given $y(x)$, again for curves in the xy-plane, we can parametrize the curve using x as the parameter:

$$\mathbf{r}(t) = \big(X(t)\,,\,Y(t)\big) \qquad \text{with } X(t) = t \text{ and } Y(t) = y(t)$$

Then

$$\frac{\mathrm{d}X}{\mathrm{d}t} = 1 \qquad \frac{\mathrm{d}^2X}{\mathrm{d}t^2} = 0 \qquad \frac{\mathrm{d}Y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x} \qquad \frac{\mathrm{d}^2Y}{\mathrm{d}t^2} = \frac{\mathrm{d}^2y}{\mathrm{d}x^2}$$

and

$$\kappa = \frac{\big|\frac{\mathrm{d}X}{\mathrm{d}t}\frac{\mathrm{d}^2Y}{\mathrm{d}t^2} - \frac{\mathrm{d}Y}{\mathrm{d}t}\frac{\mathrm{d}^2X}{\mathrm{d}t^2}\big|}{\big[\big(\frac{\mathrm{d}X}{\mathrm{d}t}\big)^2 + \big(\frac{\mathrm{d}Y}{\mathrm{d}t}\big)^2\big]^{3/2}} = \frac{\big|\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\big|}{\big[1 + \big(\frac{\mathrm{d}y}{\mathrm{d}x}\big)^2\big]^{3/2}}$$

□

Take another look at Theorem 1.3.3.c and note that

- the tangential component of acceleration, i.e. $\frac{\mathrm{d}^2s}{\mathrm{d}t^2}$, arises purely from change in speed while
- the normal component of acceleration, i.e. $\kappa\big(\frac{\mathrm{d}s}{\mathrm{d}t}\big)^2$, arises from curvature and is proportional to the square of the speed $\frac{\mathrm{d}s}{\mathrm{d}t}$. Think about what you feel when you are driving. That's why velodromes and (car) race tracks often have banked corners.

Example 1.3.4

As a warm up example, and also a check that our formulae make sense, we'll find the curvature κ, radius of curvature, ρ, unit tangent vector, $\hat{\mathbf{T}}$, unit normal vector, $\hat{\mathbf{N}}$, and centre of curvature of the parametrized curve

$$\mathbf{r}(t) = a\cos t\,\hat{\imath} + a\sin t\,\hat{\jmath}$$

with the constant $a > 0$. This is, of course, the circle of radius a centred on the origin. As

$$\mathbf{v}(t) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t) = -a\sin t\,\hat{\imath} + a\cos t\,\hat{\jmath} \implies \frac{\mathrm{d}s}{\mathrm{d}t}(t) = |\mathbf{v}(t)| = a$$

we have that the unit tangent vector

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = -\sin t\,\hat{\imath} + \cos t\,\hat{\jmath}$$

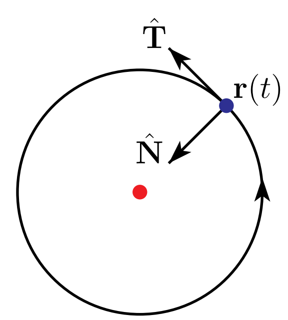
Note, as a check, that this is indeed a vector of length one and is perpendicular to the radius vector (as expected — the curve is a circle). As

$$\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}t}(t) = -\cos t\,\hat{\imath} - \sin t\,\hat{\jmath}$$

we have that

$$\hat{\mathbf{N}}(t) = \frac{\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}t}(t)}{\left|\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}t}(t)\right|} = -\cos t\,\hat{\imath} - \sin t\,\hat{\jmath} \quad \kappa(t) = \frac{\left|\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}t}(t)\right|}{\frac{\mathrm{d}s}{\mathrm{d}t}(t)} = \frac{1}{a} \quad \rho(t) = \frac{1}{\kappa(t)} = a$$

Now look at the figure.



To get to the centre of curvature we should start from $\mathbf{r}(t)$ and walk a distance $\rho(t)$, which after all is the radius of curvature, in the direction $\hat{\mathbf{N}}(T)$, which is pointing towards the centre of curvature. So the centre of curvature is

$$\mathbf{r}(t) + \rho(t)\hat{\mathbf{N}}(t) = \left[a\cos t\,\hat{\imath} + a\sin t\,\hat{\jmath}\right] + a\left[-\cos t\,\hat{\imath} - \sin t\,\hat{\jmath}\right] = \mathbf{0}$$

This makes perfectly good sense — the radius of curvature is the radius of the original circle and the centre of curvature is the centre of the original circle.

One alternative calculation of the curvature, using $x(t) = a\cos t$, $y(t) = a\sin t$, is

$$\begin{aligned}\kappa(t) &= \frac{\left|\frac{dx}{dt}(t)\frac{d^2y}{dt^2}(t) - \frac{dy}{dt}(t)\frac{d^2x}{dt^2}(t)\right|}{\left[\left(\frac{dx}{dt}(t)\right)^2 + \left(\frac{dy}{dt}(t)\right)^2\right]^{3/2}} \\ &= \frac{\left|-a\sin t\big(-a\sin t\big) - a\cos t\big(-a\cos t\big)\right|}{\left[\big(-a\sin t\big)^2 + \big(a\cos t\big)^2\right]^{3/2}} \\ &= \frac{1}{a}\end{aligned}$$

Another alternative calculation of the curvature, using $y(x) = \sqrt{a^2 - x^2}$ (for the part of the circle with $y > 0$),

$$y'(x) = -\frac{x}{\sqrt{a^2 - x^2}} = -\frac{x}{y(x)} \qquad y''(x) = -\frac{y(x) - xy'(x)}{y(x)^2} = -\frac{y(x)^2 + x^2}{y(x)^3} = -\frac{a^2}{y(x)^3}$$

is

$$\kappa(x) = \frac{\left|\frac{d^2y}{dx^2}(x)\right|}{\left[1 + \left(\frac{dy}{dx}(x)\right)^2\right]^{3/2}} = \frac{\frac{a^2}{y(x)^3}}{\left[1 + \frac{x^2}{y(x)^2}\right]^{3/2}} = \frac{a^2}{\left[y(x)^2 + x^2\right]^{3/2}} = \frac{1}{a}$$

Example 1.3.4

Example 1.3.5

As a more computationally involved example, we'll analyze

$$\begin{aligned}\mathbf{r}(t) &= \big(\cos t + t\sin t\big)\hat{\boldsymbol{\imath}} + \big(\sin t - t\cos t\big)\hat{\boldsymbol{\jmath}} \qquad t > 0 \\ \mathbf{v}(t) &= t\cos t\,\hat{\boldsymbol{\imath}} + t\sin t\,\hat{\boldsymbol{\jmath}} \\ \mathbf{a}(t) &= \big(\cos t - t\sin t\big)\hat{\boldsymbol{\imath}} + \big(\sin t + t\cos t\big)\hat{\boldsymbol{\jmath}}\end{aligned}$$

We can read off from $\mathbf{v}(t)$, that

$$\begin{aligned}\frac{ds}{dt}(t) &= |\mathbf{v}(t)| = t \\ \frac{d^2s}{dt^2}(t) &= 1 \\ \mathbf{T}(t) &= \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \cos t\,\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}}\end{aligned}$$

Next, from $\mathbf{a}(t)$, we read off that

$$\begin{aligned}\mathbf{a}(t) &= \big(\cos t - t\sin t\big)\hat{\boldsymbol{\imath}} + \big(\sin t + t\cos t\big)\hat{\boldsymbol{\jmath}} \qquad \text{and} \\ \mathbf{a}(t) &= \frac{d^2s}{dt^2}(t)\,\hat{\mathbf{T}}(t) + \kappa(t)\left(\frac{ds}{dt}(t)\right)^2\hat{\mathbf{N}}(t) \qquad \text{(by Theorem 1.3.3.c)} \\ &= \cos t\,\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}} + t^2\kappa(t)\hat{\mathbf{N}}(t) \\ \implies t^2\kappa(t)\hat{\mathbf{N}}(t) &= -t\sin t\,\hat{\boldsymbol{\imath}} + t\cos t\,\hat{\boldsymbol{\jmath}}\end{aligned}$$

so that $t^2\kappa(t)$ is the length of $-t\sin t\,\hat{\imath}+t\cos t\,\hat{\jmath}$, which is t. Thus

$$\kappa(t)=\frac{1}{t}\quad\text{and}\quad\hat{\mathbf{N}}(t)=\frac{-t\sin t\,\hat{\imath}+t\cos t\,\hat{\jmath}}{t^2\kappa(t)}=-\sin t\,\hat{\imath}+\cos t\,\hat{\jmath}$$

As an alternative calculation of the curvature, we have

$$\begin{aligned}\kappa(t)&=\frac{|\mathbf{v}(t)\times\mathbf{a}(t)|}{(\frac{ds}{dt}(t))^3}\\&=\frac{\big|[t\cos t\,\hat{\imath}+t\sin t\,\hat{\jmath}]\times\big[(\cos t-t\sin t)\hat{\imath}+(\sin t+t\cos t)\hat{\jmath}\big]\big|}{(\frac{ds}{dt}(t))^3}\\&=\frac{\big|[t\cos t(\sin t+t\cos t)-t\sin t(\cos t-t\sin t)]\hat{\mathbf{k}}\big|}{(\frac{ds}{dt}(t))^3}\\&=\frac{|t^2\hat{\mathbf{k}}|}{t^3}=\frac{1}{t}\end{aligned}$$

It pays to think before you calculate!

Example 1.3.5

1.4▲ Curves in Three Dimensions

So far, we have developed formulae for the curvature, unit tangent vector, etc., at a point $\mathbf{r}(t)$ on a curve that lies in the xy-plane. We now extend our discussion to curves in $\mathbb{R}^3$. Fix any t. For t' very close to t, $\mathbf{r}(t')$, will, by the Taylor expansion to second order, be very close to $\mathbf{r}(t)+\mathbf{r}'(t)\,(t'-t)+\frac{1}{2}\mathbf{r}''(t)\,(t'-t)^2$, so that $\mathbf{r}(t')$ almost lies in the plane through $\mathbf{r}(t)$ that is determined by the two vectors $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$. Thus, if we restrict our attention to a very small part of the curve near the point of interest $\mathbf{r}(t)$, the curve will, to a very good approximation lie in some plane. So we can still define, for example, the osculating circle to the curve at $\mathbf{r}(t)$ to be the circle in that plane that fits the curve best near $\mathbf{r}(t)$. And we still have the formulae[16]

$$\begin{aligned}\mathbf{v}&=\frac{d\mathbf{r}}{dt}=\frac{ds}{dt}\hat{\mathbf{T}}\\\frac{d\hat{\mathbf{T}}}{ds}&=\kappa\hat{\mathbf{N}}\\\frac{d\hat{\mathbf{T}}}{dt}&=\kappa\frac{ds}{dt}\hat{\mathbf{N}}\\\mathbf{a}&=\frac{d^2\mathbf{r}}{dt^2}=\frac{d^2s}{dt^2}\hat{\mathbf{T}}+\kappa\Big(\frac{ds}{dt}\Big)^2\hat{\mathbf{N}}\\\mathbf{v}\times\mathbf{a}&=\kappa\Big(\frac{ds}{dt}\Big)^3\hat{\mathbf{T}}\times\hat{\mathbf{N}}\end{aligned}$$

16 The arguments in the proof of Theorem 1.3.3 that we used to verify these formulae work in any plane, not just the xy-plane. Just choose $\hat{\imath}$ and $\hat{\jmath}$ to be any two mutually perpendicular unit vectors in the plane.

The only[17] difference is that $\mathbf{v}$, $\mathbf{a}$, $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$ are now three component vectors rather than two component vectors.

If we are lucky and our curve happens to lie completely in a single plane, the vectors $\hat{\mathbf{T}}(s)$ and $\hat{\mathbf{N}}(s)$ are mutually perpendicular unit vectors that lie in the same plane, so that their cross product $\hat{\mathbf{B}}(s) = \hat{\mathbf{T}}(s) \times \hat{\mathbf{N}}(s)$ is a unit vector that is perpendicular to the plane. By continuity, $\hat{\mathbf{B}}(s)$ has to be a constant vector, i.e. be independent of s.

If, on the other hand, $\hat{\mathbf{B}}(s)$ is not constant, then our curve doesn't lie in a single plane, and we can use the derivative

$$\begin{aligned}\frac{\mathrm{d}\hat{\mathbf{B}}}{\mathrm{d}s} &= \frac{\mathrm{d}}{\mathrm{d}s}(\hat{\mathbf{T}} \times \hat{\mathbf{N}}) = \frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s} \times \hat{\mathbf{N}} + \hat{\mathbf{T}} \times \frac{\mathrm{d}\hat{\mathbf{N}}}{\mathrm{d}s} \\ &= \hat{\mathbf{T}} \times \frac{\mathrm{d}\hat{\mathbf{N}}}{\mathrm{d}s} \qquad \left(\text{since } \frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s} \text{ is parallel to } \hat{\mathbf{N}}\right)\end{aligned}$$

as a measure

- of how badly the curve fails to lie in a plane,
- i.e. how much the plane that fits the curve best near $\mathbf{r}(s)$ twists as s increases,

The cross product in $\frac{\mathrm{d}\hat{\mathbf{B}}}{\mathrm{d}s} = \hat{\mathbf{T}} \times \frac{\mathrm{d}\hat{\mathbf{N}}}{\mathrm{d}s}$ implies that $\frac{\mathrm{d}\hat{\mathbf{B}}}{\mathrm{d}s}$ is perpendicular to $\hat{\mathbf{T}}$. In addition, $\frac{\mathrm{d}\hat{\mathbf{B}}}{\mathrm{d}s}$ must be perpendicular to $\hat{\mathbf{B}}$ because

$$|\hat{\mathbf{B}}| = 1 \implies 1 = \hat{\mathbf{B}} \cdot \hat{\mathbf{B}} \implies 0 = \frac{\mathrm{d}}{\mathrm{d}s}\left[\hat{\mathbf{B}} \cdot \hat{\mathbf{B}}\right] = 2\hat{\mathbf{B}} \cdot \frac{\mathrm{d}\hat{\mathbf{B}}}{\mathrm{d}s}$$

So $\frac{\mathrm{d}\hat{\mathbf{B}}}{\mathrm{d}s}(s)$ must be parallel to $\hat{\mathbf{N}}(s)$.

Definition 1.4.1.

(a) The *binormal vector* at $\mathbf{r}(s)$ is $\hat{\mathbf{B}}(s) = \hat{\mathbf{T}}(s) \times \hat{\mathbf{N}}(s)$. The normal vector $\hat{\mathbf{N}}(s)$ is sometimes called the unit *principal normal vector* to distinguish it from the binormal vector.

(b) We define the *torsion* $\tau(s)$ by

$$\frac{\mathrm{d}\hat{\mathbf{B}}}{\mathrm{d}s}(s) = -\tau(s)\hat{\mathbf{N}}(s)$$

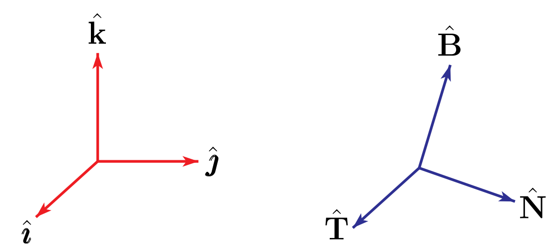
The negative sign is included so that $\tau(s) > 0$ indicates "right handed twisting". There will be an explanation of what this means in Example 1.4.4 below.

(c) The *osculating plane* at $\mathbf{r}(s)$ (the plane that fits the curve best at $\mathbf{r}(s)$) is the plane through $\mathbf{r}(s)$ with normal vector $\hat{\mathbf{B}}(s)$. The equation of the plane is

$$\hat{\mathbf{B}}(s) \cdot \{(x, y, z) - \mathbf{r}(s)\} = 0$$

17 However this can be a significant difference.

For each s, $\hat{\mathbf{T}}(s)$, $\hat{\mathbf{N}}(s)$ and $\hat{\mathbf{B}}(s)$ are mutually perpendicular unit vectors. They form an orthonormal basis for $\mathbb{R}^3$, just as $\hat{\imath}, \hat{\jmath}$ and $\hat{\mathbf{k}}$ form an orthonormal basis for $\mathbb{R}^3$. Furthermore both $(\hat{\mathbf{T}}(s), \hat{\mathbf{N}}(s), \hat{\mathbf{B}}(s))$ and $(\hat{\imath}, \hat{\jmath}, \hat{\mathbf{k}})$ are "right handed triples[18]", meaning that $\hat{\mathbf{B}}(s) = \hat{\mathbf{T}}(s) \times \hat{\mathbf{N}}(s)$ and $\hat{\mathbf{k}} = \hat{\imath} \times \hat{\jmath}$.



We have already computed $\frac{d\hat{\mathbf{T}}}{ds}$ and $\frac{d\hat{\mathbf{B}}}{ds}$. It is now an easy matter to compute

$$\begin{aligned}\frac{d\hat{\mathbf{N}}}{ds} &= \frac{d}{ds}\big(\hat{\mathbf{B}}(s) \times \hat{\mathbf{T}}(s)\big)\\ &= -\tau(s)\hat{\mathbf{N}}(s) \times \hat{\mathbf{T}}(s) + \hat{\mathbf{B}}(s) \times \big(\kappa(s)\hat{\mathbf{N}}(s)\big)\\ &= \tau(s)\hat{\mathbf{B}}(s) - \kappa(s)\hat{\mathbf{T}}(s)\end{aligned}$$

To see that $\hat{\mathbf{N}}(s) \times \hat{\mathbf{T}}(s) = -\hat{\mathbf{B}}(s)$ and $\hat{\mathbf{B}}(s) \times \hat{\mathbf{N}}(s) = -\hat{\mathbf{T}}(s)$, just look at the right hand figure above.

Now suppose that we have a curve that is parametrized by t rather than s. How do we find the torsion τ? The most obvious method is to

- recall that $\mathbf{v} \times \mathbf{a} = \kappa\big(\frac{ds}{dt}\big)^3 \hat{\mathbf{T}} \times \hat{\mathbf{N}} = \kappa\big(\frac{ds}{dt}\big)^3 \hat{\mathbf{B}}$ and that $\hat{\mathbf{B}}(t)$ is a unit vector. So

$$\hat{\mathbf{B}}(t) = \frac{\mathbf{v}(t) \times \mathbf{a}(t)}{|\mathbf{v}(t) \times \mathbf{a}(t)|}$$

- Having found $\mathbf{B}(t)$ we can differentiate it and use $\frac{d\hat{\mathbf{B}}}{ds}(s) = -\tau(s)\hat{\mathbf{N}}(s)$ and the chain rule to give

$$\frac{d\mathbf{B}}{dt} = \frac{d\mathbf{B}}{ds}\frac{ds}{dt} = -\tau\frac{ds}{dt}\hat{\mathbf{N}}$$

  from which we can read off τ, provided we know $\frac{ds}{dt}$ and $\hat{\mathbf{N}}$.

There is another, often more efficient, method to find the torsion τ that uses

$$\begin{aligned}\frac{d\mathbf{a}}{dt} &= \frac{d}{dt}\Big(\frac{d^2s}{dt^2}\hat{\mathbf{T}} + \kappa\Big(\frac{ds}{dt}\Big)^2\hat{\mathbf{N}}\Big)\\ &= \frac{d^3s}{dt^3}\hat{\mathbf{T}} + \frac{d^2s}{dt^2}\frac{ds}{dt}\kappa\hat{\mathbf{N}} + \frac{d}{dt}\Big(\kappa\Big(\frac{ds}{dt}\Big)^2\Big)\hat{\mathbf{N}} + \kappa\Big(\frac{ds}{dt}\Big)^3(\tau\hat{\mathbf{B}} - \kappa\hat{\mathbf{T}})\end{aligned}$$

18 We shall stick to "right handed triples" to make it easier to get various signs right.

While this looks a little complicated, notice that, with just one exception, namely $\kappa\left(\frac{ds}{dt}\right)^3\tau(s)\hat{\mathbf{B}}(s)$, every term on the right hand side is either in the direction $\hat{\mathbf{T}}$ or in the direction $\hat{\mathbf{N}}$ and so is perpendicular to $\hat{\mathbf{B}}$. So, dotting with $\mathbf{v}\times\mathbf{a} = \kappa\left(\frac{ds}{dt}\right)^3\hat{\mathbf{B}}$ gives

$$(\mathbf{v}\times\mathbf{a})\cdot\frac{d\mathbf{a}}{dt} = \kappa^2\left(\frac{ds}{dt}\right)^6\tau = |\mathbf{v}\times\mathbf{a}|^2\,\tau$$

and hence

$$\tau = \frac{(\mathbf{v}\times\mathbf{a})\cdot\frac{d\mathbf{a}}{dt}}{|\mathbf{v}\times\mathbf{a}|^2}$$

If the curvature[19] $\kappa(s) > 0$ and the torsion $\tau(s)$ are known, then the system of equations[20]

Equation 1.4.2 (Frenet–Serret Formulae).

$$\begin{aligned}
\frac{d\hat{\mathbf{T}}}{ds}(s) &= \kappa(s)\,\hat{\mathbf{N}}(s)\\
\frac{d\hat{\mathbf{N}}}{ds}(s) &= \tau(s)\,\hat{\mathbf{B}}(s) - \kappa(s)\,\hat{\mathbf{T}}(s)\\
\frac{d\hat{\mathbf{B}}}{ds}(s) &= -\tau(s)\,\hat{\mathbf{N}}(s)
\end{aligned}$$

is a first order linear system of ordinary differential equations

$$\frac{d}{ds}\begin{bmatrix}\hat{\mathbf{T}}(s)\\ \hat{\mathbf{N}}(s)\\ \hat{\mathbf{B}}(s)\end{bmatrix} = \begin{bmatrix}0 & \kappa(s) & 0\\ -\kappa(s) & 0 & \tau(s)\\ 0 & -\tau(s) & 0\end{bmatrix}\begin{bmatrix}\hat{\mathbf{T}}(s)\\ \hat{\mathbf{N}}(s)\\ \hat{\mathbf{B}}(s)\end{bmatrix}$$

for the 9 component vector valued function $(\hat{\mathbf{T}}(s), \hat{\mathbf{N}}(s), \hat{\mathbf{B}}(s))$.

Any first order linear initial value problem

$$\frac{d}{ds}\mathbf{x}(s) = M(s)\mathbf{x}(s) \qquad \mathbf{x}(0) = \mathbf{x}_0$$

where $\mathbf{x}$ is an n-component vector and $M(s)$ is an $n\times n$ matrix with continuous entries, has exactly one solution. If $n = 1$, so that $\mathbf{x}(s)$ and $M(s)$ are just functions, this is easy to see. Just let $\mathcal{M}(s)$ be the antiderivative of $M(s)$ that obeys $\mathcal{M}(0) = 0$. Then

$$\begin{aligned}
\frac{d}{ds}\mathbf{x}(s) = M(s)\mathbf{x}(s) &\iff e^{-\mathcal{M}(s)}\frac{d}{ds}\mathbf{x}(s) - M(s)e^{-\mathcal{M}(s)}\mathbf{x}(s) = 0\\
&\iff \frac{d}{ds}\left(e^{-\mathcal{M}(s)}\mathbf{x}(s)\right) = 0
\end{aligned}$$

19 As in two dimensions, if $\kappa(s) = 0$, then $\hat{\mathbf{N}}(s)$ is not defined. This makes even more sense in three dimensions than in two dimensions: if the curve is a straight line, there are infinitely many unit vectors perpendicular to it and there is no way to distinguish between them.

20 The equations are named after the two French mathematicians who independently discovered them: Jean Frédéric Frenet (1816–1900, the son of a wig maker), in his thesis of 1847 (actually he only gave two of the three equations), and Joseph Alfred Serret (1819–1885) in 1851.

by the product rule. So $e^{-\mathcal{M}(s)}\mathbf{x}(s)$ is a constant independent of s. In particular $e^{-\mathcal{M}(s)}\mathbf{x}(s) = e^{-\mathcal{M}(0)}\mathbf{x}(0) = \mathbf{x}_0$ so that $\mathbf{x}(s) = \mathbf{x}_0 e^{\mathcal{M}(s)}$. This argument can be generalized to any natural number n. But that is beyond the scope of this book.

Since the Frenet-Serret formulae constitute a first order system of ordinary differential equations for the vector $(\hat{\mathbf{T}}(s), \hat{\mathbf{N}}(s), \hat{\mathbf{B}}(s))$ and since any first order linear initial value problem has a exactly one solution,

- the vector valued function $(\hat{\mathbf{T}}(s), \hat{\mathbf{N}}(s), \hat{\mathbf{B}}(s))$ is determined by the functions $\kappa(s)$ and $\tau(s)$ (assuming that they are continuous) together with the initial condition $(\hat{\mathbf{T}}(0), \hat{\mathbf{N}}(0), \hat{\mathbf{B}}(0))$.
- Furthermore, once you know $\hat{\mathbf{T}}(s)$, then $\mathbf{r}(s)$ is determined by $\mathbf{r}(0)$ and $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}(s) = \hat{\mathbf{T}}(s)$.
- So any smooth curve $\mathbf{r}(s)$ is completely determined by $\mathbf{r}(0)$, $(\hat{\mathbf{T}}(0), \hat{\mathbf{N}}(0), \hat{\mathbf{B}}(0))$, $\kappa(s)$ and $\tau(s)$.
- That is, up to translations (you can move between any two possible choices of $\mathbf{r}(0)$ by a translation) and rotations (you can move between any two possible choices of $(\hat{\mathbf{T}}(0), \hat{\mathbf{N}}(0), \hat{\mathbf{B}}(0))$ by a rotation) a curve is completely determined by the curvature $\kappa(s) > 0$ and the torsion $\tau(s)$. This result is called "The fundamental theorem of space curves".

Theorem 1.4.3 (The Fundamental Theorem of Space Curves)**.**

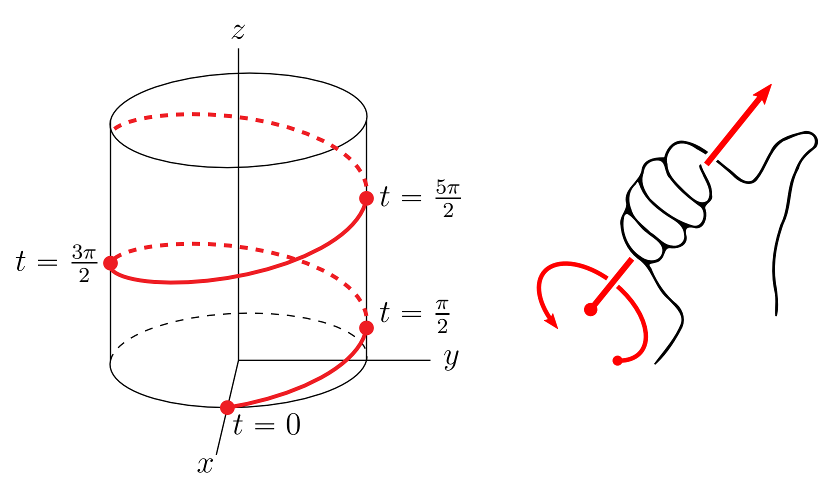
Let $\kappa(s) > 0$ and $\tau(s)$ be continuous. Then up to translations and rotations, there is a unique curve with curvature $\kappa(s)$ and torsion $\tau(s)$.

Example 1.4.4 (Right circular helix)

The right circular helix is the curve

$$\mathbf{r}(t) = a\cos t\,\hat{\boldsymbol{\imath}} + a\sin t\,\hat{\boldsymbol{\jmath}} + bt\,\hat{\mathbf{k}}$$

with $a, b > 0$ as in the figure on the left below.



Here is why it is called a *right* helix rather than a left helix. If the helix is the thread of a bolt that you are screwing into a nut, and you turn the bolt in the direction of the (curled) fingers of your right hand (as in the figure[21] on the right above), then it moves in the direction of your thumb (as in the long straight arrow of the figure on the right above).

To determine the curvature and torsion of this curve we compute

$$\begin{aligned}\mathbf{v}(t) &= -a\sin t\,\hat{\boldsymbol{\imath}} + a\cos t\,\hat{\boldsymbol{\jmath}} + b\,\hat{\mathbf{k}}\\ \mathbf{a}(t) &= -a\cos t\,\hat{\boldsymbol{\imath}} - a\sin t\,\hat{\boldsymbol{\jmath}}\\ \frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t}(t) &= a\sin t\,\hat{\boldsymbol{\imath}} - a\cos t\,\hat{\boldsymbol{\jmath}}\end{aligned}$$

From $\mathbf{v}(t)$ we read off

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{a^2+b^2} \qquad \hat{\mathbf{T}}(t) = -\frac{a}{\sqrt{a^2+b^2}}\sin t\,\hat{\boldsymbol{\imath}} + \frac{a}{\sqrt{a^2+b^2}}\cos t\,\hat{\boldsymbol{\jmath}} + \frac{b}{\sqrt{a^2+b^2}}\,\hat{\mathbf{k}}$$

From $\mathbf{a} = \frac{\mathrm{d}^2s}{\mathrm{d}t^2}\,\hat{\mathbf{T}} + \kappa\big(\frac{\mathrm{d}s}{\mathrm{d}t}\big)^2\hat{\mathbf{N}} = \kappa(a^2+b^2)\hat{\mathbf{N}}$, we read off that

$$\kappa(t) = \frac{a}{a^2+b^2} \qquad \hat{\mathbf{N}}(t) = -\cos t\,\hat{\boldsymbol{\imath}} - \sin t\,\hat{\boldsymbol{\jmath}}$$

From

$$\begin{aligned}\mathbf{v}(t)\times\mathbf{a}(t) &= \det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}}\\ -a\sin t & a\cos t & b\\ -a\cos t & -a\sin t & 0\end{bmatrix} = ab\sin t\,\hat{\boldsymbol{\imath}} - ab\cos t\,\hat{\boldsymbol{\jmath}} + a^2\,\hat{\mathbf{k}}\\ |\mathbf{v}(t)\times\mathbf{a}(t)|^2 &= a^2b^2 + a^4 = a^2(a^2+b^2)\end{aligned}$$

we read off

$$\hat{\mathbf{B}}(t) = \frac{\mathbf{v}(t)\times\mathbf{a}(t)}{|\mathbf{v}(t)\times\mathbf{a}(t)|} = \frac{b}{\sqrt{a^2+b^2}}\sin t\,\hat{\boldsymbol{\imath}} - \frac{b}{\sqrt{a^2+b^2}}\cos t\,\hat{\boldsymbol{\jmath}} + \frac{a}{\sqrt{a^2+b^2}}\,\hat{\mathbf{k}}$$

and

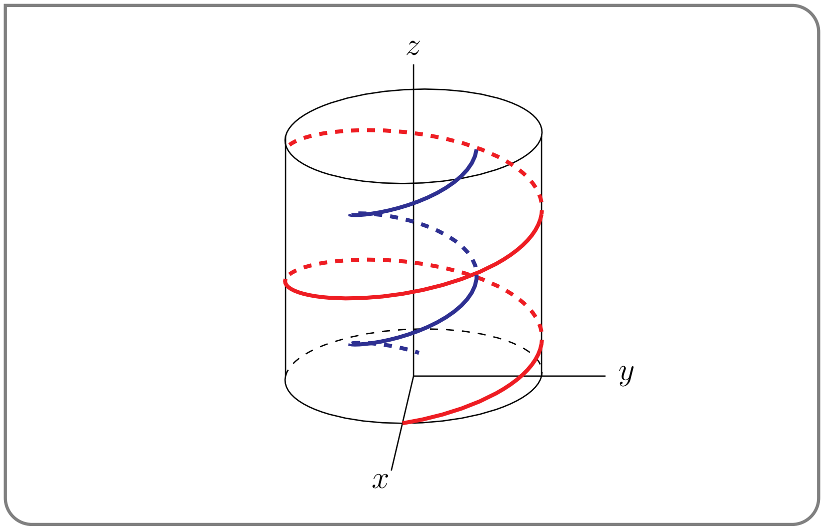
$$\tau(t) = \frac{(\mathbf{v}\times\mathbf{a})\cdot\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t}}{|\mathbf{v}\times\mathbf{a}|^2} = \frac{a^2b}{a^2(a^2+b^2)} = \frac{b}{a^2+b^2}$$

Note that, for the right handed helix, $\tau > 0$. Finally the centre of curvature is

$$\begin{aligned}\mathbf{r}(t) + \frac{1}{\kappa(t)}\hat{\mathbf{N}}(t) &= \Big(a - \frac{a^2+b^2}{a}\Big)\cos t\,\hat{\boldsymbol{\imath}} + \Big(a - \frac{a^2+b^2}{a}\Big)\sin t\,\hat{\boldsymbol{\jmath}} + bt\,\hat{\mathbf{k}}\\ &= -\frac{b^2}{a}\cos t\,\hat{\boldsymbol{\imath}} - \frac{b^2}{a}\sin t\,\hat{\boldsymbol{\jmath}} + bt\,\hat{\mathbf{k}}\end{aligned}$$

which is another helix. In the figure below, the red curve is the original helix and the blue curve is the helix traced by the centre of curvature.

21 This figure is a variant of https://commons.wikimedia.org/wiki/File:Right_hand_rule_simple.png



Example 1.4.4

1.5▲ A Compendium of Curve Formula

In the following $\mathbf{r}(t) = \big(x(t)\,,\, y(t)\,,\, z(t)\big)$ is a parametrization of some curve. The vectors $\hat{\mathbf{T}}(t)$, $\hat{\mathbf{N}}(t)$, and $\hat{\mathbf{B}}(t)$ are the unit tangent, normal and binormal vectors, respectively, at $\mathbf{r}(t)$. The tangent vector points in the direction of travel (i.e. direction of increasing t) and the normal vector points toward the centre of curvature. The arc length from time 0 to time t is denoted $s(t)$. The binormal $\hat{\mathbf{B}}(t) = \hat{\mathbf{T}}(t) \times \hat{\mathbf{N}}(t)$ is perpendicular to the plane that fits the curve best at $\mathbf{r}(t)$. Some formulae use an arc length parametrization, which is denoted $\mathbf{r}(s)$.

- the velocity $\qquad \mathbf{v}(t) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t) = \frac{\mathrm{d}s}{\mathrm{d}t}(t)\,\hat{\mathbf{T}}(t)$
- the unit tangent vector $\qquad \hat{\mathbf{T}}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$ (general parametrization)
  $\hat{\mathbf{T}}(s) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}(s)$ (arc length parametrization)
- the acceleration $\qquad \mathbf{a}(t) = \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}(t) = \frac{\mathrm{d}^2s}{\mathrm{d}t^2}(t)\,\hat{\mathbf{T}}(t) + \kappa(t)\big(\frac{\mathrm{d}s}{\mathrm{d}t}(t)\big)^2\hat{\mathbf{N}}(t)$
- the speed $\qquad \frac{\mathrm{d}s}{\mathrm{d}t}(t) = |\mathbf{v}(t)| = \big|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\big|$
- the arc length $\qquad s(T) = \int_0^T \frac{\mathrm{d}s}{\mathrm{d}t}(t)\,\mathrm{d}t = \int_0^T \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,\mathrm{d}t$
- the curvature $\qquad \kappa(t) = \big|\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}t}(t)\big| / \frac{\mathrm{d}s}{\mathrm{d}t}(t) = \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{(\frac{\mathrm{d}s}{\mathrm{d}t}(t))^3}$
  $\kappa(s) = \big|\frac{\mathrm{d}\phi}{\mathrm{d}s}(s)\big| = \big|\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s}(s)\big|$
- the unit normal vector $\qquad \hat{\mathbf{N}}(t) = \frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}t}(t) / \big|\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}t}(t)\big| \qquad \hat{\mathbf{N}}(s) = \frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s}(s)/\kappa(s)$
- the radius of curvature $\qquad \rho(t) = \frac{1}{\kappa(t)}$
- the centre of curvature $\qquad \mathbf{r}(t) + \rho(t)\hat{\mathbf{N}}(t)$

- the torsion $$\tau(t) = \frac{\big(\mathbf{v}(t)\times\mathbf{a}(t)\big)\cdot\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t}(t)}{|\mathbf{v}(t)\times\mathbf{a}(t)|^2}$$
- the binormal $$\hat{\mathbf{B}}(t) = \hat{\mathbf{T}}(t)\times\hat{\mathbf{N}}(t) = \frac{\mathbf{v}(t)\times\mathbf{a}(t)}{|\mathbf{v}(t)\times\mathbf{a}(t)|}$$

Under arclength parametrization (i.e. if $t = s$) we have $\hat{\mathbf{T}}(s) = \frac{d\mathbf{r}}{ds}(s)$ and the Frenet-Serret formulae

$$\begin{aligned}
\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s}(s) &= \quad \kappa(s)\,\hat{\mathbf{N}}(s) \\
\frac{\mathrm{d}\hat{\mathbf{N}}}{\mathrm{d}s}(s) &= \quad \tau(s)\,\hat{\mathbf{B}}(s) - \kappa(s)\,\hat{\mathbf{T}}(s) \\
\frac{\mathrm{d}\hat{\mathbf{B}}}{\mathrm{d}s}(s) &= -\tau(s)\,\hat{\mathbf{N}}(s)
\end{aligned}$$

which in matrix form is

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{bmatrix}\hat{\mathbf{T}}(s)\\ \hat{\mathbf{N}}(s)\\ \hat{\mathbf{B}}(s)\end{bmatrix} = \begin{bmatrix}0 & \kappa(s) & 0\\ -\kappa(s) & 0 & \tau(s)\\ 0 & -\tau(s) & 0\end{bmatrix}\begin{bmatrix}\hat{\mathbf{T}}(s)\\ \hat{\mathbf{N}}(s)\\ \hat{\mathbf{B}}(s)\end{bmatrix}$$

When the curve lies entirely in the xy-plane the curvature is given by

$$\kappa(t) = \frac{\left|\frac{\mathrm{d}x}{\mathrm{d}t}(t)\,\frac{\mathrm{d}^2y}{\mathrm{d}t^2}(t) - \frac{\mathrm{d}y}{\mathrm{d}t}(t)\,\frac{\mathrm{d}^2x}{\mathrm{d}t^2}(t)\right|}{\left[\left(\frac{\mathrm{d}x}{\mathrm{d}t}(t)\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}(t)\right)^2\right]^{3/2}}$$

When the curve lies entirely in the xy-plane and the y-coordinate is given as a function, $y(x)$, of the x-coordinate, the curvature is

$$\kappa(x) = \frac{\left|\frac{\mathrm{d}^2y}{\mathrm{d}x^2}(x)\right|}{\left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}(x)\right)^2\right]^{3/2}}$$

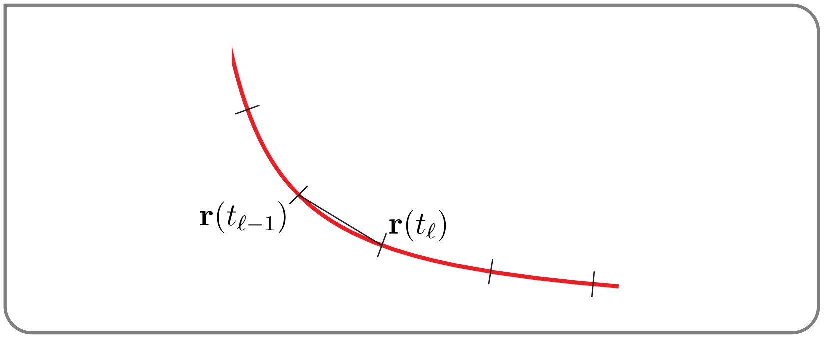
Notice that this follows from the previous formula since $\frac{\mathrm{d}x}{\mathrm{d}x} = 1$ and $\frac{\mathrm{d}^2x}{\mathrm{d}x^2} = 0$.

1.6▲ Integrating Along a Curve

Suppose that we have a curve $\mathcal{C}$ that is parametrized as $\mathbf{r}(t)$ with $a \leqslant t \leqslant b$. Suppose further that $\mathcal{C}$ is actually a piece of wire and that the density (i.e. mass per unit length) of the wire at the point $\mathbf{r}$ is $\rho(\mathbf{r})$. How do we figure out the mass of $\mathcal{C}$? Of course we use the standard Calculus divide and conquer strategy. We select a natural number n and

- divide the interval $a \leqslant t \leqslant b$ into n equal subintervals, each of length $\Delta t = \frac{b-a}{n}$. We denote by $t_\ell = a + \ell\Delta t$ the right hand end of interval number ℓ.

◦ Then we approximate the length of the part of the curve between $\mathbf{r}(t_{\ell-1})$ and $\mathbf{r}(t_\ell)$ by $|\mathbf{r}(t_\ell) - \mathbf{r}(t_{\ell-1})|$ and the mass of the part of the curve between $\mathbf{r}(t_{\ell-1})$ and $\mathbf{r}(t_\ell)$ by $\rho(\mathbf{r}(t_\ell))|\mathbf{r}(t_\ell) - \mathbf{r}(t_{\ell-1})|$.



◦ This gives us, as an approximate mass for $\mathcal{C}$ of

$$\sum_{\ell=1}^{n} \rho(\mathbf{r}(t_\ell))|\mathbf{r}(t_\ell) - \mathbf{r}(t_{\ell-1})| = \sum_{\ell=1}^{n} \rho(\mathbf{r}(t_\ell))\left|\frac{\mathbf{r}(t_\ell) - \mathbf{r}(t_{\ell-1})}{t_\ell - t_{\ell-1}}\right|\Delta t$$

Then we take the limit as $n \to \infty$. Assuming[22] that $\mathbf{r}(t)$ is continuously differentiable and that $\rho(\mathbf{r})$ is continuous we get

$$\text{Mass of } \mathcal{C} = \int_a^b \rho(\mathbf{r}(t)) \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\right| \mathrm{d}t$$

which we take to be a definition.

Definition 1.6.1.

(a) For a parametrized curve $(x(t), y(t), z(t))$, $a \leqslant t \leqslant b$, in $\mathbb{R}^3$ that we call $\mathcal{C}$, and for a function $f(x, y, z)$, we define

$$\int_{\mathcal{C}} f(x, y, z)\, \mathrm{d}s = \int_a^b f(x(t), y(t), z(t))\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, \mathrm{d}t$$

In this notation the subscript $\mathcal{C}$ specifies the curve, and $\mathrm{d}s$ signifies arc length.

(b) For a curve $y = f(x)$, $a \leqslant x \leqslant b$, in $\mathbb{R}^2$ that we call C, and for a function $g(x, y)$, we define

$$\int_C g(x, y)\, \mathrm{d}s = \int_a^b g(x, f(x))\sqrt{1 + f'(x)^2}\, \mathrm{d}x$$

22 We could relax these conditions somewhat by instead assuming that $\mathbf{r}'(t)$ and $\rho(t)$ are bounded and are continuous except at a finite number of points. ($\mathbf{r}'(t)$ need not exist at all at those points.)

Example 1.6.2

Suppose that we have a helical wire[23]

$$\mathbf{r}(t) = \big(x(t)\,,\, y(t)\,,\, z(t)\big) = \big(a\cos t\,,\, a\sin t\,,\, bt\big) \qquad 0 \leqslant t \leqslant 2\pi$$

and that this wire has constant mass density ρ. Let's find the centre of mass of the wire. Recall that the centre of mass is $(\bar{x}, \bar{y}, \bar{z})$ with, for example, $\bar{x}$ being the weighted average

$$\bar{x} = \frac{\int x\rho\,\mathrm{d}s}{\int \rho\,\mathrm{d}s} = \frac{\int x\,\mathrm{d}s}{\int \mathrm{d}s} \qquad \text{(since } \rho \text{ is constant)}$$

of x over the wire. Similarly $\bar{y} = \frac{\int y\,\mathrm{d}s}{\int \mathrm{d}s}$ and $\bar{z} = \frac{\int z\,\mathrm{d}s}{\int \mathrm{d}s}$. For the given curve

$$\begin{aligned}
\big(x(t)\,,\, y(t)\,,\, z(t)\big) &= \big(a\cos t\,,\, a\sin t\,,\, bt\big)\\
\big(x'(t)\,,\, y'(t)\,,\, z'(t)\big) &= \big(-a\sin t\,,\, a\cos t\,,\, b\big)\\
\frac{\mathrm{d}s}{\mathrm{d}t}(t) &= \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2+b^2}
\end{aligned}$$

so that

$$\begin{aligned}
\bar{x} &= \frac{\int x\,\mathrm{d}s}{\int \mathrm{d}s} = \frac{\int_0^{2\pi} x(t)\sqrt{a^2+b^2}\,\mathrm{d}t}{\int_0^{2\pi}\sqrt{a^2+b^2}\,\mathrm{d}t} = \frac{\int_0^{2\pi} a\cos(t)\,\mathrm{d}t}{2\pi} = 0\\
\bar{y} &= \frac{\int y\,\mathrm{d}s}{\int \mathrm{d}s} = \frac{\int_0^{2\pi} y(t)\sqrt{a^2+b^2}\,\mathrm{d}t}{\int_0^{2\pi}\sqrt{a^2+b^2}\,\mathrm{d}t} = \frac{\int_0^{2\pi} a\sin(t)\,\mathrm{d}t}{2\pi} = 0\\
\bar{z} &= \frac{\int z\,\mathrm{d}s}{\int \mathrm{d}s} = \frac{\int_0^{2\pi} z(t)\sqrt{a^2+b^2}\,\mathrm{d}t}{\int_0^{2\pi}\sqrt{a^2+b^2}\,\mathrm{d}t} = \frac{\int_0^{2\pi} bt\,\mathrm{d}t}{2\pi} = \frac{b}{2\pi}\left[\frac{t^2}{2}\right]_0^{2\pi} = b\pi
\end{aligned}$$

So the centre of mass is right on the axis of the helix, half way up, which makes perfect sense.

Example 1.6.2

1.7▲ Sliding on a Curve

We are going to investigate the motion of a particle of mass m sliding on a frictionless[24], smooth curve that lies in a vertical plane. We will consider three scenarios:

- First, to set things up we'll look at a bead sliding on a stiff wire.
- Then, we will imagine that we are skiing straight downhill and ask "Where on the hill can we become airborne?".
- Then we will imagine that we are skateboarding in a culvert (a large pipe) and ask "When is it safe?".

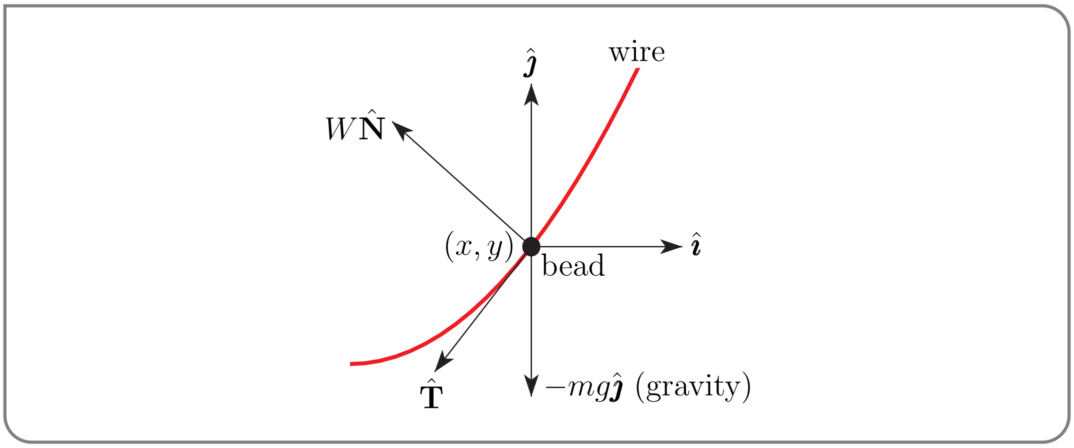
23 For example, your favourite solenoid or spring or slinky.
24 We are mathematicians — we like idealized situations.

▸▸ The Sliding Bead

First, consider a bead of mass m that is sliding, without friction, on a stiff wire. According to Newton's law of motion

$$m\mathbf{a} = \mathbf{F}$$

where $\mathbf{F}$ is the net force being applied to the bead. The bead is subject to two forces. The gravitational force is $-mg\hat{\boldsymbol{\jmath}}$. By definition, absence of friction means that the wire is does not apply any force that is in the direction tangential to the wire. But, because it is stiff, the wire never changes shape and instead applies just the right amount of force, in the direction normal to the wire, that is needed to keep the bead on the wire[25] without bending the wire. Call this normal force $W\hat{\mathbf{N}}$.



So, by Newton's law,

$$m\,\mathbf{a} = -mg\,\hat{\boldsymbol{\jmath}} + W\,\hat{\mathbf{N}}$$

We'll analyse this equation by splitting it into its tangential and normal components.

To extract the tangential component of Newton's law, we dot it with $\mathbf{v} = |\mathbf{v}|\hat{\mathbf{T}}$. Since $\hat{\mathbf{T}} \cdot \hat{\mathbf{N}} = 0$ this kills all normal components.

$$\begin{aligned} m\mathbf{v} \cdot \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} &= -mg\hat{\boldsymbol{\jmath}} \cdot \mathbf{v} + W\hat{\mathbf{N}} \cdot \mathbf{v} \\ \frac{1}{2}m\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v} \cdot \mathbf{v}) &= -mg\frac{\mathrm{d}y}{\mathrm{d}t} \end{aligned}$$

Here we have used

- Theorem 1.1.3.c on the left hand side and
- that $\hat{\boldsymbol{\jmath}} \cdot \mathbf{v}$ is just the y component of $\mathbf{v}$ and
- that $\hat{\mathbf{N}}$ and $\mathbf{v} = |\mathbf{v}|\hat{\mathbf{T}}$ are perpendicular.

Moving everything to the left hand side of the equation gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m|\mathbf{v}|^2 + mgy\right) = 0$$

and we conclude that the quantity

25 This force is required to keep the bead from either passing through the wire or flying off the wire.

Equation 1.7.1 (Conservation of Energy).

$$E = \frac{1}{2}m|\mathbf{v}|^2 + mgy$$

is a constant, independent of time. This is, of course, the principle of conservation of energy. It determines the speed $|\mathbf{v}| = \sqrt{\frac{2E}{m} - 2gy}$ of the bead as a function of the height y (and of the energy E, which is determined by the initial conditions).

To extract the normal component of Newton's law, we dot it with $\hat{\mathbf{N}}$:

$$m\mathbf{a} \cdot \hat{\mathbf{N}} = -mg\hat{\boldsymbol{\jmath}} \cdot \hat{\mathbf{N}} + W$$

Since

$$\mathbf{a} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\hat{\mathbf{T}} + \kappa\Big(\frac{\mathrm{d}s}{\mathrm{d}t}\Big)^2\hat{\mathbf{N}} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\hat{\mathbf{T}} + \kappa|\mathbf{v}|^2\hat{\mathbf{N}}$$

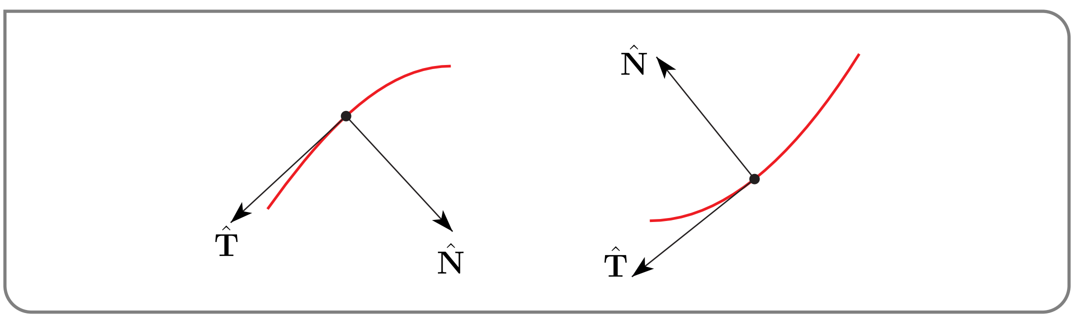
and $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$ are perpendicular, this gives, after a little rearrangement,

Equation 1.7.2 (Normal Force).

$$W = m\kappa|\mathbf{v}|^2 + mg\hat{\boldsymbol{\jmath}} \cdot \hat{\mathbf{N}} = 2\kappa(E - mgy) + mg\hat{\boldsymbol{\jmath}} \cdot \hat{\mathbf{N}}$$

▸▸ The Skier

The difference between the bead on the wire and the skier on the hill is that while the hill is capable of applying an upward normal force (i.e. it can push you upward to keep you from falling to the centre of the Earth), it is not capable of applying a downward normal force. That is the hill cannot pull down on you to keep you on the hill. Only gravity can keep you grounded. There are two main possibilities[26].



- If the hill is concave downward as in the figure on the left above, then $\hat{\mathbf{N}}$ points downward and the hill is allowed to have $W \leqslant 0$ (which corresponds to the normal force $W\hat{\mathbf{N}}$ pushing upward). If ever $W > 0$, the hill would have to pull on you to keep you on hill. It can't, so you become airborne. Since $\hat{\boldsymbol{\jmath}} \cdot \hat{\mathbf{N}} < 0$, this happens whenever

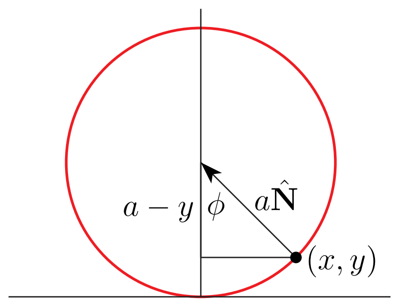
$$W > 0 \iff m\kappa|\mathbf{v}|^2 + mg\hat{\boldsymbol{\jmath}} \cdot \hat{\mathbf{N}} > 0 \iff |\mathbf{v}| > \sqrt{\frac{g}{\kappa}|\hat{\boldsymbol{\jmath}} \cdot \hat{\mathbf{N}}|}$$

26 We assume that you are going downhill and that the curvature $\kappa > 0$.

- If the hill is concave upward as in the figure on the right above, then $\hat{\mathbf{N}}$ points upward and the hill is allowed to have $W \geqslant 0$ (which corresponds to the normal force $W\hat{\mathbf{N}}$ pushing upward). Since $\hat{\boldsymbol{\jmath}} \cdot \hat{\mathbf{N}} > 0$ we always have $W = m\kappa|\mathbf{v}|^2 + mg\hat{\boldsymbol{\jmath}} \cdot \hat{\mathbf{N}} > 0$. You never become airborne. On the other hand your knees may complain.

▸▸ The Skate Boarder

So far, Equations (1.7.1) and (1.7.2) apply to any stiff frictionless "wire". We now specialize to the special case of a skateboarder inside a circular culvert of radius a. Let's put the bottom of the circle at the origin $(0, 0)$, so that the centre of the circle is at $(0, a)$.



In this case the curvature is $\kappa = \frac{1}{a}$ and $\hat{\boldsymbol{\jmath}} \cdot \hat{\mathbf{N}} = \cos\phi = \frac{a-y}{a}$ so (1.7.1) and (1.7.2) simplify to

$$|\mathbf{v}| = \sqrt{\frac{2}{m}(E - mgy)} = \sqrt{2g\Big(\frac{E}{mg} - y\Big)}$$
$$W = \frac{2}{a}(E - mgy) + \frac{mg}{a}(a - y) = \frac{3mg}{a}\Big(\frac{2}{3mg}E + \frac{a}{3} - y\Big)$$

Imagine now that you start at the bottom of the culvert, that is at $y = 0$, with energy $E > 0$. As time progresses, y increases and consequently $|\mathbf{v}|$ and W both decrease, as, of course, they should. This continues until one of the following three things happen.

(i) $|\mathbf{v}|$ hits 0, in which case you stop rising and start descending. The speed $|\mathbf{v}|$ is zero when $y = y_S = \frac{E}{mg}$. (The subscript "S" stands for "stop".) Physicists say that when you reach y_S all of your kinetic energy ($\frac{1}{2}m|\mathbf{v}|^2$) has been converted into potential energy (mgy).

(ii) W hits zero. When you get higher than this, W becomes negative and the culvert would have to pull on you to keep your feet on the culvert. As the culvert can only push on you, you become airborne. The normal force W is zero when $y = y_A = \frac{2}{3}\frac{E}{mg} + \frac{a}{3}$. (The subscript "$A$" stands for "airborne".)

(iii) y hits $2a$. This is the summit of the culvert. You descend on the other side.

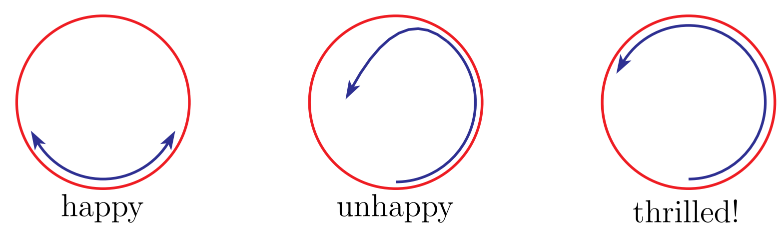
Which case actually happens is determined by the relative sizes of y_S, y_A and $2a$.

- Comparing $y_S = \frac{2}{3}\frac{E}{mg} + \frac{1}{3}\frac{E}{mg}$ and $y_A = \frac{2}{3}\frac{E}{mg} + \frac{a}{3}$, we see that $y_S \leqslant y_A \iff \frac{E}{mg} \leqslant a$.

- Comparing $y_A = \frac{2}{3}\frac{E}{mg} + \frac{a}{3}$ and $a = \frac{2}{3}a + \frac{a}{3}$, we see that $y_A \leqslant a \iff \frac{E}{mg} \leqslant a$.
- Comparing $y_A = \frac{2}{3}\frac{E}{mg} + \frac{a}{3}$ and $2a = \frac{5}{3}a + \frac{a}{3}$, we see that $y_A \leqslant 2a \iff \frac{E}{mg} \leqslant \frac{5}{2}a$.

So the conclusions are:

- **If** $\mathbf{0 \leqslant \frac{E}{mg} \leqslant a}$ then $0 \leqslant y_S \leqslant y_A \leqslant a$. In this case you just oscillate between heights 0 and $y_S \leqslant a$ in the bottom half of the culvert, as in the figure on the left below.
- **If** $\mathbf{a \leqslant \frac{E}{mg} \leqslant \frac{5}{2}a}$ then $a \leqslant y_A \leqslant y_S, 2a$. In this case you make it more than half way to the top. But you become airborne at $y = y_A$ which is somewhere between the half way mark $y = a$ and the top $y = 2a$. At this point our model breaks down because you are no longer in contact with the culvert. You just freely follow a parabolic arc until you crash back into the culvert, as in the figure in the centre below.
- **If** $\mathbf{\frac{5}{2}a < \frac{E}{mg}}$ then $2a < y_A < y_S$. In this case you successfully go all the way around the culvert, looping the loop, as in the figure on the right below. Note that, as $\frac{E}{mg} > \frac{5}{2}a > 2a$, this requires significantly more energy than that required to reach the top, i.e. to reach height $2a$.

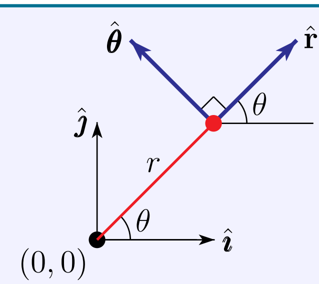


1.8▲ Optional — Polar Coordinates

So far we have always written vectors in two dimensions in terms of the basis vectors $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$. This is not always convenient. For example, when working in polar coordinates it is often convenient to use basis vectors $\hat{\mathbf{r}}(\theta)$, $\hat{\boldsymbol{\theta}}(\theta)$ which depend on the value of the current polar coordinate θ — though one usually just writes $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, suppressing the dependence on θ from the notation. When one is at the point with polar coordinates (r, θ), these basis vectors are defined by

Equation 1.8.1.

$$\hat{\mathbf{r}}(\theta) = \cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}$$
$$\hat{\boldsymbol{\theta}}(\theta) = -\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}}$$



Note that this basis has two very nice properties.

1. $|\hat{\mathbf{r}}(\theta)| = |\hat{\boldsymbol{\theta}}(\theta)| = 1,\ \hat{\mathbf{r}}(\theta) \perp \hat{\boldsymbol{\theta}}(\theta)$ (orthonormality)
2. $\frac{d\hat{\mathbf{r}}}{d\theta}(\theta) = \hat{\boldsymbol{\theta}}(\theta),\ \frac{d\hat{\boldsymbol{\theta}}}{d\theta}(\theta) = -\hat{\mathbf{r}}(\theta)$

That $\frac{d\hat{\mathbf{r}}}{d\theta}(\theta)$ is some scalar multiple of $\hat{\boldsymbol{\theta}}(\theta)$ follows just from the fact that $|\hat{\mathbf{r}}(\theta)| = 1$.

$$\begin{aligned} |\hat{\mathbf{r}}(\theta)| = 1 &\implies \hat{\mathbf{r}}(\theta) \cdot \hat{\mathbf{r}}(\theta) = 1 \\ &\implies \hat{\mathbf{r}}(\theta) \cdot \frac{d\hat{\mathbf{r}}}{d\theta}(\theta) = \frac{1}{2}\frac{d}{dt}\big(\hat{\mathbf{r}}(\theta) \cdot \hat{\mathbf{r}}(\theta)\big) = 0 \\ &\implies \frac{d\hat{\mathbf{r}}}{d\theta}(\theta) \perp \hat{\mathbf{r}}(\theta) \implies \frac{d\hat{\mathbf{r}}}{d\theta}(\theta) \parallel \hat{\boldsymbol{\theta}}(\theta) \end{aligned}$$

Similarly, that $\frac{d\hat{\boldsymbol{\theta}}}{d\theta}(\theta)$ is some scalar multiple of $\hat{\mathbf{r}}(\theta)$ follows just from the fact that $|\hat{\boldsymbol{\theta}}(\theta)| = 1$.

Lemma 1.8.2.

If we parametrize a curve by giving its polar coordinates[27] $\big(r(t)\,,\,\theta(t)\big)$, then

(a) $\mathbf{r}(t) = r(t)\,\hat{\mathbf{r}}\big(\theta(t)\big)$

(b) $\mathbf{v}(t) = \frac{dr}{dt}(t)\,\hat{\mathbf{r}}\big(\theta(t)\big) + r(t)\,\frac{d\theta}{dt}(t)\,\hat{\boldsymbol{\theta}}\big(\theta(t)\big)$

(c) $\mathbf{a}(t) = \Big(\frac{d^2r}{dt^2}(t) - r(t)\big(\frac{d\theta}{dt}(t)\big)^2\Big)\hat{\mathbf{r}}\big(\theta(t)\big) + \Big(r(t)\,\frac{d^2\theta}{dt^2}(t) + 2\frac{dr}{dt}(t)\frac{d\theta}{dt}(t)\Big)\hat{\boldsymbol{\theta}}\big(\theta(t)\big)$

It is standard to suppress the arguments t and $\theta(t)$ and write, for example,

$$\mathbf{v} = \frac{dr}{dt}\,\hat{\mathbf{r}} + r\,\frac{d\theta}{dt}\,\hat{\boldsymbol{\theta}}$$

But it is important to remember that the arguments really are there.

Proof. The vector from the origin to the point whose polar coordinates are (r, θ) is $\mathbf{r} = r\,\hat{\mathbf{r}}(\theta)$. So if we parametrize a curve by giving the polar coordinates at time t,

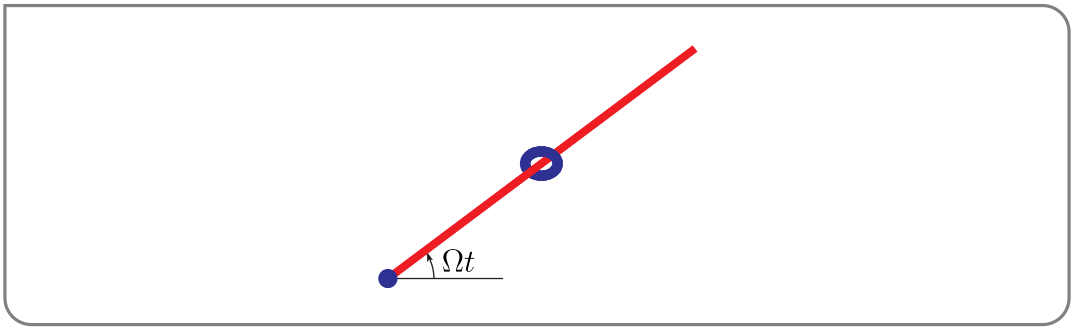
$$\begin{aligned} \mathbf{r}(t) &= r(t)\,\hat{\mathbf{r}}\big(\theta(t)\big) \\ \mathbf{v}(t) &= \frac{dr}{dt}(t)\,\hat{\mathbf{r}}\big(\theta(t)\big) + r(t)\,\frac{d\hat{\mathbf{r}}}{d\theta}\big(\theta(t)\big)\,\frac{d\theta}{dt}(t) \\ &= \frac{dr}{dt}(t)\,\hat{\mathbf{r}}\big(\theta(t)\big) + r(t)\,\frac{d\theta}{dt}(t)\,\hat{\boldsymbol{\theta}}\big(\theta(t)\big) \\ \mathbf{a}(t) &= \frac{d^2r}{dt^2}\,\hat{\mathbf{r}} + \frac{dr}{dt}\,\frac{d\hat{\mathbf{r}}}{d\theta}\,\frac{d\theta}{dt} + \frac{dr}{dt}\,\frac{d\theta}{dt}\,\hat{\boldsymbol{\theta}} + r\,\frac{d^2\theta}{dt^2}\,\hat{\boldsymbol{\theta}} + r\left(\frac{d\theta}{dt}\right)^2\frac{d\hat{\boldsymbol{\theta}}}{d\theta} \\ &= \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\hat{\mathbf{r}} + \left(r\,\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\,\frac{d\theta}{dt}\right)\hat{\boldsymbol{\theta}} \end{aligned}$$

□

27 As usual r is the distance from the origin to the point and θ is angle between the x-axis and the vector from the origin to the point. The symbols r, θ are the standard mathematics symbols for the polar coordinates. Appendix G gives another set of symbols that is commonly used in the physical sciences and engineering.

Example 1.8.3

As an example, consider a bead that is sliding on a frictionless rod that has one end fixed at the origin and that is rotating about the origin at a constant Ω rad/sec.



Because the rod is frictionless, it is incapable of applying to the bead any force parallel to the rod. So under Newton's law, $m\mathbf{a} = \mathbf{F}$, the radial[28] component of the acceleration of the particle is exactly zero. So, if the polar coordinates of the bead at time t are $\big(r(t), \theta(t)\big)$, then, by Lemma 1.8.2.c,

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = 0$$

As the rod is rotating at Ω rad/sec, $\frac{d\theta}{dt} = \Omega$ and

$$\frac{d^2r}{dt^2} - \Omega^2\, r = 0$$

The general solution to this constant coefficient second order ordinary differential equation is[29]

$$r(t) = Ae^{\Omega t} + Be^{-\Omega t}$$

where A and B are arbitrary constants that are determined by initial conditions. Just as an example, if $r(0) = 1$ and $r'(0) = 0$, then $A + B = 1$ and $A\Omega - B\Omega = 0$, so that $A = B = \frac{1}{2}$ and

$$r(t) = \frac{1}{2}\left(e^{\Omega t} + e^{-\Omega t}\right)$$

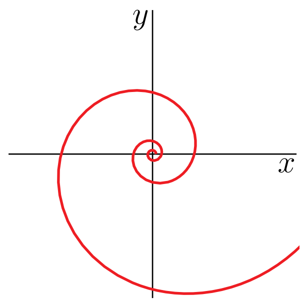
If, again for example, $\theta(0) = 0$, then $\theta(t) = \Omega t$ and the bead follows the polar coordinate curve

$$r(\theta) = \frac{1}{2}\left(e^{\theta} + e^{-\theta}\right)$$

Observe that $r(\theta)$ is 1 when $\theta = 0$, increases as θ increases, and tends to ∞ as $\theta \to +\infty$. The curve is a spiral.

28 The $\hat{\boldsymbol{\theta}}$ component of the acceleration just tells us how much normal force the rod is applying to the bead to keep it on the rod.

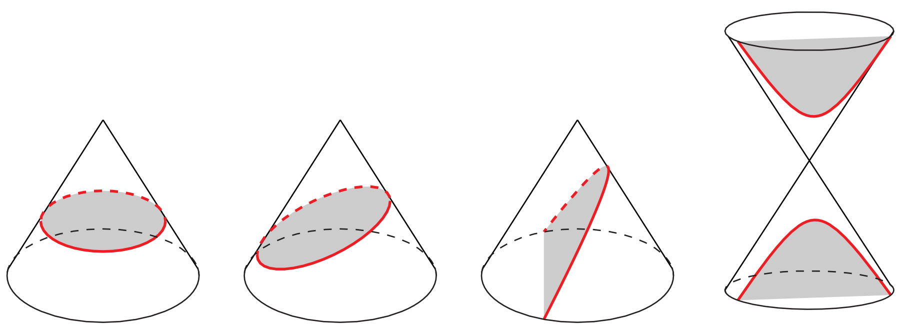
29 A review of the technique used to find this solution is given in Appendix I. In any event, it is easy to check that $r(t) = Ae^{\Omega t} + Be^{-\Omega t}$ really does obey $\frac{d^2r}{dt^2} - \Omega^2\, r = 0$.



Example 1.8.3

Example 1.8.4 (Conic sections in polar coordinates)

In this example, we derive the equation of a general conic section in polar coordinates. A conic section is the intersection of a plane with a cone. This is illustrated in the figures below. For our current purposes, it is convenient to use the equivalent[30] (and often used)

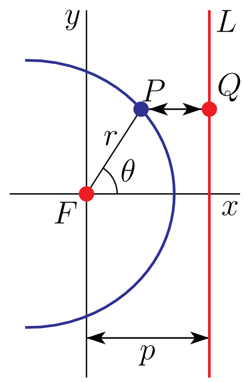


definition that a conic section is the set of points P in the xy-plane

- whose distance from a fixed point F (called the *focus* of the conic)
- is a constant multiple $\varepsilon \geqslant 0$ (called the *eccentricity* of the conic)
- of the distance from P to a fixed line L (called the *directrix* of the conic).

Choose a coordinate system with the focus F of the conic being the origin and with the directrix L being $x = p$ for some $p > 0$.

30 It is outside our scope to prove this equivalence.



If P has polar coordinates (r, θ), then P has x-coordinate $r\cos\theta$. The point Q on the line L in the figure above has x-coordinate p. So the distance from P to L, which is also the distance from P to Q, is $p - r\cos\theta$. The distance from P to F is r. We require that the distance from P to F is ε times the distance from P to L. So

$$r = \varepsilon\big(p - r\cos\theta\big) \iff r = \frac{\varepsilon p}{1 + \varepsilon\cos\theta}$$

The numerator εp is usually renamed to ℓ giving the equation

$$r = \frac{\ell}{1 + \varepsilon\cos\theta}$$

Example 1.8.4

Example 1.8.5 (Conic sections in polar coordinates, again)

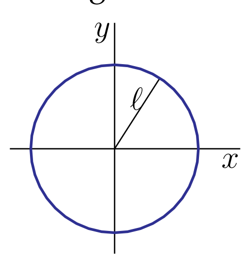
We'll now take the equation $r = \frac{\ell}{1+\varepsilon\cos\theta}$ for a conic section in polar coordinates, from the last example, and convert it to the more familiar Cartesian coordinates. Just by the definition of polar coordinates

$$\begin{aligned} r\big(1 + \varepsilon\cos\theta\big) = \ell &\iff r = \ell - \varepsilon x \\ &\iff x^2 + y^2 = \ell^2 - 2\varepsilon\ell x + \varepsilon^2 x^2 \\ &\iff (1-\varepsilon^2)x^2 + 2\varepsilon\ell x + y^2 = \ell^2 \qquad \text{(C)} \end{aligned}$$

Now consider separately four different cases, depending on the value of $\varepsilon \geqslant 0$.

- If $\varepsilon = 0$, (C) reduces to

  $$x^2 + y^2 = \ell^2$$

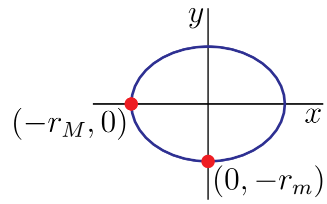
  which is of course a circle of radius ℓ.

- If $0 < \varepsilon < 1$, completing the square in (C) gives

$$(1-\varepsilon^2)\left(x+\frac{\varepsilon\ell}{1-\varepsilon^2}\right)^2 + y^2 = \ell^2 + \frac{\varepsilon^2\ell^2}{1-\varepsilon^2} = \frac{\ell^2}{1-\varepsilon^2}$$

which is equivalent to

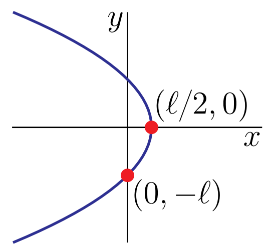
$$\frac{\left(x+\frac{\varepsilon\ell}{1-\varepsilon^2}\right)^2}{\frac{\ell^2}{(1-\varepsilon^2)^2}} + \frac{y^2}{\frac{\ell^2}{1-\varepsilon^2}} = 1$$



and is of course an ellipse with semi-major axis $r_M = \frac{\ell}{1-\varepsilon^2}$ and semi-minor axis $r_m = \frac{\ell}{\sqrt{1-\varepsilon^2}}$.

- If $\varepsilon = 1$, (C) reduces to

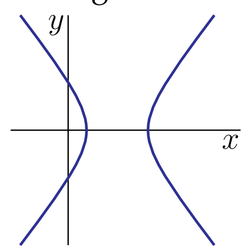
$$y^2 = \ell^2 - 2\ell x$$



which is of course a parabola.

- If $\varepsilon > 1$, the same computation as in the $0 < \varepsilon < 1$ case gives

$$\frac{\left(x-\frac{\varepsilon\ell}{\varepsilon^2-1}\right)^2}{\frac{\ell^2}{(\varepsilon^2-1)^2}} - \frac{y^2}{\frac{\ell^2}{\varepsilon^2-1}} = 1$$



and is of course a hyperbola.

Example 1.8.5

1.9▲ Optional — Central Forces

One of the great triumphs of Newtonian mechanics was the explanation of Kepler's laws[31], which said

1. The planets trace out ellipses about the sun as focus.
2. The radius vector **r** sweeps out equal areas in equal times.

31 The German astronomer Johannes Kepler (1571–1630) developed these laws during the course of an attempt to relate the five extraterrestrial planets then known to the five Platonic solids. He based the laws on a great number of careful measurements made by the Danish Astronomer Tycho Brahe (1546–1601). Then Isaac Newton (English, 1642–1727) provided the explanation in 1687. Kepler also wrote a paper entitled "On the Six-Cornered Snowflake". Tycho Brahe lost his nose in a sword duel and wore a prosthetic nose from then on. The story is that Brahe died from a burst bladder that resulted from his refusing to leave the dinner table before his host.

3. The square of the period of each planet is proportional to the cube of the major axis of the planet's orbit.

Newton showed that all of these behaviours follow from the assumption that the acceleration $\mathbf{a}(t)$ of each planet obeys the law of motion $m\mathbf{a} = \mathbf{F}$ where m is the mass of the planet and

$$\mathbf{F} = -\frac{GMm}{r^3}\mathbf{r}$$

is the "gravitational force" applied on the planet by the sun. Here G is a constant[32], called the "gravitational constant" or the "universal gravitational constant", M is the mass of the sun, $\mathbf{r}$ is the vector from the sun to the planet and $r = |\mathbf{r}|$.

In this section, we'll show that some of these properties follow from the weaker assumption that the acceleration $\mathbf{a}(t)$ of each planet obeys the law of motion $m\mathbf{a} = \mathbf{F}$ with $\mathbf{F}$ being a central force. That is, the assumption that $\mathbf{F}$ is parallel to $\mathbf{r}$. The verification that the other properties follow from the specific form of the gravitational force, proportional to r^{-2}, will be delayed until the optional §1.10.

So, in this section, we assume that we have a parametrized curve $\mathbf{r}(t)$ and that this curve obeys

$$m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}(t) = \mathbf{F}\big(\mathbf{r}(t)\big)$$

where, for all $\mathbf{r} \in \mathbb{R}^3$, $\mathbf{F}(\mathbf{r})$ is parallel to $\mathbf{r}$. We shall show that

1. The path $\mathbf{r}(t)$ lies in a plane through the origin and that
2. the radius vector $\mathbf{r}$ sweeps out equal areas in equal times.

We'll start by trying to guess what the plane is. Pretend that we know that $\mathbf{r}(t)$ lies in a fixed plane through the origin. Then $\mathbf{v}(t) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)$ lies in the same plane and $\mathbf{r}(t) \times \mathbf{v}(t)$ is perpendicular to the plane. If our path really does lie in a fixed plane, $\mathbf{r}(t) \times \mathbf{v}(t)$ cannot change direction — it must always be parallel to the normal vector to the plane. So let's define

$$\mathbf{\Omega}(t) = \mathbf{r}(t) \times \mathbf{v}(t)$$

and check how it depends on time. By the product rule,

$$\frac{\mathrm{d}\mathbf{\Omega}}{\mathrm{d}t}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\big(\mathbf{r}(t) \times \mathbf{v}(t)\big) = \mathbf{v}(t) \times \mathbf{v}(t) + \mathbf{r}(t) \times \mathbf{a}(t) = \frac{1}{m}\mathbf{r}(t) \times \mathbf{F}\big(\mathbf{r}(t)\big) = \mathbf{0} \qquad \text{(A)}$$

because $\mathbf{r}(t)$ and $\mathbf{F}\big(\mathbf{r}(t)\big)$ are parallel. So $\mathbf{\Omega}(t)$ is[33] in fact independent of t. It is a constant vector that we'll just denote $\mathbf{\Omega}$.

As $\mathbf{r}(t) \times \mathbf{v}(t) = \mathbf{\Omega}$, we have that $\mathbf{r}(t)$ is always perpendicular to $\mathbf{\Omega}$ and

$$\mathbf{r}(t) \cdot \mathbf{\Omega} = 0$$

- If $\mathbf{\Omega} \neq \mathbf{0}$, this is exactly the statement that $\mathbf{r}(t)$ always lies in the plane through the origin with normal vector $\mathbf{\Omega}$.

32 Its value is about $6.67408 \times 10^{-11}\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{sec}^{-2}$.
33 Physicists call $m\mathbf{\Omega}(t)$ the angular momentum at time t and refer to (A) as (an example of) conservation of angular momentum. Conservation of angular momentum is exploited in gyro-compasses and by ice skaters (to spin faster/slower).

- If $\mathbf{\Omega} = \mathbf{0}$, then $\mathbf{r}(t)$ is always parallel to $\mathbf{v}(t)$ and there is some function $\alpha(t)$ such that

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t) = \mathbf{v}(t) = \alpha(t)\,\mathbf{r}(t)$$

  This is a first order, linear, ordinary differential equation that we can solve by using an integrating factor. Set

$$\beta(t) = \int_0^t \alpha(t)\,\mathrm{d}t$$

  Then

$$\begin{aligned}\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t) = \alpha(t)\,\mathbf{r}(t) &\iff e^{-\beta(t)}\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t) - \alpha(t)e^{-\beta(t)}\,\mathbf{r}(t) = 0\\ &\iff \frac{\mathrm{d}}{\mathrm{d}t}\big[e^{-\beta(t)}\mathbf{r}(t)\big] = 0\\ &\iff e^{-\beta(t)}\mathbf{r}(t) = \mathbf{r}(0)\\ &\iff \mathbf{r}(t) = e^{\beta(t)}\mathbf{r}(0)\end{aligned}$$

  so that $\mathbf{r}(t)$ lies on a line through the origin. This makes sense — the particle is always moving parallel to its radius vector.

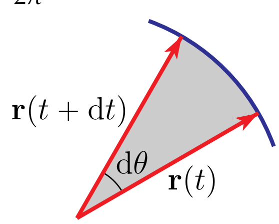
This completes the verification that $\mathbf{r}(t)$ lies in a plane through the origin.

Now we show that the radius vector $\mathbf{r}(t)$ sweeps out equal areas in equal times. In other words, we now verify that the rate at which $\mathbf{r}(t)$ sweeps out area is independent of time. To do so we rewrite the statement that $|\mathbf{r}(t) \times \mathbf{v}(t)|$ is constant in polar coordinates. Writing $\mathbf{r}(t) = r(t)\hat{\mathbf{r}}\big(\theta(t)\big)$ and then applying Lemma 1.8.2.b gives that

$$\text{constant} = \big|\mathbf{r}\times\mathbf{v}\big| = \Big|r\hat{\mathbf{r}} \times \Big(\frac{\mathrm{d}r}{\mathrm{d}t}\,\hat{\mathbf{r}} + r\,\frac{\mathrm{d}\theta}{\mathrm{d}t}\,\hat{\boldsymbol{\theta}}\Big)\Big| = r^2\frac{\mathrm{d}\theta}{\mathrm{d}t} \qquad \text{since} \quad |\hat{\mathbf{r}}\times\hat{\mathbf{r}}| = 0,\ |\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}}| = 1$$

is constant. It now suffices to observe that $r(t)^2\frac{\mathrm{d}\theta}{\mathrm{d}t}(t)$ is exactly twice the rate at which $\mathbf{r}(t)$ sweeps out area. To see this, just look at the figure below. The shaded area is essentially a wedge of a circular disk of radius r. (If $r(t)$ were independent of t, it would be exactly a wedge of a circular disk.) Its area is the fraction $\frac{\mathrm{d}\theta}{2\pi}$ of the area of the full disk, which is

$$\frac{\mathrm{d}\theta}{2\pi}\,\pi r^2 = \frac{1}{2}r^2\,\mathrm{d}\theta$$



1.10▲ Optional — Planetary Motion

We now return to the claim, made in §1.9 on central forces, that if $\mathbf{r}(t)$ obeys Newton's inverse square law

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = -\frac{GM}{r^3}\mathbf{r} = -\frac{GM}{r^2}\hat{\mathbf{r}}$$

then the curve obeys Kepler's laws

1. $\mathbf{r}(t)$ runs over an ellipse having one focus at the origin and
2. $\mathbf{r}(t)$ sweeps out equal areas in equal times and
3. the square of the period is proportional to the cube of the major axis of the ellipse.

We just showed, in §1.9, that the fact that $-\frac{GM}{r^3}\mathbf{r}$ is parallel to $\mathbf{r}$ implies that $\mathbf{r}(t)$ lies in a plane through the origin and sweeps out equal area in equal times. We now verify the remaining Kepler laws.

We start by just rewriting Newton's laws above in polar coordinates. We saw in Lemma 1.8.2.c, that if we write $\mathbf{r}(t) = r(t)\,\hat{\mathbf{r}}(t)$, then

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = \left(\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - r\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2\right)\hat{\mathbf{r}} + \left(r\,\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + 2\frac{\mathrm{d}r}{\mathrm{d}t}\,\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)\hat{\boldsymbol{\theta}} = -\frac{GM}{r^3}\mathbf{r} = -\frac{GM}{r^2}\hat{\mathbf{r}}$$

The $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ components of this equation are

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - r\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 = -\frac{GM}{r^2}$$

$$r\,\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + 2\frac{\mathrm{d}r}{\mathrm{d}t}\,\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0$$

The second of these two equations only tells us that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{r^2\,\frac{\mathrm{d}\theta}{\mathrm{d}t}\right\} = r\left\{r\,\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + 2\frac{\mathrm{d}r}{\mathrm{d}t}\,\frac{\mathrm{d}\theta}{\mathrm{d}t}\right\} = 0 \implies r^2\,\frac{\mathrm{d}\theta}{\mathrm{d}t} = h, \quad \text{a constant}$$

which we already knew. Substituting $\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{h}{r^2}$ into the first equation gives

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - \frac{h^2}{r^3} = -\frac{GM}{r^2} \tag{1.10.1}$$

This equations contains a lot of $\frac{1}{r}$'s. So let's set $u = \frac{1}{r}$. Furthermore, for the first of Kepler's laws, we really want r as a function of θ rather than t. So let's make u a function of θ and write

$$r(t) = \frac{1}{u(\theta(t))}$$

Then

$$\frac{\mathrm{d}r}{\mathrm{d}t}(t) = -\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\theta}\big(\theta(t)\big)\frac{\mathrm{d}\theta}{\mathrm{d}t}(t) = -h\frac{\mathrm{d}u}{\mathrm{d}\theta}\big(\theta(t)\big) \qquad \text{since } \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{h}{r^2} = hu^2$$
$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2}(t) = -h\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2}\big(\theta(t)\big)\frac{\mathrm{d}\theta}{\mathrm{d}t}(t) = -h^2 u\big(\theta(t)\big)^2\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2}\big(\theta(t)\big)$$

and our equation becomes

$$-h^2u^2\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} - h^2u^3 = -GMu^2 \qquad \text{or} \qquad \frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u = \frac{GM}{h^2} \tag{1.10.2}$$

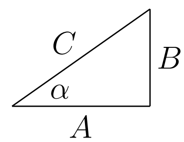
This is a second order, linear, ordinary differential equation with constant coefficients. Recall[34] that the general solution of such an equation is the sum of a "particular solution"

34 See Appendix I.

(i.e. any one solution, which in this case we can take to be the constant function $\frac{GM}{h^2}$) plus the general solution of the homogeneous equation $u'' + u = 0$, which one often writes as

$$A\cos\theta + B\sin\theta$$

with A and B arbitrary constants. In this particular application it is more convenient to write the solution in a different, standard but less commonly used, form. Namely, we can use the triangle



to write $A = C\cos\alpha$ and $B = C\sin\alpha$ so that the general solution of the homogeneous equation $u'' + u = 0$ becomes

$$C\cos\alpha\cos\theta + C\sin\alpha\sin\theta = C\cos(\theta - \alpha)$$

with C and α being arbitrary constants. So the general solution to (1.10.2) is

$$u(\theta) = \frac{GM}{h^2} + C\cos(\theta - \alpha)$$

and the general solution to (1.10.1) is

$$r(t) = \frac{1}{\frac{GM}{h^2} + C\cos(\theta(t) - \alpha)}$$

The angle α just shifts the zero point of our coordinate θ. By rotating our coordinate system by α, we can arrange that $\alpha = 0$ and then

$$r(t) = \frac{1}{\frac{GM}{h^2} + C\cos(\theta(t))} = \frac{\ell}{1 + \varepsilon\cos\theta} \qquad \text{with} \quad \ell = \frac{h^2}{GM}, \ \varepsilon = \frac{Ch^2}{GM}$$

As we saw in Example 1.8.4, this is exactly the equation of a conic section with eccentricity ε.

That leaves only the last of Kepler's laws, relating the period to the semi-major axis. As we are talking about planets, whose orbits remain bounded, our conic section must be a circle or ellipse, rather than a parabola or hyperbola. Looking back at Example 1.8.5, we see that the semi-major and semi-minor axes of our ellipse are

$$a = \frac{\ell}{1 - \varepsilon^2} \qquad b = \frac{\ell}{\sqrt{1 - \varepsilon^2}}$$

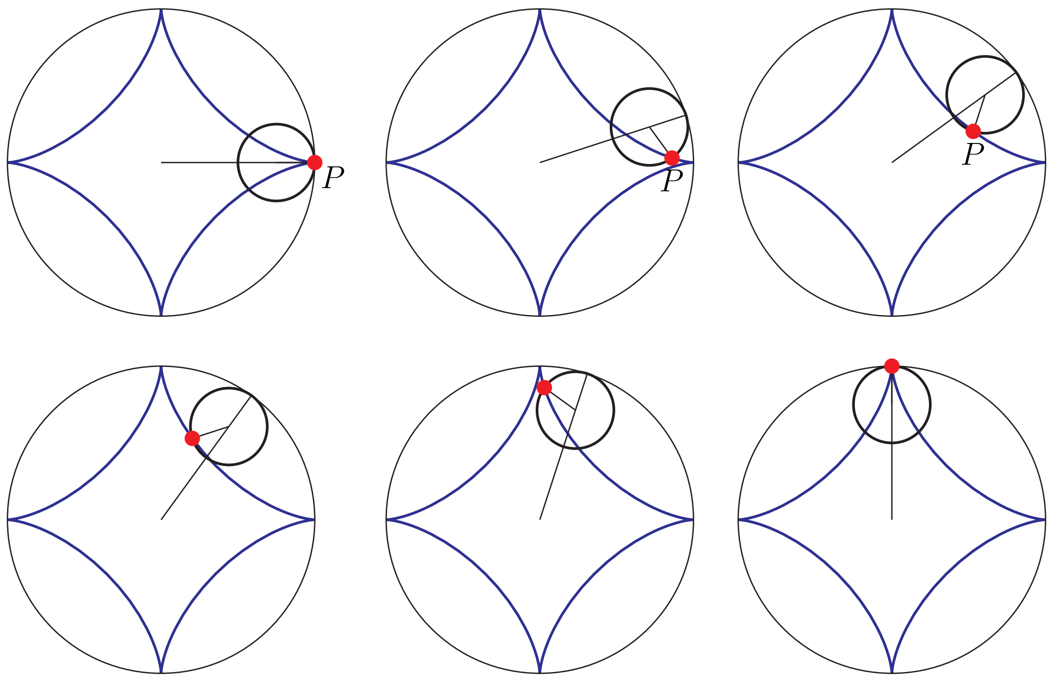
The period T of our orbit is just the length of time it takes the radius vector $\mathbf{r}(t)$ to sweep out the area of the ellipse[35], which is πab. As the rate at which the radius vector is sweeping out area is $\frac{1}{2}r^2\frac{d\theta}{dt} = \frac{h}{2}$, we have

$$T^2 = \left(\frac{\pi ab}{h/2}\right)^2 = \frac{4\pi^2a^2b^2}{h^2} = \frac{4\pi^2a^2b^2}{GM\ell} = \frac{4\pi^2}{GM}a^3 \qquad \text{since } \ell = \frac{b^2}{a}$$

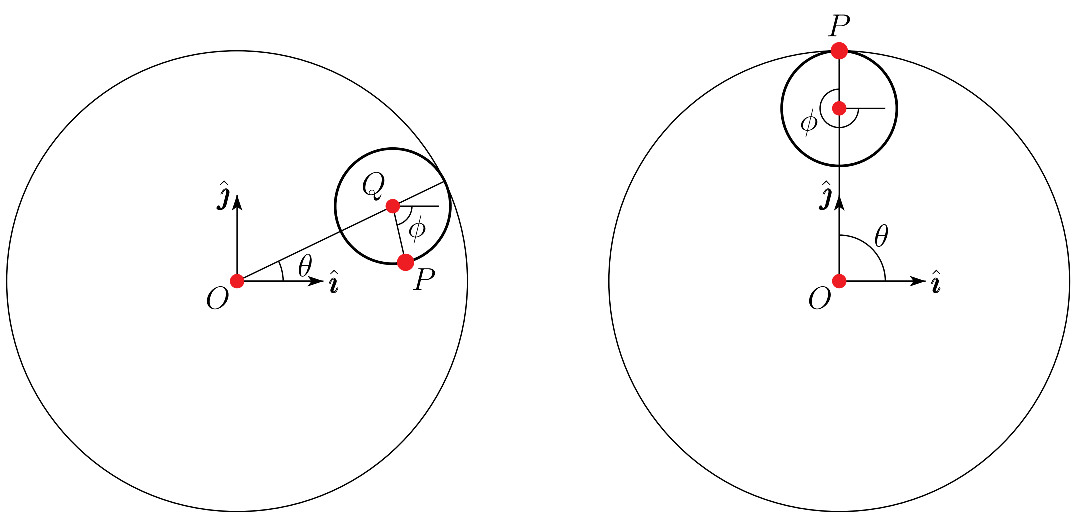
35 You probably computed the area of an ellipse in first year calculus. If not, you should be able to do it now in a few lines.

1.11▲ Optional — The Astroid

Imagine a ball of radius $a/4$ rolling around the inside of a circle of radius a. The curve traced by a point P painted on the inner circle (that's the blue curve in the figures below) is called an astroid[36]. We shall find its equation.



Define the angles θ and ϕ as in the figure in the left below.



36 The name "astroid" comes from the Greek word "aster", meaning star, with the suffix "oid" meaning "having the shape of". The curve was first discussed by Johann Bernoulli in 1691–92.

That is

- the vector from the centre, O, of the circle of radius a to the centre, Q, of the ball of radius $a/4$ is $\frac{3}{4}a\big(\cos\theta, \sin\theta\big)$ and
- the vector from the centre, Q, of the ball of radius $a/4$ to the point P is $\frac{1}{4}a\big(\cos\phi, -\sin\phi\big)$

As θ runs from 0 to $\frac{\pi}{2}$, the point of contact between the two circles travels through one quarter of the circumference of the circle of radius a, which is a distance $\frac{1}{4}(2\pi a)$, which, in turn, is exactly the circumference of the inner circle. Hence if $\phi = 0$ for $\theta = 0$ (i.e. if P starts on the x-axis), then for $\theta = \frac{\pi}{2}$, P is back in contact with the big circle at the north pole of both the inner and outer circles. That is, $\phi = \frac{3\pi}{2}$ when $\theta = \frac{\pi}{2}$. (See the figure on the right above.) So $\phi = 3\theta$ and P has coordinates

$$\frac{3}{4}a\big(\cos\theta, \sin\theta\big) + \frac{1}{4}a\big(\cos\phi, -\sin\phi\big) = \frac{a}{4}\big(3\cos\theta + \cos 3\theta, 3\sin\theta - \sin 3\theta\big)$$

As, recalling your double angle, or even better your triple angle, trig identities,

$$\begin{aligned}
\cos 3\theta &= \cos\theta\cos 2\theta - \sin\theta\sin 2\theta \\
&= \cos\theta[\cos^2\theta - \sin^2\theta] - 2\sin^2\theta\cos\theta \\
&= \cos\theta[\cos^2\theta - 3\sin^2\theta] \\
\sin 3\theta &= \sin\theta\cos 2\theta + \cos\theta\sin 2\theta \\
&= \sin\theta[\cos^2\theta - \sin^2\theta] + 2\sin\theta\cos^2\theta \\
&= \sin\theta[3\cos^2\theta - \sin^2\theta]
\end{aligned}$$

we have

$$\begin{aligned}
3\cos\theta + \cos 3\theta &= \cos\theta[3 + \cos^2\theta - 3\sin^2\theta] = \cos\theta[3 + \cos^2\theta - 3(1 - \cos^2\theta)] = 4\cos^3\theta \\
3\sin\theta - \sin 3\theta &= \sin\theta[3 - 3\cos^2\theta + \sin^2\theta] = \sin\theta[3 - 3(1 - \sin^2\theta) + \sin^2\theta] = 4\sin^3\theta
\end{aligned}$$

and the coordinates of P simplify to

$$x(\theta) = a\cos^3\theta \qquad y(\theta) = a\sin^3\theta$$

Oof! As $x^{2/3} + y^{2/3} = a^{2/3}\cos^2\theta + a^{2/3}\sin^2\theta$, the path traced by P obeys the equation

$$x^{2/3} + y^{2/3} = a^{2/3}$$

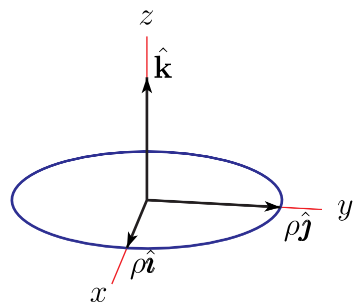
which is surprisingly simple, considering what we went through to get here.

There remains the danger that there could exist points (x, y) obeying the equation $x^{2/3} + y^{2/3} = a^{2/3}$ that are not of the form $x = a\cos^3\theta$, $y = a\sin^3\theta$ for any θ. That is, there is a danger that the parametrized curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ covers only a portion of $x^{2/3} + y^{2/3} = a^{2/3}$. We now show that the parametrized curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ in fact covers all of $x^{2/3} + y^{2/3} = a^{2/3}$ as θ runs from 0 to 2π.

First, observe that $x^{2/3} = \big(\sqrt[3]{x}\big)^2 \geqslant 0$ and $y^{2/3} = \big(\sqrt[3]{y}\big)^2 \geqslant 0$. Hence, if (x, y) obeys $x^{2/3} + y^{2/3} = a^{2/3}$, then necessarily $0 \leqslant x^{2/3} \leqslant a^{2/3}$ and so $-a \leqslant x \leqslant a$. As θ runs from 0 to 2π, $a\cos^3\theta$ takes all values between $-a$ and a and hence takes all possible values of x. For each $x \in [-a, a]$, y takes two values, namely $\pm[a^{2/3} - x^{2/3}]^{3/2}$. If $x = a\cos^3\theta_0 = a\cos^3(2\pi - \theta_0)$, the two corresponding values of y are precisely $a\sin^3\theta_0$ and $-a\sin^3\theta_0 = a\sin^3(2\pi - \theta_0)$.

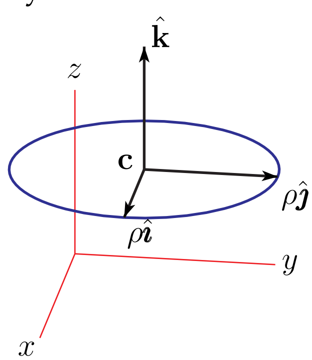
1.12▲ Optional — Parametrizing Circles

We now discuss a simple strategy for parametrizing circles in three dimensions, starting with the circle in the xy-plane that has radius ρ and is centred on the origin. This is easy to parametrize:



$$\mathbf{r}(t) = \rho\cos t\,\hat{\boldsymbol{\imath}} + \rho\sin t\,\hat{\boldsymbol{\jmath}} \qquad 0 \leqslant t < 2\pi$$

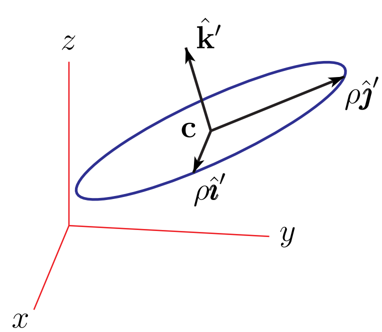
Now let's move the circle so that its centre is at some general point $\mathbf{c}$. To parametrize this new circle, which still has radius ρ and which is still parallel to the xy-plane, we just translate by $\mathbf{c}$:



$$\mathbf{r}(t) = \mathbf{c} + \rho\cos t\,\hat{\boldsymbol{\imath}} + \rho\sin t\,\hat{\boldsymbol{\jmath}} \qquad 0 \leqslant t < 2\pi$$

Finally, let's consider a circle in general position. The secret to parametrizing a general circle is to replace $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ by two new vectors $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$ which

(a) are unit vectors,
(b) are parallel to the plane of the desired circle and
(c) are mutually perpendicular.



$$\mathbf{r}(t) = \mathbf{c} + \rho\cos t\,\hat{\boldsymbol{\imath}}' + \rho\sin t\,\hat{\boldsymbol{\jmath}}' \qquad 0 \leqslant t < 2\pi$$

To check that this is correct, observe that

- $\mathbf{r}(t) - \mathbf{c}$ is parallel to the plane of the desired circle because both $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$ are parallel to the plane of the desired circle and $\mathbf{r}(t) - \mathbf{c} = \rho\cos t\,\hat{\boldsymbol{\imath}}' + \rho\sin t\,\hat{\boldsymbol{\jmath}}'$

- $\mathbf{r}(t) - \mathbf{c}$ is of length ρ for all t because

$$\begin{aligned}|\mathbf{r}(t) - \mathbf{c}\,|^2 &= (\mathbf{r}(t) - \mathbf{c}\,) \cdot (\mathbf{r}(t) - \mathbf{c}\,) \\ &= (\rho\cos t\,\hat{\boldsymbol{\imath}}' + \rho\sin t\,\hat{\boldsymbol{\jmath}}') \cdot (\rho\cos t\,\hat{\boldsymbol{\imath}}' + \rho\sin t\,\hat{\boldsymbol{\jmath}}') \\ &= \rho^2\cos^2 t\,\hat{\boldsymbol{\imath}}' \cdot \hat{\boldsymbol{\imath}}' + \rho^2\sin^2 t\,\hat{\boldsymbol{\jmath}}' \cdot \hat{\boldsymbol{\jmath}}' + 2\rho\cos t\sin t\,\hat{\boldsymbol{\imath}}' \cdot \hat{\boldsymbol{\jmath}}' \\ &= \rho^2(\cos^2 t + \sin^2 t) = \rho^2\end{aligned}$$

since $\hat{\boldsymbol{\imath}}' \cdot \hat{\boldsymbol{\imath}}' = \hat{\boldsymbol{\jmath}}' \cdot \hat{\boldsymbol{\jmath}}' = 1$ ($\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$ are both unit vectors) and $\hat{\boldsymbol{\imath}}' \cdot \hat{\boldsymbol{\jmath}}' = 0$ ($\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$ are perpendicular).

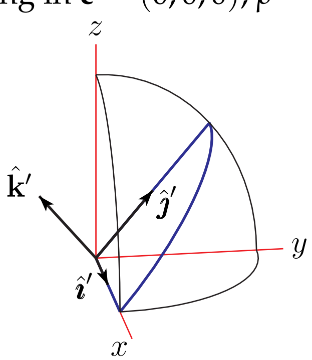
To find such a parametrization in practice, we need to find the centre $\mathbf{c}$ of the circle, the radius ρ of the circle and two mutually perpendicular unit vectors, $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$, in the plane of the circle. It is often easy to find at least one point $\mathbf{p}$ on the circle. Then we can take $\hat{\boldsymbol{\imath}}' = \frac{\mathbf{p}-\mathbf{c}}{|\mathbf{p}-\mathbf{c}|}$. It is also often easy to find a unit vector, $\hat{\mathbf{k}}'$, that is normal to the plane of the circle. Then we can choose $\hat{\boldsymbol{\jmath}}' = \hat{\mathbf{k}}' \times \hat{\boldsymbol{\imath}}'$. We'll illustrate this now.

Example 1.12.1

Let C be the intersection of the sphere $x^2 + y^2 + z^2 = 4$ and the plane $z = y$.

- The intersection of any plane with any sphere is a circle. The plane in question passes through the centre of the sphere, so C has the same centre and same radius as the sphere. So C has radius 2 and centre $(0, 0, 0)$.
- Notice that the point $(2, 0, 0)$ satisfies both $x^2 + y^2 + z^2 = 4$ and $z = y$ and so is on C. We may choose $\hat{\boldsymbol{\imath}}'$ to be the unit vector in the direction from the centre $(0, 0, 0)$ of the circle towards $(2, 0, 0)$. Namely $\hat{\boldsymbol{\imath}}' = (1, 0, 0)$.
- Since the plane of the circle is $z - y = 0$, the vector $\boldsymbol{\nabla}(z - y) = (0, -1, 1)$ is perpendicular to the plane of C. So we may take $\hat{\mathbf{k}}' = \frac{1}{\sqrt{2}}(0, -1, 1)$.
- Then $\hat{\boldsymbol{\jmath}}' = \hat{\mathbf{k}}' \times \hat{\boldsymbol{\imath}}' = \frac{1}{\sqrt{2}}(0, -1, 1) \times (1, 0, 0) = \frac{1}{\sqrt{2}}(0, 1, 1)$.

Substituting in $\mathbf{c} = (0, 0, 0)$, $\rho = 2$, $\hat{\boldsymbol{\imath}}' = (1, 0, 0)$ and $\hat{\boldsymbol{\jmath}}' = \frac{1}{\sqrt{2}}(0, 1, 1)$ gives



$$\begin{aligned}\mathbf{r}(t) &= 2\cos t\,(1, 0, 0) + 2\sin t\,\frac{1}{\sqrt{2}}(0, 1, 1) \\ &= 2\big(\cos t, \frac{\sin t}{\sqrt{2}}, \frac{\sin t}{\sqrt{2}}\big) \qquad 0 \leqslant t < 2\pi\end{aligned}$$

To check this, note that $x = 2\cos t$, $y = \sqrt{2}\sin t$, $z = \sqrt{2}\sin t$ satisfies both $x^2 + y^2 + z^2 = 4$ and $z = y$.

Example 1.12.1

Example 1.12.2

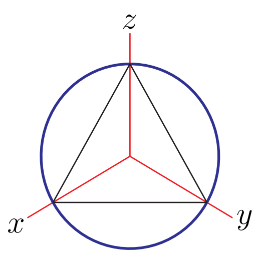
Let C be the circle that passes through the three points $(3, 0, 0)$, $(0, 3, 0)$ and $(0, 0, 3)$.

- All three points obey $x+y+z=3$. So the circle lies in the plane $x+y+z=3$. We guess, by symmetry, or by looking at the figure below, that the centre of the circle is at the centre of mass of the three points, which is $\frac{1}{3}[(3,0,0)+(0,3,0)+(0,0,3)] = (1,1,1)$. We must check this and can do so by checking that $(1,1,1)$ is equidistant from the three points:

$$\begin{aligned}
\big|(3,0,0)-(1,1,1)\big| &= \big|(2,-1,-1)\big| = \sqrt{6}\\
\big|(0,3,0)-(1,1,1)\big| &= \big|(-1,2,-1)\big| = \sqrt{6}\\
\big|(0,0,3)-(1,1,1)\big| &= \big|(-1,-1,2)\big| = \sqrt{6}
\end{aligned}$$

  This tells us both that $(1,1,1)$ is indeed the centre (as only the centre is equidistant from any three distinct points on a circle) and that the radius of C is $\sqrt{6}$.
- We may choose $\hat{\boldsymbol{\imath}}'$ to be the unit vector in the direction from the centre $(1,1,1)$ of the circle towards $(3,0,0)$. Namely $\hat{\boldsymbol{\imath}}' = \frac{1}{\sqrt{6}}(2,-1,-1)$.
- Since the plane of the circle is $x+y+z=3$, the vector $\boldsymbol{\nabla}(x+y+z) = (1,1,1)$ is perpendicular to the plane of C. So we may take $\hat{\mathbf{k}}' = \frac{1}{\sqrt{3}}(1,1,1)$.
- Then $\hat{\boldsymbol{\jmath}}' = \hat{\mathbf{k}}' \times \hat{\boldsymbol{\imath}}' = \frac{1}{\sqrt{18}}(1,1,1)\times(2,-1,-1) = \frac{1}{\sqrt{18}}(0,3,-3) = \frac{1}{\sqrt{2}}(0,1,-1)$.

Substituting in $\mathbf{c}=(1,1,1)$, $\rho=\sqrt{6}$, $\hat{\boldsymbol{\imath}}' = \frac{1}{\sqrt{6}}(2,-1,-1)$ and $\hat{\boldsymbol{\jmath}}' = \frac{1}{\sqrt{2}}(0,1,-1)$ gives



$$\begin{aligned}
\mathbf{r}(t) &= (1,1,1) + \sqrt{6}\cos t\,\frac{1}{\sqrt{6}}(2,-1,-1) + \sqrt{6}\sin t\,\frac{1}{\sqrt{2}}(0,1,-1)\\
&= \big(1+2\cos t, 1-\cos t+\sqrt{3}\sin t, 1-\cos t-\sqrt{3}\sin t\big)
\end{aligned}$$

To check this, note that $\mathbf{r}(0) = (3,0,0)$, $\mathbf{r}\big(\frac{2\pi}{3}\big) = (0,3,0)$ and $\mathbf{r}\big(\frac{4\pi}{3}\big) = (0,0,3)$ since $\cos\frac{2\pi}{3} = \cos\frac{4\pi}{3} = -\frac{1}{2}$, $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$ and $\sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$.

Example 1.12.2

Chapter 2

Vector Fields

2.1 ▲ Definitions and First Examples

In the last chapter, we studied vector valued functions of a single variable, like, for example, the velocity $\mathbf{v}(t)$ of a particle at time t. Suppose however that we are interested in a fluid. There is a, possibly different, velocity at each point in the fluid. So the velocity of a fluid is really a vector valued function of several variables. Such a function is called a vector field.

Definition 2.1.1.

(a) A vector field in the plane is a rule which assigns to each point (x, y) in a subset, D, of the xy-plane, a two component vector $\mathbf{v}(x, y)$.

(b) A vector field in space is a rule which assigns to each point (x, y, z) in a subset of $\mathbb{R}^3$, a three component vector $\mathbf{v}(x, y, z)$.

Here are two typical applications that naturally involve vector fields.

- If $\mathbf{v}(x, y, z)$ is the velocity of a moving fluid at position (x, y, z), then $\mathbf{v}$ is called a *velocity field*.
- If $\mathbf{F}(x, y, z)$ is the force at position (x, y, z), then $\mathbf{F}$ is called[1] a *force field*.

Example 2.1.2 (The Point Source)

Imagine

- ○ The whole world is filled with an incompressible fluid. Call it water.

1 No, force fields are not only a sci-fi trope. Gravity is an example of a force field.

- Somehow you find a way to produce still more water at the origin. Say you create $4\pi m$ litres per second.
- This forces the water to flow outward. Let's suppose that it flows symmetrically outward from the origin.

Let's find the resulting vector field $\mathbf{v}(x, y, z)$. As the flow is to be symmetric, the velocity of the water at the point (x, y, z)

- has to be pointing radially outward from the origin. That is, the direction of the velocity vector $\mathbf{v}(x, y, z)$ has to be the unit radial vector

$$\hat{\mathbf{r}}(x, y, z) = \frac{x\hat{\mathbf{\imath}} + y\hat{\mathbf{\jmath}} + z\hat{\mathbf{k}}}{\sqrt{x^2 + y^2 + z^2}}$$

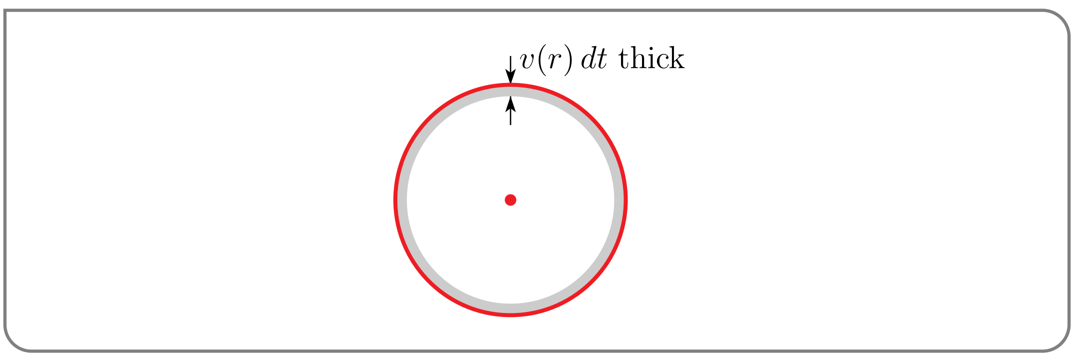
- The magnitude of the velocity, i.e. the speed $|\mathbf{v}(x, y, z)|$ of the water, has to depend only on the distance from the origin. That is, the speed can only be some function of

$$r(x, y, z) = \sqrt{x^2 + y^2 + z^2}$$

Thus the velocity field is of the form

$$\mathbf{v}(x, y, z) = v\big(r(x, y, z)\big)\,\hat{\mathbf{r}}(x, y, z)$$

We just have to determine the function $v(r)$. Fix any $r > 0$ and concentrate on the sphere $x^2 + y^2 + z^2 = r^2$. It is sketched in red in the figure below. During a very short time



interval dt seconds, $4\pi m\,dt$ litres of water is created at the origin (which is the red dot). As the water is incompressible, $4\pi m\,dt$ litres of water must exit through the sphere during the same time interval to make room for the newly created water.

But, at the surface of the sphere the water is flowing radially outward with speed $v(r)$. So during the time interval in question the water near the surface of the sphere moves outward a distance $v(r)\,dt$, and in particular the water that was in the thin spherical shell $r - v(r)\,dt \leqslant \sqrt{x^2 + y^2 + z^2} \leqslant r$ at the beginning of the time interval exits through the sphere $\sqrt{x^2 + y^2 + z^2} = r$ during the time interval. The shell is sketched in gray in the figure above. The volume of water in the gray shell is essentially the surface area of the shell, which is $4\pi r^2$, times the thickness of the shell, which is $v(r)\,dt$. So, equating the volume of water created inside the sphere with the volume of water that exited the sphere,

$$4\pi m\,dt = (4\pi r^2)\big(v(r)\,dt\big) \implies v(r) = \frac{4\pi m}{4\pi r^2} = \frac{m}{r^2}$$

Thus our vector field is

$$\mathbf{v}(x,y,z) = \frac{m}{r(x,y,z)^2}\,\hat{\mathbf{r}}(x,y,z)$$

If the world were two, rather than three dimensional[2], and the source created $2\pi m$ litres per second, the same argument leads to

$$2\pi m\,dt = (2\pi r)\big(v(r)\,dt\big) \implies v(r) = \frac{2\pi m}{2\pi r} = \frac{m}{r}$$

and to the vector field

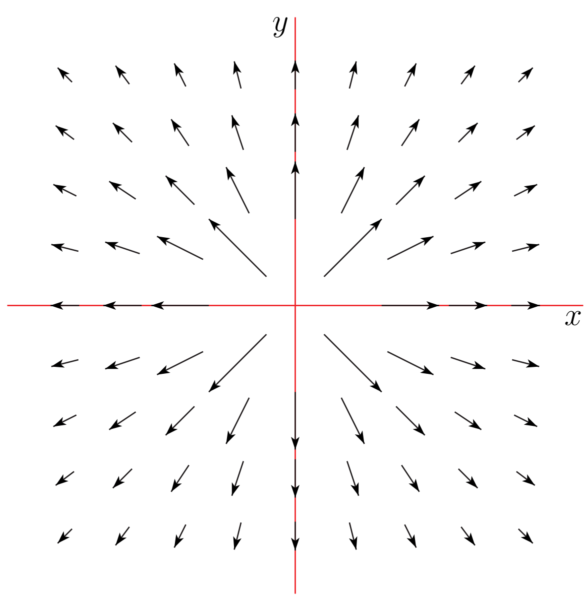
$$\mathbf{v}(x,y) = \frac{m}{r(x,y)}\,\hat{\mathbf{r}}(x,y) \qquad r(x,y) = \sqrt{x^2+y^2} \qquad \hat{\mathbf{r}}(x,y) = \frac{x\hat{\imath}+y\hat{\jmath}}{\sqrt{x^2+y^2}}$$

To get a mental image of what this field looks like, imagine sketching, for each point (x,y), the vector $\frac{m}{r(x,y)}\,\hat{\mathbf{r}}(x,y)$ with its tail at (x,y). Note that the vector $\frac{m}{r(x,y)}\,\hat{\mathbf{r}}(x,y)$

- points radially outward and
- has length $\frac{m}{r(x,y)}$ which
  - depends only on $r = |(x,y)|$ and
  - is very long when (x,y) is near the origin and
  - decreases in length like $\frac{1}{r}$ as $r = |(x,y)|$ increases.

Here is a sketch of a bunch of such vectors.

Figure 2.1.1.



vector field $\vec{v} = m\frac{\hat{\mathbf{r}}}{r}$

2 You might want to think about what happens in d dimensions for general d.

Note that as $|(x,y)| \to 0$, the magnitude of the velocity $|\mathbf{v}(x,y)| \to \infty$. This is a consequence of our idealized assumption that we are producing water at a single point (the origin).

Example 2.1.2

Example 2.1.3 (The Vortex)

In this example, we sketch the vector field

$$\mathbf{v}(x,y) = \Omega\big(-y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}}\big)$$

where Ω is just a strictly positive constant. We give an efficient procedure for getting a rough sketch, which still provides a pretty realistic picture of the vector field, and which also generalises to other vector fields. First concentrate on the horizontal component $\hat{\boldsymbol{\imath}} \cdot \mathbf{v}(x,y)$ of the vector field and determine in which part of the xy-plane it is zero, in which part it is positive and in which part it is negative.

$$\hat{\boldsymbol{\imath}} \cdot \mathbf{v}(x,y) = -\Omega y \begin{cases} = 0 & \text{if } y = 0 \\ < 0 & \text{if } y > 0 \\ > 0 & \text{if } y < 0 \end{cases}$$

Next repeat with the vertical component.

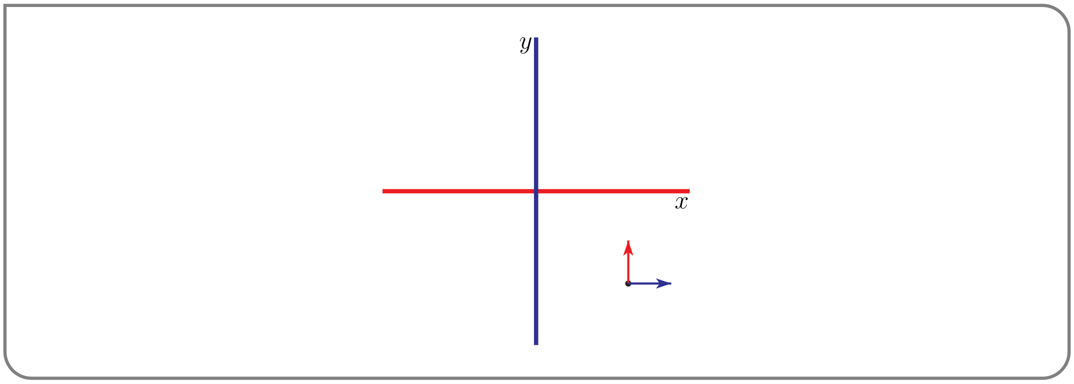
$$\hat{\boldsymbol{\jmath}} \cdot \mathbf{v}(x,y) = \Omega x \begin{cases} = 0 & \text{if } x = 0 \\ < 0 & \text{if } x < 0 \\ > 0 & \text{if } x > 0 \end{cases}$$

This naturally divides the xy-plane into nine parts according to whether each of the components is positive, 0 or negative —

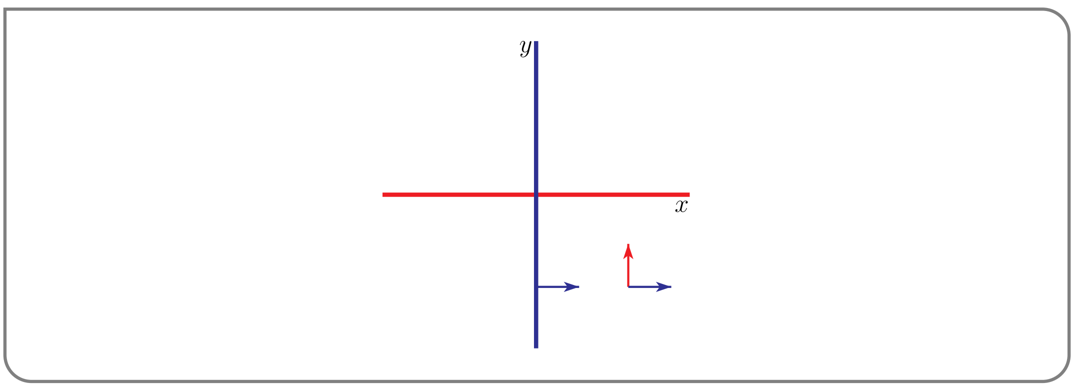
- $\hat{\boldsymbol{\imath}} \cdot \mathbf{v} > 0$ and $\hat{\boldsymbol{\jmath}} \cdot \mathbf{v} > 0$ in $\big\{ (x,y) \in \mathbb{R}^2 \,\big|\, y < 0,\ x > 0 \big\}$
- $\hat{\boldsymbol{\imath}} \cdot \mathbf{v} > 0$ and $\hat{\boldsymbol{\jmath}} \cdot \mathbf{v} = 0$ in $\big\{ (x,y) \in \mathbb{R}^2 \,\big|\, y < 0,\ x = 0 \big\}$
- $\hat{\boldsymbol{\imath}} \cdot \mathbf{v} > 0$ and $\hat{\boldsymbol{\jmath}} \cdot \mathbf{v} < 0$ in $\big\{ (x,y) \in \mathbb{R}^2 \,\big|\, y < 0,\ x < 0 \big\}$
- $\hat{\boldsymbol{\imath}} \cdot \mathbf{v} = 0$ and $\hat{\boldsymbol{\jmath}} \cdot \mathbf{v} > 0$ in $\big\{ (x,y) \in \mathbb{R}^2 \,\big|\, y = 0,\ x > 0 \big\}$
- and so on

Now think of $\mathbf{v}(x,y)$ as being the velocity at (x,y) of a flowing fluid.

- Look at the first bullet point above. It says that in the first of the nine parts, namely $\big\{ (x,y) \in \mathbb{R}^2 \,\big|\, y < 0,\ x > 0 \big\}$, which is the fourth quadrant, the horizontal component $\hat{\boldsymbol{\imath}} \cdot \mathbf{v} > 0$ signifying that the fluid is flowing rightwards. Indicate this in the sketch by drawing a rightward pointing horizontal arrow at some generic point in the middle of the fourth quadrant. (It's the blue arrow in the figure below.) The vertical component $\hat{\boldsymbol{\jmath}} \cdot \mathbf{v} > 0$ signifying that the fluid is also moving upwards. Indicate this in the sketch by drawing an upward pointing vertical arrow at the same generic point in the fourth quadrant. (It's the red arrow in the figure below.)



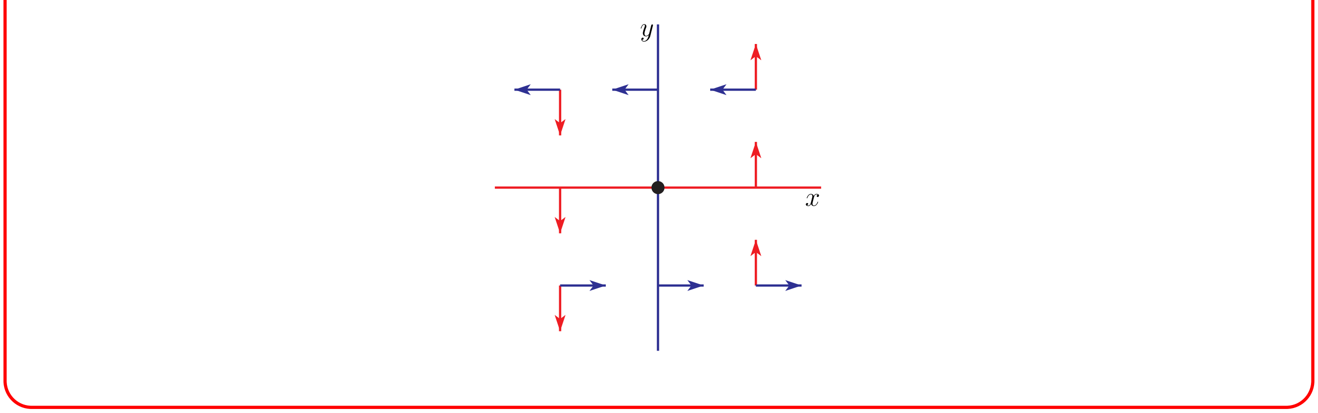
- Next, look at the second bullet point above. It says that on the second of the nine parts, namely $\{ (x,y) \in \mathbb{R}^2 \mid y < 0,\ x = 0 \}$, which is the bottom half of the y-axis, the horizontal component $\hat{\imath} \cdot \mathbf{v} > 0$, signifying that the fluid is moving rightwards. Indicate this in the sketch by drawing a rightward pointing horizontal arrow at some generic point in the middle of the bottom half of the y-axis. (It's the second blue arrow in the figure below.) The vertical component $\hat{\jmath} \cdot \mathbf{v} = 0$ signifying that the fluid has no vertical motion at all. Indicate this in the sketch by not drawing any vertical arrow on the bottom half of the y-axis.



- and so on

By the time we have looked at all nine regions we will have built up the following sketch.

Figure 2.1.2.

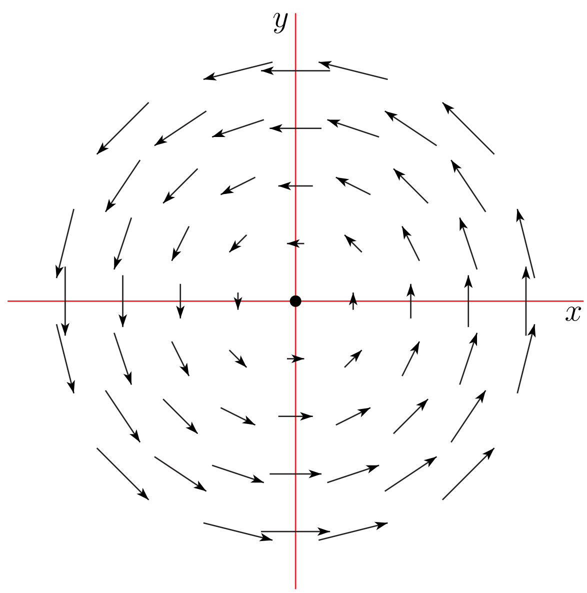


From this sketch we see that, for example, in the first quadrant,

- the fluid is moving upwards and to the left and
- the fluid crosses the x-axis vertically (so that close to the x-axis, the arrows will be almost vertical) and
- the fluid crosses the y-axis horizontally (so that close to the y-axis, the arrows will be almost horizontal) and
- there is one point, namely $(0,0)$, where the vector field is exactly zero. It's the black dot in the centre of the figure above. Furthermore $\mathbf{v}(x,y) = \Omega(-y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}})$ is smaller when (x,y) is closer to $(0,0)$ and $\mathbf{v}(x,y)$ is larger when (x,y) is farther from $(0,0)$,

Putting all of this accumulated wisdom together, we come up with this better sketch of the vector field.

Figure 2.1.3.



vector field $\mathbf{v} = \Omega(-y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}})$

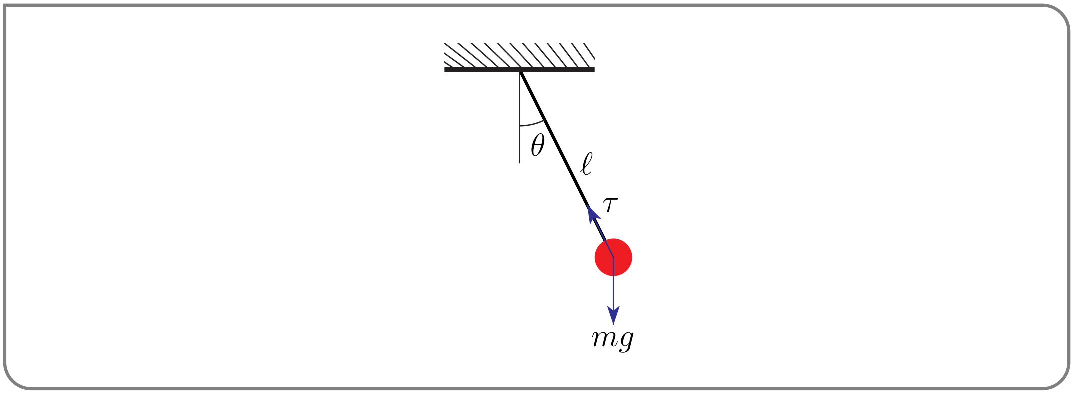
This shows the field swirling around the origin in a counterclockwise direction. Hence the name "vortex".

Example 2.1.3

Example 2.1.4 (The Undamped Nonlinear Pendulum)

In this example, we illustrate another way in which vector fields arise. Model a pendulum by a mass m that is connected to a hinge by an idealized rod that is massless[3] and of fixed length ℓ. Denote by θ the angle between the rod and vertical. The forces acting on the

3 While we are idealizing, let's put everything in a vacuum.



mass are

- gravity and
- the tension in the rod, whose magnitude, τ, automatically adjusts itself so that the distance between the mass and the hinge is fixed at ℓ.

In the optional[4] Section 2.5, we show that the angle $\theta(t)$ obeys the second order nonlinear[5] differential equation

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{\ell}\sin\theta = 0$$

It is often much more convenient to deal with first order, rather than second order, differential equations. The second order pendulum equation above may be reformulated[6] as a system of first order ordinary differential equations, by the simple expedient of defining

$$x(t) = \theta(t) \qquad y(t) = \theta'(t)$$

So $x(t)$ is the angle at time t and $y(t)$ is the angular velocity at time t. Then,

$$\begin{aligned} x'(t) &= \theta'(t) = y(t) \\ y'(t) &= \theta''(t) = -\frac{g}{\ell}\sin x(t) \end{aligned}$$

Usually, one does not write in the (t) dependence explicitly.

$$\begin{aligned} x' &= y \\ y' &= -\frac{g}{\ell}\sin x \end{aligned}$$

The right hand sides form the vector field

$$\mathbf{v}\big((x,y)\big) = \Big(y\,,\,-\frac{g}{\ell}\sin x\Big)$$

4 In the optional Section 2.5 we also include frictional forces. In this example, we do not, so set the β of Section 2.5 to zero here.
5 It is common, when considering only small amplitude oscillations, to approximate $\sin\theta$ by θ. This converts our nonlinear differential equation into a linear differential equation.
6 This "hack" generalizes easily and is commonly used when generating, by computer, approximate solutions to higher order ordinary differential equations.

We can sketch this vector field, just as we sketched the vector field of Example 2.1.3. Noting that the horizontal component

$$\hat{\boldsymbol{\imath}}\cdot\mathbf{v}(x,y) = y \ \begin{cases} = 0 & \text{if } y = 0 \\ > 0 & \text{if } y > 0 \\ < 0 & \text{if } y < 0 \end{cases}$$

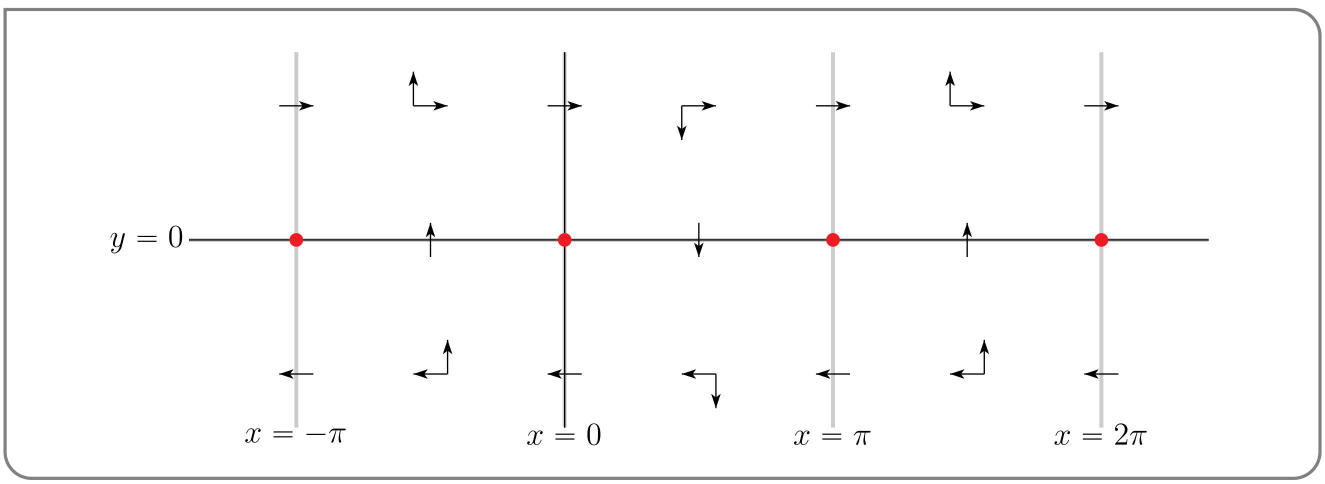
and the vertical component.

$$\hat{\boldsymbol{\jmath}}\cdot\mathbf{v}(x,y) = -\frac{g}{\ell}\sin x \ \begin{cases} = 0 & \text{if } x = 0,\ \pm\pi, \pm 2\pi, \cdots \\ > 0 & \text{if } -\pi < x < 0,\ \pi < x < 2\pi, \text{etc.} \\ < 0 & \text{if } 0 < x < \pi,\ \ 2\pi < x < 3\pi, \text{etc.} \end{cases}$$

we have

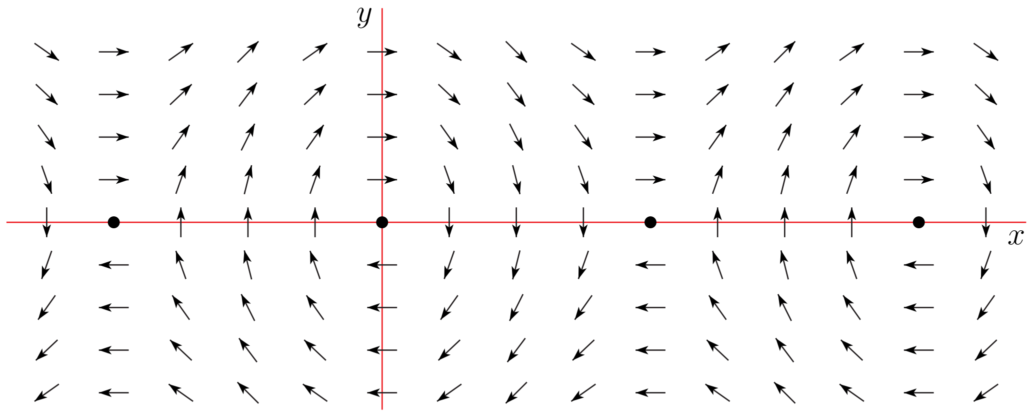
- ◦ rightward motion[7] when $y > 0$
- ◦ leftward motion when $y < 0$
- ◦ downward motion when $0 < x < \pi,\ 2\pi < x < 3\pi, \cdots$ and
- ◦ upward motion when $-\pi < x < 0,\ \pi < x < 2\pi, \cdots$.

This gives us the collection of arrows in the figure



Our full sketch will be less cluttered if we make all arrows the same length. This gives

7 Note that this is rightward motion of the point (x, y), not of the pendulum itself.



direction field for $x' = y,\ y' = -2\sin x$

which is a sketch of what is called the direction field of our vector field (see below).

In the next section, we'll learn how to use vector field sketches to sketch solution trajectories.

Example 2.1.4

Definition 2.1.5.

The direction field of a vector field $\mathbf{v}(x,y,z)$ is the vector field

$$\mathbf{V}(x,y,z) = \begin{cases} \frac{\mathbf{v}(x,y,z)}{|\mathbf{v}(x,y,z)|} & \text{if } \mathbf{v}(x,y,z) \neq \mathbf{0} \\ \mathbf{0} & \text{if } \mathbf{v}(x,y,z) = \mathbf{0} \end{cases}$$

2.2▲ Optional — Field Lines

Suppose that we drop a tiny stick into a river[8] with the velocity field of the flowing water being $\mathbf{v}(x,y)$. We are assuming, for simplicity, that the velocity field does not depend[9] on time t. The stick will move along with the water[10]. When the stick is at $\mathbf{r}$, its velocity will be the same as the velocity of the water at $\mathbf{r}$, which is $\mathbf{v}(\mathbf{r})$. Thus if the stick is at $\mathbf{r}(t)$ at time t, we will have

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}\big(\mathbf{r}(t)\big)$$

The stick will trace out a path, parametrized by $\mathbf{r}(t)$.

8 Think Poohsticks.
9 This is not such an unreasonable assumption. The flow often changes on a larger time scale.
10 This is also not an unreasonable approximation.

Definition 2.2.1.

A path that is parametrized by a function $\mathbf{r}(t)$ that obeys

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{v}\big(\mathbf{r}(t)\big)$$

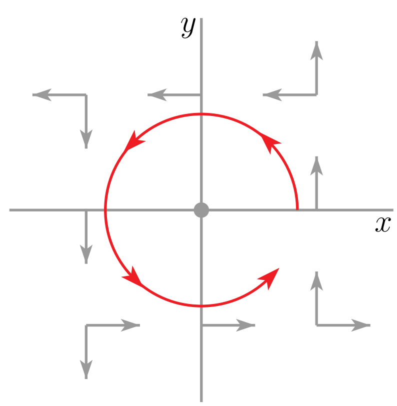
is called a

- field line or integral curve (for general vector fields) or a
- stream line or flow line (when the vector field $\mathbf{v}$ is being thought of as a velocity field) or a
- line of force (when the vector field $\mathbf{v}$ is being thought of as a force field)

of the vector field $\mathbf{v}$.

Example 2.2.2 (Flow Line Sketch for the Vortex of Example 2.1.3)

Consider the vortex vector field, $\mathbf{v}(x, y) = \Omega\big(-y\hat{\mathbf{i}} + x\hat{\mathbf{j}}\big)$ of Example 2.1.3. Once we sketched the vector field, as in Figure 2.1.3, or even made the "skeleton sketch" of Figure 2.1.2, we can get rough idea of what the stream lines look like just by following the arrows. For example, suppose that we start a stream line (i.e. drop the stick into the stream) on the positive x-axis. Looking at Figure 2.1.2, which is repeated here,



the stick

- starts by moving in the $+y$ direction, i.e. straight upward.
- As it moves farther into the first quadrant it develops a larger and larger negative x-component of velocity. So it also moves leftwards toward the y-axis.
- Eventually it crosses the positive y-axis moving in the $-x$ direction, i.e. to the left.
- As it moves farther into the second quadrant it develops a larger and larger negative y-component of velocity. So it also moves downwards toward the x-axis.
- Eventually it crosses the negative x-axis moving in the $-y$ direction, i.e. straight downward.

- As it moves farther into the third quadrant it develops a larger and larger positive x-component of velocity. So it also moves rightward towards the y-axis.
- Eventually it crosses the negative y-axis moving in the $+x$ direction, i.e. to the right.
- As it moves farther into the fourth quadrant it develops a larger and larger positive y-component of velocity. So it also moves upwards toward the x-axis.

With this type of analysis we cannot tell if the streamline, which is the red line in the figure above, will return to the x-axis

- exactly at its starting point, forming a closed curve, or
- inside its starting point, spiralling inwards, or
- outside its starting point, spiralling outwards.

Example 2.2.2

While the above procedure is a good way to get a qualitative feel for trajectories, we can develop more precise, detailed descriptions of field lines by working analytically. As we saw above, thinking of $\mathbf{r}(t)$ as the position at time t of a stick dropped into water whose velocity at (x, y) is $\mathbf{v}(x, y)$, the velocity of the stick at time t will be the same as the velocity of the water at $\mathbf{r}(t)$, which is $\mathbf{v}\big(\mathbf{r}(t)\big)$. Thus $\mathbf{r}(t)$ will obey the system of first order differential equations

Equation 2.2.3.

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t) = \mathbf{v}\big(\mathbf{r}(t)\big)$$

Notice that if we reparametrize $\mathbf{r}(t)$, say to $\mathbf{R}(u) = \mathbf{r}\big(t(u)\big)$, then $\mathbf{R}'(u) = \mathbf{r}'\big(t(u)\big)\, t'(u)$ is parallel to (though not necessarily equal to) $\mathbf{r}'\big(t(u)\big) = \mathbf{v}\big(\mathbf{r}((t(u))\big) = \mathbf{v}\big(\mathbf{R}(u)\big)$. So if we only care about the curve traced out by the stick, and not about *when* the stick is at each point of the path, then it suffices to impose the weaker condition[11] that, when the stick is at $\mathbf{r}(t)$, its velocity $\mathbf{r}'(t)$ is parallel to (though not necessarily equal to) $\mathbf{v}\big(\mathbf{r}(t)\big)$. In three dimensions, $\mathbf{r}'(t)$ is parallel to $\mathbf{v}\big(\mathbf{r}(t)\big)$ when the cross product is zero:

Equation 2.2.4.

$$\mathbf{r}'(t) \times \mathbf{v}\big(\mathbf{r}(t)\big) = \mathbf{0}$$

In two dimensions we can still use the cross product by the simple expedient of thinking of $\mathbf{r}'(t)$ and $\mathbf{v}\big(\mathbf{r}(t)\big)$ as three component vectors whose third components are zero.

A more convenient way to implement the weaker "just parallel" condition, involves reparametrizing our streamline. Suppose that we are in two dimensions with $\mathbf{r}'(t) = \big(\frac{\mathrm{d}x}{\mathrm{d}t}(t), \frac{\mathrm{d}y}{\mathrm{d}t}(t)\big)$ and $\mathbf{v}(\mathbf{r}) = \big(v_1(\mathbf{r}), v_2(\mathbf{r})\big)$ and fix some t_0. If $\frac{\mathrm{d}x}{\mathrm{d}t}(t_0)$ is nonzero[12], we can reparametrize the curve (at least near $\mathbf{r}(t_0)$) so as to use x, rather than t as the parameter. To do so, we

11 We'll have a more careful discussion of this in the optional §2.2.1.

12 If $\frac{\mathrm{d}x}{\mathrm{d}t}(t_0) = 0$, but $\frac{\mathrm{d}y}{\mathrm{d}t}(t_0) \neq 0$, we should use y rather than x as the parameter. If $\frac{\mathrm{d}x}{\mathrm{d}t}(t_0) = \frac{\mathrm{d}y}{\mathrm{d}t}(t_0) = 0$, then $\mathbf{r}(t) = \mathbf{r}(t_0)$ for all t and the streamline doesn't move. It is just a single point.

- solve $x = x(t)$ for t as a function of x. Call the solution $T(x)$. Then
- the point on the curve which has x-coordinate x is $\mathbf{R}(x) = \big(X(x)\,,\,Y(x)\big)$ with $X(x) = x$ and $Y(x) = y\big(T(x)\big)$.

Then the condition that $\mathbf{R}'(x) = \big(1, Y'(x)\big)$ is parallel to $\mathbf{v}\big(\mathbf{R}(x)\big)$ says that $\mathbf{R}'(x)$ is a scalar multiple of $\mathbf{v}\big(\mathbf{R}(x)\big)$ so that there is a nonzero number $c(x)$ so that $\mathbf{R}'(x) = c(x)\mathbf{v}\big(\mathbf{R}(x)\big)$. That is

$$\big(1, Y'(x)\big) = \big(c(x)v_1\big(x, Y(x)\big)\,,\, c(x)v_2\big(x, Y(x)\big)\big)$$

or equivalently

$$Y'(x) = \frac{Y'(x)}{1} = \frac{c(x)\,v_2\big(x, Y(x)\big)}{c(x)\,v_1\big(x, Y(x)\big)\big)} = \frac{v_2\big(x, Y(x)\big)}{v_1\big(x, Y(x)\big)}$$

This is exactly the statement that $y = Y(x)$ is a solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}(x) = \frac{v_2(x, y)}{v_1(x, y)}$$

It is conventional to pretend[13] that $\frac{\mathrm{d}y}{\mathrm{d}x}$ is the ratio of $\mathrm{d}y$ and $\mathrm{d}x$ and rewrite the differential equation[14] as

$$\frac{\mathrm{d}x}{v_1(x, y)} = \frac{\mathrm{d}y}{v_2(x, y)}$$

Here is a summary of the discussion we have just completed. It extends to three dimensions in an obvious way.

13 Of course $\frac{\mathrm{d}y}{\mathrm{d}x}$ is not the ratio of $\mathrm{d}y$ and $\mathrm{d}x$. However pretending that it is provides a simple way to remember the technique that is used to solve the equation. You may have used this mnemonic device before when you learned how to solve separable differential equations. Section 2.4 of the CLP-2 text contains a treatment of separable differential equations, including a justification for the mnemonic device.

14 Here is another nonrigorous, but intuitive way to come up with this equation. Suppose that our stick is at (x, y) and has velocity $\big(\frac{\mathrm{d}x}{\mathrm{d}t}(t)\,,\,\frac{\mathrm{d}y}{\mathrm{d}t}(t)\big)$. In a tiny time interval $\mathrm{d}t$ the stick moves by $\big(\frac{\mathrm{d}x}{\mathrm{d}t}(t)\,,\,\frac{\mathrm{d}y}{\mathrm{d}t}(t)\big)\mathrm{d}t = (\mathrm{d}x, \mathrm{d}y)$, which is parallel to $\big(v_1(x, y)\,,\,v_2(x, y)\big)$ if $\frac{\mathrm{d}x}{v_1(x,y)} = \frac{\mathrm{d}y}{v_2(x,y)}$.

Equation 2.2.5.

Use the symbol $\parallel$ to stand for "is parallel to".
In two dimensions

$$\begin{aligned}&\Big(\frac{\mathrm{d}x}{\mathrm{d}t}(t)\,,\,\frac{\mathrm{d}y}{\mathrm{d}t}(t)\Big) \parallel \big(v_1(\mathbf{r}(t))\,,\,v_2(\mathbf{r}(t))\big)\\ &\qquad\iff \Big(\frac{\mathrm{d}x}{\mathrm{d}t}(t)\,,\,\frac{\mathrm{d}y}{\mathrm{d}t}(t)\,,\,0\Big)\times\big(v_1(\mathbf{r}(t))\,,\,v_2(\mathbf{r}(t))\,,\,0\big)=\mathbf{0}\\ &\qquad\iff \frac{dx}{v_1(x,y)}=\frac{dy}{v_2(x,y)}\end{aligned}$$

and in three dimensions

$$\begin{aligned}&\Big(\frac{\mathrm{d}x}{\mathrm{d}t}(t)\,,\,\frac{\mathrm{d}y}{\mathrm{d}t}(t)\,,\,\frac{\mathrm{d}z}{\mathrm{d}t}(t)\Big) \parallel \big(v_1(\mathbf{r}(t))\,,\,v_2(\mathbf{r}(t))\,,\,v_3(\mathbf{r}(t))\big)\\ &\qquad\iff \Big(\frac{\mathrm{d}x}{\mathrm{d}t}(t)\,,\,\frac{\mathrm{d}y}{\mathrm{d}t}(t)\,,\,\frac{\mathrm{d}z}{\mathrm{d}t}(t)\Big)\times\big(v_1(\mathbf{r}(t))\,,\,v_2(\mathbf{r}(t))\,,\,v_3(\mathbf{r}(t))\big)=\mathbf{0}\\ &\qquad\iff \frac{dx}{v_1(x,y,z)}=\frac{dy}{v_2(x,y,z)}=\frac{dz}{v_3(x,y,z)}\end{aligned}$$

Let us apply this to two examples, in which the stream lines of the vortex field of Example 2.1.3 are found by two different methods.

Example 2.2.6 (Stream lines for the vortex field using $\mathbf{r}'(t) \parallel \mathbf{v}(\mathbf{r}(t))$)

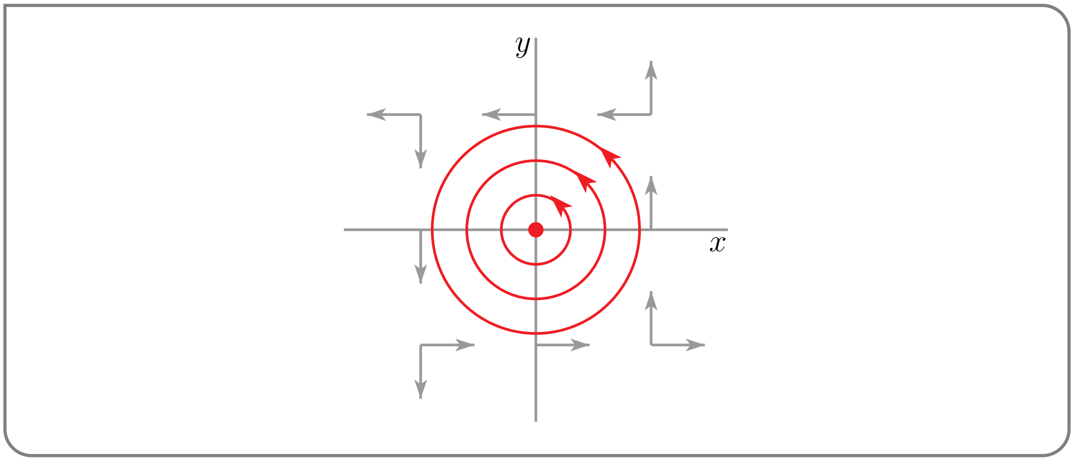
In this example we will find the stream lines for the vortex field, $\mathbf{v}(x,y)=\Omega\big(-y\hat{\boldsymbol{\imath}}+x\hat{\boldsymbol{\jmath}}\big)$ of Example 2.1.3, by using the requirement that, on a stream line, the velocity vector $\mathbf{r}'(t)$ must be parallel to $\mathbf{v}\big(\mathbf{r}(t)\big)$. By (2.2.5) one way to express this requirement mathematically is

$$\frac{dx}{-\Omega y}=\frac{dy}{\Omega x}$$

This is a simple separable differential equation. We can solve it by cross multiplying and integrating both sides. (Recall that Ω is a constant.)

$$\begin{aligned}\Omega x\,\mathrm{d}x=-\Omega y\,\mathrm{d}y &\iff \Omega\int x\,\mathrm{d}x=-\Omega\int y\,\mathrm{d}y\\ &\iff \tfrac{1}{2}\Omega x^2=-\tfrac{1}{2}\Omega y^2+C'\\ &\iff x^2+y^2=C\end{aligned}$$

where C' and $C=\frac{2}{\Omega}C'$ are just arbitrary constants. So the stream lines of the vortex field are exactly circles centred on the origin.



We can come to exactly the same conclusion by using the cross product formulation of (2.2.4).

$$\begin{aligned}
&\Big(\frac{\mathrm{d}x}{\mathrm{d}t}(t)\,,\ \frac{\mathrm{d}y}{\mathrm{d}t}(t)\,,\ 0\Big) \times \big(v_1(\mathbf{r}(t))\,,\ v_2(\mathbf{r}(t))\,,\ 0\big) = \mathbf{0} \\
&\iff \Big(\frac{\mathrm{d}x}{\mathrm{d}t}(t)\,\hat{\imath} + \frac{\mathrm{d}y}{\mathrm{d}t}(t)\,\hat{\jmath}\Big) \times \big(-\Omega y(t)\,\hat{\imath} + \Omega x(t)\,\hat{\jmath}\big) = \mathbf{0} \\
&\iff \Big(\Omega x(t)\frac{\mathrm{d}x}{\mathrm{d}t}(t) + \Omega y(t)\frac{\mathrm{d}y}{\mathrm{d}t}(t)\Big)\hat{\mathbf{k}} = \mathbf{0} \\
&\iff \Omega x(t)\frac{\mathrm{d}x}{\mathrm{d}t}(t) + \Omega y(t)\frac{\mathrm{d}y}{\mathrm{d}t}(t) = 0 \\
&\iff \frac{\mathrm{d}}{\mathrm{d}t}\Big(\tfrac{1}{2}\Omega x(t)^2 + \tfrac{1}{2}\Omega y(t)^2\Big) = 0 \qquad \text{(Go ahead and evaluate the derivative.)} \\
&\iff \tfrac{1}{2}\Omega\big(x(t)^2 + y(t)^2\big) = C' \\
&\iff x(t)^2 + y(t)^2 = C
\end{aligned}$$

Example 2.2.6

Example 2.2.7 (Stream lines for the vortex field using $\mathbf{r}'(t) = \mathbf{v}(\mathbf{r}(t))$)

This time we will find the stream lines for the vortex field, $\mathbf{v}(x, y) = \Omega\big(-y\hat{\imath} + x\hat{\jmath}\big)$ of Example 2.1.3, by using (2.2.3), which is

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= -\Omega y \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= \Omega x
\end{aligned}$$

We can convert this system of first order linear ordinary differential equations into a single second order linear constant coefficient differential equation[15], by differentiating the first

15 In Example 2.1.4 we converted a second order ordinary differential equation into a system of first order ordinary differential equations. We are now just reversing the procedure we used there.

equation, to get $\frac{d^2x}{dt^2} = -\Omega\frac{dy}{dt}$, and then substituting in the second equation to get

$$\frac{d^2x}{dt^2} + \Omega^2 x = 0$$

This equation is a special case of the ordinary differential equation treated in Example I.3 of the Appendix I, entitled "Review of Linear Ordinary Differential Equations". In fact it is exactly ($I.5_h$) with $R = 0$, $L = C = \frac{1}{\Omega}$. So the general solution is (I.7) with $\rho = 0$ and $\nu = \Omega$, which is

$$x(t) = A\cos(\Omega t - \theta)$$

with A and θ being arbitrary constants[16]. Then

$$y(t) = -\frac{1}{\Omega}\frac{dx}{dt} = A\sin(\Omega t - \theta)$$

giving us the familiar circular stream lines.

Example 2.2.7

2.2.1 ▸▸ More about $\mathbf{r}'(t) \times \mathbf{v}\big(\mathbf{r}(t)\big) = \mathbf{0}$

Here is a lemma that gives a more precise version of "if we only care about the curve traced out by the stick, and not about when the stick is at each point of the path, then it suffices to impose the weaker condition $\mathbf{r}'(t) \times \mathbf{v}\big(\mathbf{r}(t)\big) = \mathbf{0}$".

Lemma 2.2.8.

Lat $a < b$ and let $\mathbf{v}(\mathbf{r})$ be a vector field. Assume that, for all $a < u < b$, $\mathbf{R}(u)$ is defined, both $\mathbf{R}'(u)$ and $\mathbf{v}\big(\mathbf{R}(u)\big)$ are continuous and nonzero and

$$\mathbf{R}'(u) \times \mathbf{v}\big(\mathbf{R}(u)\big) = \mathbf{0}$$

Then $\big\{\, \mathbf{R}(u) \,\big|\, a < u < b \,\big\}$ is contained in a field line.

Proof. As $\mathbf{R}'(u) \times \mathbf{v}\big(\mathbf{R}(u)\big) = \mathbf{0}$ and both $\mathbf{R}'(u)$ and $\mathbf{v}\big(\mathbf{R}(u)\big)$ are nonzero, there is an $a(u)$ such that

$$\mathbf{R}'(u) = a(u)\,\mathbf{v}\big(\mathbf{R}(u)\big)$$

This $a(u) = \frac{\mathbf{R}'(u)\cdot\mathbf{v}(\mathbf{R}(u))}{\mathbf{v}(\mathbf{R}(u))\cdot\mathbf{v}(\mathbf{R}(u))}$ is necessarily nonzero and continuous. Since $a(u)$ is nonzero and continuous, it never changes sign. That is, either $a(u) > 0$ for all u, or $a(u) < 0$ for all u. Let $T(u)$ be an antiderivative of $a(u)$. Then $T(u)$ is strictly monotone (and continuous) and hence is invertible. That is, there is a continuous function $U(t)$ that obeys $U\big(T(u)\big) = u$ for all $a < u < b$ and $T\big(U(t)\big) = t$ for all t in the range of U. Differentiating

16 Even if you don't know how $x(t) = A\cos(\Omega t - \theta)$ was arrived at, you should be able to easily verify that it really does obey $x'' + \Omega^2 x = 0$.

$T\big(U(t)\big) = t$ gives $T'\big(U(t)\big)\, U'(t) = 1$ and hence $U'(t) = \frac{1}{T'(U(t))}$. Set $\mathbf{r}(t) = \mathbf{R}\big(U(t)\big)$. Then

$$\begin{aligned}\mathbf{r}'(t) &= \mathbf{R}'\big(U(t)\big)U'(t) = a\big(U(t)\big)\,\mathbf{v}\big(\mathbf{R}\big(U(t)\big)\big)\frac{1}{T'\big(U(t)\big)} = a\big(U(t)\big)\,\mathbf{v}\big(\mathbf{r}(t)\big)\frac{1}{a\big(U(t)\big)}\\ &= \mathbf{v}\big(\mathbf{r}(t)\big)\end{aligned}$$

So $\mathbf{r}(t)$ is a field line and $\mathbf{R}(u) = \mathbf{r}(T(u))$ is a reparametrization of $\mathbf{r}(t)$. □

Here are a couple of examples that show that bad things can happen if we drop the requirement that $\mathbf{v}(\mathbf{R}(u))$ is nonzero.

Example 2.2.9

Let the vector field $\mathbf{v}(x, y)$ be identically zero. Then any field line $\big(x(t)\,,\, y(t)\big)$ must obey

$$x'(t) = 0 \qquad y'(t) = 0$$

which forces both $x(t)$ and $y(t)$ to be constants. So each field line is just a single point. On the other hand every nonconstant $\mathbf{R}(u)$ obeys $\mathbf{R}'(u) \times \mathbf{v}\big(\mathbf{R}(u)\big) = \mathbf{0}$ but is not contained in a field line. (As $\mathbf{R}(u)$ is not constant, it covers more than one point, while each field line is just a single point.)

Example 2.2.9

Now here is a more interesting example.

Example 2.2.10

Consider the vector field $\mathbf{v}(x, y) = x\,\hat{\boldsymbol{\imath}}$. This vector field takes the value $\mathbf{0}$ at each point on the y-axis, is a positive multiple of $\hat{\boldsymbol{\imath}}$ at every point of the right half-plane and is a negative multiple of $\hat{\boldsymbol{\imath}}$ at every point of the left half-plane. Let's find the field lines. Any field line must obey

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = x(t) \qquad \frac{\mathrm{d}y}{\mathrm{d}t}(t) = 0$$

So $y(t)$ must be a constant. We can solve the linear ordinary differential equation $\frac{\mathrm{d}x}{\mathrm{d}t}(t) = x(t)$ by moving the $x(t)$ to the left hand side, and multiplying by the (integrating factor) e^{-t}. This gives

$$e^{-t}\frac{\mathrm{d}x}{\mathrm{d}t}(t) - e^{-t}x(t) = 0$$

By the product rule, this is the same as

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(e^{-t}x(t)\big) = 0$$

which forces $e^{-t}x(t)$ to be a constant. So our field lines are $\big(Ce^t\,,\, D\big)$, with C and D being arbitrary constants. Note that

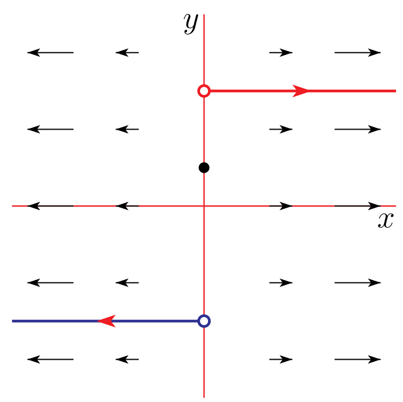
- if $C = 0$, the field line is just the single point $(0, D)$ on the y-axis. It is illustrated by the black dot in the figure below.
- If $C > 0$, then as t runs from $-\infty$ to $+\infty$, the field line covers the horizontal half-line

  $$\{ (x, D) \mid x > 0 \}$$

  in the right half-plane. It is illustrated by the red line in the figure below.
- If $C < 0$, then as t runs from $-\infty$ to $+\infty$, the field line covers the horizontal half-line

  $$\{ (x, D) \mid x < 0 \}$$

  in the left half-plane. It is illustrated by the blue line in the figure below (with a different value of D than for the red line).



vector field $\mathbf{v} = x\,\hat{\boldsymbol{\imath}}$

On the other hand, fix any constant D and set $\mathbf{R}(u) = u\hat{\boldsymbol{\imath}} + D\hat{\boldsymbol{\jmath}}$. Then

$$\mathbf{R}'(u) \times \mathbf{v}\big(\mathbf{R}(u)\big) = \hat{\boldsymbol{\imath}} \times \big(u\hat{\boldsymbol{\imath}}\big) = \mathbf{0}$$

But as u runs from $-\infty$ to $+\infty$, $\mathbf{R}(u)$ runs over the full line $\{ (x, D) \mid -\infty < x < \infty \}$. It is not contained in any single field line and, in fact, completely covers three different field lines.

Example 2.2.10

2.3▲ Conservative Vector Fields

Not all vector fields are created equal. In particular, some vector fields are easier to work with than others. One important class of vector fields that are relatively easy to work with, at least sometimes, but that still arise in many applications are "conservative vector fields".

Definition 2.3.1.

(a) The vector field $\mathbf{F}$ is said to be *conservative* if there exists a function φ such that $\mathbf{F} = \boldsymbol{\nabla}\varphi$. Then φ is called a *potential* for $\mathbf{F}$. Note that if φ is a potential for $\mathbf{F}$ and if C is a constant, then $\varphi + C$ is also a potential for $\mathbf{F}$.

(b) If $\mathbf{F} = \boldsymbol{\nabla}\varphi$ is a conservative field with potential φ and if C is a constant, then the set of points that obey $\varphi(x, y, z) = C$ is called an equipotential surface. Similarly, in two dimensions, the set of points that obey $\varphi(x, y) = C$ is called an equipotential curve.

Warning 2.3.2.

In physics, when a vector field is of the form $\mathbf{F} = -\boldsymbol{\nabla}\varphi$, then φ is called a potential for $\mathbf{F}$. *Note* the minus[17] sign in $\mathbf{F} = \underset{\uparrow}{-}\boldsymbol{\nabla}\varphi$.

Example 2.3.3 (Potential energy)

The "conservative" in "conservative vector field" has nothing to do with politics. It comes from "conservation of energy". Here is how. Suppose that you have a particle of mass m moving in a force field $\mathbf{F}$ that happens to be of the form $\mathbf{F} = \boldsymbol{\nabla}\varphi$ for some function φ. If the position of the particle a time t is $\big(x(t), y(t), z(t)\big)$, then, by Newton's law of motion,

$$
\begin{aligned}
m\mathbf{a} = \mathbf{F} \qquad &\Longrightarrow \qquad m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}(t) = \mathbf{F}\big(x(t), y(t), z(t)\big) \\
&\Longrightarrow \qquad m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}(t) = \boldsymbol{\nabla}\varphi\big(x(t), y(t), z(t)\big)
\end{aligned}
$$

Now dot both sides with $\mathbf{v}(t)$.

$$
\begin{aligned}
\Longrightarrow \qquad m\,\mathbf{v}(t)\cdot\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}(t) &= \mathbf{v}(t)\cdot\boldsymbol{\nabla}\varphi\big(x(t), y(t), z(t)\big) \\
&= x'(t)\frac{\partial\varphi}{\partial x}\big(x(t), y(t), z(t)\big) + y'(t)\frac{\partial\varphi}{\partial y}\big(x(t), y(t), z(t)\big) \\
&\qquad + z'(t)\frac{\partial\varphi}{\partial z}\big(x(t), y(t), z(t)\big)
\end{aligned}
$$

Next use $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}\cdot\mathbf{v} = 2\mathbf{v}\cdot\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$ on the left hand side and the chain rule on the right hand side.

$$
\begin{aligned}
&\Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}m\mathbf{v}(t)\cdot\mathbf{v}(t)\Big) = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\varphi\big(x(t), y(t), z(t)\big)\Big) \\
&\Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}m\mathbf{v}(t)\cdot\mathbf{v}(t) - \varphi\big(x(t), y(t), z(t)\big)\Big) = 0 \\
&\Longrightarrow \quad \frac{1}{2}m|\mathbf{v}(t)|^2 - \varphi\big(x(t), y(t), z(t)\big) = \text{const}
\end{aligned}
$$

17 Physicists introduce this minus sign in order to eliminate the minus sign in the next footnote.

So $\frac{1}{2}m|\mathbf{v}(t)|^2 - \varphi\big(x(t),y(t),z(t)\big)$, which is called the energy[18] of the particle at time t, does not actually depend on time — it is conserved. Let's call the initial energy E. That is, $E = \frac{1}{2}m|\mathbf{v}(0)|^2 - \varphi\big(x(0),y(0),z(0)\big)$. Then $\frac{1}{2}m|\mathbf{v}(t)|^2 - \varphi\big(x(t),y(t),z(t)\big) = E$ for all t and, in particular

$$\varphi\big(x(t),y(t),z(t)\big) = \frac{1}{2}m|\mathbf{v}(t)|^2 - E \geqslant -E$$

So even without having to find $\big(x(t)\,,\,y(t)\,,\,z(t)\big)$, we know that our particle can never escape the region $\{\,(x,y,z)\mid \varphi(x,y,z)\geqslant -E\,\}$.

Example 2.3.3

Example 2.3.4 (Gravity)

The gravitational force that a body of mass M at the origin exerts on a body of mass m at $\mathbf{r} = (x,y,z)$ is

$$\mathbf{F}(\mathbf{r}) = -\frac{GMm}{r^3}\mathbf{r}$$

where $r = |\mathbf{r}| = \sqrt{x^2+y^2+z^2}$ and G is the gravitational constant. This force is conservative, with potential $\varphi(\mathbf{r}) = \frac{GMm}{r}$. To verify that this is correct, observe that

$$\begin{aligned}
\frac{\partial}{\partial x}\varphi(\mathbf{r}) &= \frac{\partial}{\partial x}\frac{GMm}{\sqrt{x^2+y^2+z^2}} = -\frac{1}{2}\frac{GMm(2x)}{[x^2+y^2+z^2]^{3/2}} &= -\frac{GMm}{r^3}x\\
\frac{\partial}{\partial y}\varphi(\mathbf{r}) &= \frac{\partial}{\partial y}\frac{GMm}{\sqrt{x^2+y^2+z^2}} = -\frac{1}{2}\frac{GMm(2y)}{[x^2+y^2+z^2]^{3/2}} &= -\frac{GMm}{r^3}y\\
\frac{\partial}{\partial z}\varphi(\mathbf{r}) &= \frac{\partial}{\partial z}\frac{GMm}{\sqrt{x^2+y^2+z^2}} = -\frac{1}{2}\frac{GMm(2z)}{[x^2+y^2+z^2]^{3/2}} &= -\frac{GMm}{r^3}z
\end{aligned}$$

Example 2.3.4

We have already found conservation of energy very helpful a couple of times in Section 1.7 (Sliding on a Curve). So, at this point, there are probably several questions gnawing away at you.

- Is every vector field conservative?
- If not, is there an easy way to tell whether or not a vector field is conservative?
- If we know that a given vector field is conservative, how do you find a potential for it?

Have no fear. We will consider those questions in some detail shortly. But first, a couple of more examples.

Example 2.3.5

In this example we will show that the vector field $\mathbf{F}(x,y) = x\,\hat{\imath} - y\,\hat{\jmath}$ is conservative and find both its potential and its field lines.

18 $\frac{1}{2}m|\mathbf{v}(t)|^2$ is the kinetic energy and $-\varphi$ is the potential energy. See Warning 2.3.2.

(a) *The potential:* Our vector field $\mathbf{F}(x,y) = x\,\hat{\imath} - y\,\hat{\jmath}$ is conservative if we can find a φ obeying

$$\frac{\partial\varphi}{\partial x}(x,y) = x$$
$$\frac{\partial\varphi}{\partial y}(x,y) = -y$$

Recall that, when taking the partial derivative $\frac{\partial}{\partial x}$ the coordinate y is viewed as a constant. So the first of these equations is satisfied if and only if there is a $\psi(y)$, which does not depend on x, so that

$$\varphi(x,y) = \frac{x^2}{2} + \psi(y)$$

For this to also satisfy the second equation, we need

$$-y = \frac{\partial\varphi}{\partial y}(x,y) = \frac{\partial}{\partial y}\Big(\frac{x^2}{2} + \psi(y)\Big) = \psi'(y)$$

which is the case if and only if there is a constant C with

$$\psi(y) = -\frac{y^2}{2} + C$$

So, for any choice of the constant C,

$$\frac{x^2}{2} - \frac{y^2}{2} + C$$

is a potential. In particular, taking $C = 0$, one possible potential is

$$\varphi(x,y) = \frac{x^2}{2} - \frac{y^2}{2}$$

Some equipotential curves for this potential are sketched in (c) below. They are the blue curves.

(b) *The field lines (Optional):* Recalling (2.2.5), the field lines of the vector field $\mathbf{F}(x,y) = x\,\hat{\imath} - y\,\hat{\jmath}$ are determined by

$$\begin{aligned} \frac{\mathrm{d}x}{x} = \frac{\mathrm{d}y}{-y} &\iff -y\mathrm{d}x = x\mathrm{d}y \\ &\iff x\mathrm{d}y + y\mathrm{d}x = 0 \\ &\iff \mathrm{d}(xy) = 0 \qquad \text{by the product rule} \\ &\iff xy = C \end{aligned}$$

for some constant C. If you are not comfortable with the use of the product rule above, here is another way to write the same computation.

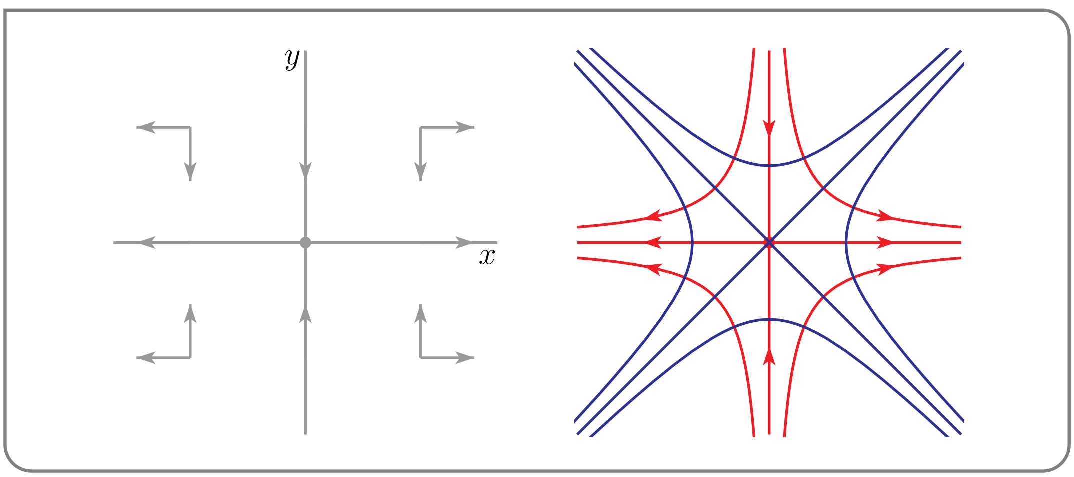
$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x} &\iff x\frac{\mathrm{d}y}{\mathrm{d}x} = -y \\
&\iff x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0 \\
&\iff \frac{\mathrm{d}}{\mathrm{d}x}(xy) = 0 \qquad \text{by the product rule} \\
&\iff xy = C
\end{aligned}$$

Some field lines are sketched in (c) below. They are the red curves. Note that they appear to cross the equipotential curves, the blue curves, at right angles. We shall see in Lemma 2.3.6, below, that this is not a coincidence. Also note that, while the above computation tells what the field lines are, it does not give us the direction of motion along the field lines. We determine the direction of motion next.

(c) *Direction of motion (Optional):* The sign data

$$\hat{\boldsymbol{\imath}} \cdot \mathbf{F}(x,y) = x \begin{Bmatrix} > 0 & \text{if } x > 0 \\ = 0 & \text{if } x = 0 \\ < 0 & \text{if } x < 0 \end{Bmatrix} \qquad \hat{\boldsymbol{\jmath}} \cdot \mathbf{F}(x,y) = -y \begin{Bmatrix} > 0 & \text{if } y < 0 \\ = 0 & \text{if } y = 0 \\ < 0 & \text{if } y > 0 \end{Bmatrix}$$

is visually displayed in the figure on the left below. The arrows in the figure on the left gives us the direction of motion along the field lines (in red) in the figure on the right below. Some equipotential curves are also sketched (in blue) in the figure on the right below.



Example 2.3.5

We have just seen one example of a conservative vector field for which the field lines appear to cross the equipotential curves at right angles. Here is a result which says that that was no accident. The field lines of conservative fields always cross the equipotential surfaces at right angles.

Lemma 2.3.6 (Optional).

If **F** is a conservative vector field, then the field lines of **F** are perpendicular to the equipotential surfaces of **F**.

Proof. Let $\mathbf{F} = \nabla\varphi$. Pick any point $\mathbf{r}_0$ and any nonzero vector $\mathbf{T}$ that is tangent to the equipotential surface at $\mathbf{r}_0$. That equipotential surface is $\varphi(x,y,z) = \varphi(\mathbf{r}_0)$. Consider any curve $\mathbf{r}(t) = \big(x(t), y(t), z(t)\big)$ that

- lies in the equipotential surface of **F** through $\mathbf{r}_0$, so that $\varphi\big(\mathbf{r}(t)\big) = \varphi(\mathbf{r}_0)$ for all t, and also obeys
- $\mathbf{r}(0) = \mathbf{r}_0$ and
- $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(0) = \mathbf{T}$.

Differentiating $\varphi\big(\mathbf{r}(t)\big) = \varphi(\mathbf{r}_0)$ with respect to t and applying the chain rule gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[\varphi\big(x(t),y(t),z(t)\big)\big] = 0$$

$$\frac{\partial\varphi}{\partial x}\big(x(t),y(t),z(t)\big)\frac{\mathrm{d}x}{\mathrm{d}t}(t) + \frac{\partial\varphi}{\partial y}\big(x(t),y(t),z(t)\big)\frac{\mathrm{d}y}{\mathrm{d}t}(t) + \frac{\partial\varphi}{\partial z}\big(x(t),y(t),z(t)\big)\frac{\mathrm{d}z}{\mathrm{d}t}(t) = 0$$

Notice that the left hand side is exactly the dot product of $\big(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\big) = \nabla\varphi$ with $\big(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t}\big) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$. So

$$\nabla\varphi\big(\mathbf{r}(t)\big)\cdot\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t) = 0$$

$$\mathbf{F}\big(\mathbf{r}(t)\big)\cdot\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t) = 0$$

Then set $t = 0$ to get

$$\mathbf{F}\big(\mathbf{r}_0\big)\cdot\mathbf{T} = 0$$

This says that the vector **T** (which is tangent to the equipotential surface at $\mathbf{r}_0$) is perpendicular to the vector field at $\mathbf{r}_0$ (which is a tangent vector to the field line of **F** through $\mathbf{r}_0$). □

Here is another example in which we try to find a potential for a vector field.

Example 2.3.7

Let's try to find a potential for the vortex vector field $\mathbf{v}(x,y) = \Omega\big(-y\hat{\imath} + x\hat{\jmath}\big)$ of Example 2.1.3. The potential would have to obey

$$\frac{\partial\varphi}{\partial x}(x,y) = -\Omega y$$

$$\frac{\partial\varphi}{\partial y}(x,y) = \Omega x$$

We proceed just as we did in Example 2.3.5. The first of these equations is satisfied if and only if there is a $\psi(y)$, which does not depend on x, so that

$$\varphi(x,y) = -\Omega xy + \psi(y)$$

For this to also satisfy the second equation, we need

$$\Omega x = \frac{\partial \varphi}{\partial y}(x,y) = \frac{\partial}{\partial y}\Big(-\Omega xy + \psi(y)\Big) = -\Omega x + \psi'(y) \iff \psi'(y) = 2\Omega x$$

If $\Omega \neq 0$, the right hand side of this equation depends on x while the left hand side in independent of x, no matter what ψ is. So no ψ can work, and $\mathbf{v}(x,y) = \Omega\big(-y\hat{\imath} + x\hat{\jmath}\big)$ is not conservative.

Example 2.3.7

The previous example shows that not all vector fields are conservative. That answers the first of the questions that we posed just before Example 2.3.5. The next theorem provides a simple screening test for conservativeness, which partially answers the second question. The easy way to remember the screening test uses the curl, which we now define.

Definition 2.3.8.

The curl of a vector field $\mathbf{F}(x,y,z)$ is denoted by $\boldsymbol{\nabla}\times\mathbf{F}(x,y,z)$ and is defined by

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{F} &= \Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)\hat{\imath} + \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big)\hat{\jmath} + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\hat{\mathbf{k}} \\ &= \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1(x,y,z) & F_2(x,y,z) & F_3(x,y,z)\end{bmatrix}\end{aligned}$$

The determinant in the second row is really just a mnemonic device used to make it easy to remember the expression after the equals sign in the first row. One must be careful about the signs in this definition — the determinant helps with that.

Theorem 2.3.9 (Screening test for conservative vector fields.).

(a) Assume that $F_1(x,y)$ and $F_2(x,y)$ are continuously differentiable. If the vector field $F_1(x,y)\hat{\imath} + F_2(x,y)\hat{\jmath}$ is conservative, then we must have

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

(b) Assume that $F_1(x,y,z)$, $F_2(x,y,z)$ and $F_3(x,y,z)$ are continuously differentiable. If the vector field $F_1(x,y,z)\hat{\imath} + F_2(x,y,z)\hat{\jmath} + F_3(x,y,z)\hat{\mathbf{k}}$ is conservative, then

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x} \qquad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$$

Equivalently,

$$\nabla \times \mathbf{F} = \Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)\hat{\imath} + \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big)\hat{\jmath} + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\hat{\mathbf{k}} = \mathbf{0}$$

That is, **F** is curl free.

Proof. (a) If the vector field $F_1(x,y)\hat{\imath} + F_2(x,y)\hat{\jmath}$ is conservative, then there is a potential $\varphi(x,y)$ such that

$$\frac{\partial \varphi}{\partial x}(x,y) = F_1(x,y)$$
$$\frac{\partial \varphi}{\partial y}(x,y) = F_2(x,y)$$

Applying $\frac{\partial}{\partial y}$ to the first equation and $\frac{\partial}{\partial x}$ to the second equation gives

$$\frac{\partial^2 \varphi}{\partial y \partial x} = \frac{\partial F_1}{\partial y}$$
$$\frac{\partial^2 \varphi}{\partial x \partial y} = \frac{\partial F_2}{\partial x}$$

Recall that, for any twice continuously differentiable function, $\frac{\partial^2 \varphi}{\partial y \partial x} = \frac{\partial^2 \varphi}{\partial x \partial y}$. So the two left hand sides are equal, and the two right hand sides must also be equal.
(b) If the vector field $F_1(x,y,z)\hat{\imath} + F_2(x,y,z)\hat{\jmath} + F_3(x,y,z)\hat{\mathbf{k}}$ is conservative, then there is a potential $\varphi(x,y,z)$ such that

$$\frac{\partial \varphi}{\partial x}(x,y,z) = F_1(x,y,z)$$
$$\frac{\partial \varphi}{\partial y}(x,y,z) = F_2(x,y,z)$$
$$\frac{\partial \varphi}{\partial z}(x,y,z) = F_3(x,y,z)$$

We proceed just as in (a).

- Applying $\frac{\partial}{\partial y}$ to the first equation and $\frac{\partial}{\partial x}$ to the second equation gives

$$\left\{\begin{aligned} \frac{\partial^2\varphi}{\partial y\partial x} &= \frac{\partial F_1}{\partial y} \\ \frac{\partial^2\varphi}{\partial x\partial y} &= \frac{\partial F_2}{\partial x} \end{aligned}\right\} \implies \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

- Applying $\frac{\partial}{\partial z}$ to the first equation and $\frac{\partial}{\partial x}$ to the third equation gives

$$\left\{\begin{aligned} \frac{\partial^2\varphi}{\partial z\partial x} &= \frac{\partial F_1}{\partial z} \\ \frac{\partial^2\varphi}{\partial x\partial z} &= \frac{\partial F_3}{\partial x} \end{aligned}\right\} \implies \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$$

- Applying $\frac{\partial}{\partial z}$ to the second equation and $\frac{\partial}{\partial y}$ to the third equation gives

$$\left\{\begin{aligned} \frac{\partial^2\varphi}{\partial z\partial y} &= \frac{\partial F_2}{\partial z} \\ \frac{\partial^2\varphi}{\partial y\partial z} &= \frac{\partial F_3}{\partial y} \end{aligned}\right\} \implies \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$$

Combining the three bullet points gives $\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0}$. ☐

Warning 2.3.10.

As always, we have to be careful with the flow of logic[19]. The screening test in Theorem 2.3.9 is a one-way test. If, for example, $\frac{\partial F_1}{\partial y} \neq \frac{\partial F_2}{\partial x}$ then the vector field $\mathbf{F}$ cannot be conservative. But if $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$ Theorem 2.3.9 does *not* guarantee that $\mathbf{F}$ is conservative. In fact there are fields that are not conservative but do obey $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$. We'll see one in Example 2.3.14, below. We shall later find some additional regularity conditions which, when combined with $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$, do imply conservativeness. See Theorem 2.4.8, below.

Example 2.3.11 (Example 2.3.7 revisited)

In Example 2.3.7, we attempted to find a potential for the vector field

$$\mathbf{v}(x,y) = \Omega\big(-y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}}\big)$$

In the end we showed that, if $\Omega \neq 0$, no potential could exist, i.e. $\mathbf{v}(x,y)$ is not conservative. Had we known the screening test of Theorem 2.3.9.a, we could have concluded that $\mathbf{v}(x,y)$ is not conservative by simply observing that

$$\begin{aligned} \frac{\partial \mathbf{v}_1}{\partial y} &= \frac{\partial}{\partial y}\big(-\Omega y\big) = -\Omega \\ \frac{\partial \mathbf{v}_2}{\partial x} &= \frac{\partial}{\partial x}\big(\Omega x\big) \ = +\Omega \end{aligned}$$

19 Use your favourite search engine to look up a list of common logical errors. One is "affirming the consequent". An example would be concluding that because Shakespeare is dead, Elvis, who is also dead, must also be Shakespeare.

are not equal, unless $\Omega = 0$. But $\Omega = 0$ makes a rather boring vector field.

Example 2.3.11

Example 2.3.12

Determine whether or not the vector field

$$\mathbf{F}(x, y, z) = y\hat{\boldsymbol{\imath}} - z\hat{\boldsymbol{\jmath}} + x\hat{\mathbf{k}}$$

is conservative. If it is conservative, find a potential.

Solution. Let's start by applying the screening test Theorem 2.3.9.b. Since

$$\nabla \times \mathbf{F} = \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -z & x \end{bmatrix} = \hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} - \hat{\mathbf{k}}$$

is not $\mathbf{0}$, the vector field $\mathbf{F}$ cannot be conservative.

Example 2.3.12

Example 2.3.13

Determine whether or not the vector field

$$\mathbf{F}(x, y, z) = (y^2 + 2xz^2 - 1)\hat{\boldsymbol{\imath}} + (2x + 1)y\,\hat{\boldsymbol{\jmath}} + (2x^2z + z^3)\hat{\mathbf{k}}$$

is conservative. If it is conservative, find a potential.

Solution. Again start by applying the screening test Theorem 2.3.9.b. This time

$$\begin{aligned} \nabla \times \mathbf{F} &= \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 + 2xz^2 - 1 & (2x+1)y & 2x^2z + z^3 \end{bmatrix} = 0\hat{\boldsymbol{\imath}} - (4xz - 4xz)\hat{\boldsymbol{\jmath}} + (2y - 2y)\hat{\mathbf{k}} \\ &= \mathbf{0} \end{aligned}$$

So $\mathbf{F}$ passes the screening test. Let's look for a function $\varphi(x, y, z)$ obeying

$$\begin{aligned} \frac{\partial \varphi}{\partial x}(x, y, z) &= y^2 + 2xz^2 - 1 \\ \frac{\partial \varphi}{\partial y}(x, y, z) &= (2x + 1)y \\ \frac{\partial \varphi}{\partial z}(x, y, z) &= 2x^2z + z^3 \end{aligned} \tag{$*$}$$

The partial derivative $\frac{\partial}{\partial x}$ treats y and z as constants. So $\varphi(x, y, z)$ obeys the first equation if and only if there is a function $\psi(y, z)$ with

$$\varphi(x, y, z) = xy^2 + x^2z^2 - x + \psi(y, z)$$

This $\varphi(x,y,z)$ will also obey the second equation if and only if

$$\begin{aligned}\frac{\partial}{\partial y}\big(xy^2+x^2z^2-x+\psi(y,z)\big)=(2x+1)y &\iff 2xy+\frac{\partial\psi}{\partial y}(y,z)=(2x+1)y\\ &\iff \frac{\partial\psi}{\partial y}(y,z)=y\\ &\iff \psi(y,z)=\frac{y^2}{2}+\zeta(z)\end{aligned}$$

for some function $\zeta(z)$ which depends only on z. At this stage we know that

$$\varphi(x,y,z)=xy^2+x^2z^2-x+\psi(y,z)=xy^2+x^2z^2-x+\frac{y^2}{2}+\zeta(z)$$

obeys the first two equations of $(*)$, for any function $\zeta(z)$. Finally to have the third equation of $(*)$ also satisfied, we also need to chose $\zeta(z)$ to obey

$$\begin{aligned}\frac{\partial}{\partial z}\left(xy^2+x^2z^2-x+\frac{y^2}{2}+\zeta(z)\right)=2x^2z+z^3 &\iff 2x^2z+\zeta'(z)=2x^2z+z^3\\ &\iff \zeta'(z)=z^3\\ &\iff \zeta(z)=\frac{z^4}{4}+C\end{aligned}$$

for any constant C. So one possible potential, namely that with $C=0$, is

$$\varphi(x,y,z)=xy^2+x^2z^2-x+\frac{y^2}{2}+\frac{z^4}{4}$$

Note, as a check[20], that

$$\nabla\varphi(x,y,z)=\big(y^2+2xz^2-1\big)\hat{\imath}+\big(2xy+y\big)\hat{\jmath}+\big(2x^2z+z^3\big)\hat{\mathbf{k}}$$

as desired.

Example 2.3.13

Example 2.3.14 $\left(\text{Optional: First look at } -\frac{y}{x^2+y^2}\hat{\imath}+\frac{x}{x^2+y^2}\hat{\jmath}\right)$

Now is a good time to reread Warning 2.3.10. In this example we will show that the vector field

$$\mathbf{F}(x,y)=-\frac{y}{x^2+y^2}\hat{\imath}+\frac{x}{x^2+y^2}\hat{\jmath}\qquad\text{defined for all }(x,y)\text{ in }\mathbb{R}^2\text{ except }(x,y)=(0,0)$$

passes the screening test of Theorem 2.3.9.a. We will also begin to see why it is *not* conservative on the domain $\mathbb{R}^2\setminus\{(0,0)\}$. To verify the screening test, we compute

$$\begin{aligned}\frac{\partial}{\partial y}\Big(-\frac{y}{x^2+y^2}\Big)&=-\frac{(x^2+y^2)-y(2y)}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2}\\ \frac{\partial}{\partial x}\Big(\frac{x}{x^2+y^2}\Big)&=\frac{(x^2+y^2)-x(2x)}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2}\end{aligned}$$

20 It is always worth doing this check.

and observe that the two right hand sides are identical. So the screening test is passed.

In order for **F** to be conservative on the domain $\mathbb{R}^2\backslash\{(0,0\}$, there must exist a function $\varphi(x,y)$, that, together with both partial derivatives $\frac{\partial\varphi}{\partial x}(x,y)$ and $\frac{\partial\varphi}{\partial y}(x,y)$, is defined for all (x,y) in $\mathbb{R}^2$ except $(x,y)=(0,0)$, and obeys

$$\frac{\partial\varphi}{\partial x}(x,y) = -\frac{y}{x^2+y^2} = \frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2} = \frac{\partial}{\partial x}\Big(\arctan\frac{y}{x}\Big)$$

$$\frac{\partial\varphi}{\partial y}(x,y) = \frac{x}{x^2+y^2} = \frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2} = \frac{\partial}{\partial y}\Big(\arctan\frac{y}{x}\Big)$$

by the chain rule, because

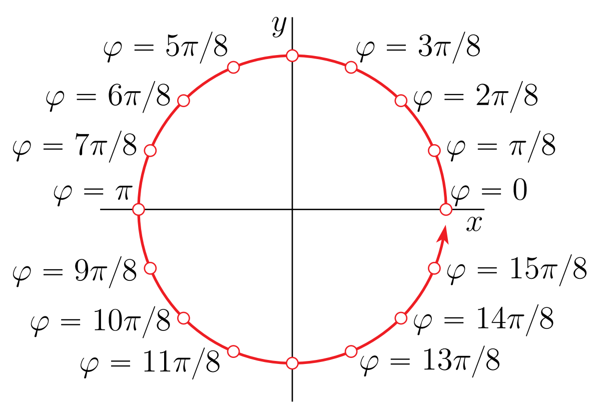
$$\frac{\partial}{\partial x}\Big(\frac{y}{x}\Big) = -\frac{y}{x^2} \qquad \frac{\partial}{\partial y}\Big(\frac{y}{x}\Big) = \frac{1}{x}$$

It looks like we have found a potential, namely $\arctan\frac{y}{x}$. But there is a problem. Recall that, by definition, $\arctan\frac{y}{x}$ is an angle $\theta(x,y)$ that obeys $\tan\theta(x,y) = \arctan\frac{y}{x}$; but for any $(x,y)\in\mathbb{R}^2\backslash\{(0,0\}$ there are infinitely many angles having the tangent $\frac{y}{x}$. To define $\varphi(x,y)$ we have to select exactly one such angle. It is impossible to do so in such a way that $\varphi(x,y)$ is continuous on all of $\mathbb{R}^2\backslash\{(0,0\}$.

To see why, fix any $r>0$, and imagine that you are walking on the circle $x^2+y^2=r^2$ in the xy-plane. At time θ, you are at $x=r\cos\theta$, $y=r\sin\theta$ and then $\frac{y}{x}=\tan\theta$ and you are allowed to define $\varphi(x,y)=\theta+k\pi$, for any integer k.

Suppose that at time $\theta=0$ you choose $k=0$. That is, you choose $\varphi(r,0)=0$. Now start walking, choosing an allowed $\varphi(x,y)$, i.e. choosing a k, for each point (x,y) that you cross. Because $\varphi(x,y)$ has to vary continuously[21] with (x,y), you have to continue choosing $k=0$. But you run off a cliff as θ approaches 2π, because then

- you are approaching $(r,0)$ from below, as in the figure below, and
- because you are choosing $k=0$, $\varphi(x,y)$ is just a little less than 2π, but
- you have already chosen $\varphi(r,0)=0$, not 2π. So $\varphi(x,y)$ has a jump discontinuity[22] along the positive x-axis.



21 If $\varphi(x,y)$ is not continuous, its gradient does not exist, and φ cannot be a potential.
22 Those who have taken some complex analysis may recognize this as the branch cut in $\ln z$.

If you are having trouble following this argument, don't worry about it. We will return with a less hand-wavy argument later.

Example 2.3.14

2.4▲ Line Integrals

We have already seen, in §1.6, one type of integral along curves. We are now going to see a second, that turns out to have significant connections to conservative vector fields. It arose from the concept of "work" in classical mechanics.

Suppose that we wish to find the work done by a force $\mathbf{F}(\mathbf{r})$ moving a particle along a path $\mathbf{r}(t)$. During the "infinitesimal time interval[23]" from t to $t + \mathrm{d}t$ the particle moves from $\mathbf{r}(t)$ to $\mathbf{r}(t) + \mathrm{d}\mathbf{r}$ with $\mathrm{d}\mathbf{r} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\,\mathrm{d}t$. By definition, the work done during that infinitesimal time interval is

$$\mathbf{F}\big(\mathbf{r}(t)\big) \cdot \mathrm{d}\mathbf{r} = \mathbf{F}\big(\mathbf{r}(t)\big) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\,\mathrm{d}t$$

The total work done during the time interval from t_0 to t_1 is then

$$\text{Work} = \int_{t_0}^{t_1} \mathbf{F}\big(\mathbf{r}(t)\big) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\,\mathrm{d}t$$

There are some useful shorthand notations for this work.

Notation 2.4.1.

Denote by $\mathcal{C}$ the parametrized path $\mathbf{r}(t)$ with $t_0 \leqslant t \leqslant t_1$. Then

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{\mathcal{C}} \big(\mathbf{F}_1 \mathrm{d}x + \mathbf{F}_2 \mathrm{d}y + \mathbf{F}_3 \mathrm{d}z\big) = \int_{t_0}^{t_1} \mathbf{F}\big(\mathbf{r}(t)\big) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\,\mathrm{d}t$$

If $\mathcal{C}$ is a closed path, the notation $\oint_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is also used.

In the event that $\mathbf{F}$ is conservative, and we know the potential φ, the following theorem provides a really easy way to compute "work integrals". The theorem is a generalization of the fundamental theorem of calculus, and indeed some people call it the fundamental theorem of line integrals.

23 Yes, yes. We should first consider short time intervals $\Delta t > 0$ and then take the limit $\Delta t \to 0$ at the end. But you have undoubtedly used this type of argument so many times before that you would be thoroughly bored by it.

Theorem 2.4.2.

Let $\mathbf{F} = \nabla\varphi$ be a conservative vector field. Then if $\mathcal{C}$ is any curve that starts at P_0 and ends at P_1, we have[24]

$$\int_{\mathcal{C}} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \varphi(P_1) - \varphi(P_0)$$

Proof. Let $\mathbf{r}(t) = \big(x(t), y(t), z(t)\big)$, $t_0 \leqslant t \leqslant t_1$, be any parametrization of $\mathcal{C}$ with $\mathbf{r}(t_0) = P_0$ and $\mathbf{r}(t_1) = P_1$. Then, by definition,

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \int_{t_0}^{t_1} \mathbf{F}\big(\mathbf{r}(t)\big)\cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\,\mathrm{d}t = \int_{t_0}^{t_1} \boldsymbol{\nabla}\varphi\big(\mathbf{r}(t)\big)\cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\,\mathrm{d}t \\ &= \int_{t_0}^{t_1} \Big[\frac{\partial\varphi}{\partial x}\big(x(t),y(t),z(t)\big)\frac{\mathrm{d}x}{\mathrm{d}t}(t) + \frac{\partial\varphi}{\partial y}\big(x(t),y(t),z(t)\big)\frac{\mathrm{d}y}{\mathrm{d}t}(t) \\ &\qquad\qquad + \frac{\partial\varphi}{\partial z}\big(x(t),y(t),z(t)\big)\frac{\mathrm{d}z}{\mathrm{d}t}(t)\Big]\mathrm{d}t \\ &= \int_{t_0}^{t_1} \frac{\mathrm{d}}{\mathrm{d}t}\Big[\varphi\big(x(t),y(t),z(t)\big)\Big]\mathrm{d}t \qquad \text{by the chain rule in reverse} \\ &= \varphi\big(\mathbf{r}(t_1)\big) - \varphi\big(\mathbf{r}(t_0)\big) = \varphi(P_1) - \varphi(P_0)\end{aligned}$$

by the fundamental theorem of calculus. □

Observe that, in Theorem 2.4.2, the value, $\varphi(P_1) - \varphi(P_0)$, of the integral $\int_{\mathcal{C}} \mathbf{F}\cdot \mathrm{d}\mathbf{r}$ depended only on the endpoints P_0 and P_1 of the curve, not on the path that the curve followed to get to P_0 from P_1. We shall see, in Theorem 2.4.7, below, that this happens only for conservative vector fields. Here are several examples of line integrals of vector fields that are not conservative.

Example 2.4.3

Set $P_0 = (0,0)$, $P_1 = (1,1)$ and[25]

$$\mathbf{F}(x,y) = xy\,\hat{\boldsymbol{\imath}} + (y^2+1)\,\hat{\boldsymbol{\jmath}}$$

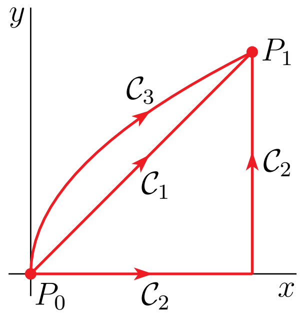
We shall consider three curves, all starting at P_0 and ending at P_1.

(a) Let $\mathcal{C}_1$ be the straight line from P_0 to P_1.

(b) Let $\mathcal{C}_2$ be the path, made from two straight lines, which follows the x-axis from P_0 to $(1,0)$ and then follows the line $x=1$ from $(1,0)$ to P_1.

(c) Let $\mathcal{C}_3$ be the part of the parabola $x = y^2$ from P_0 to P_1.

24 So φ acts a bit like the antiderivative of first year calculus.
25 The reader should check that this vector field is not conservative.



We shall calculate the work $\int_{\mathcal{C}_i} \mathbf{F}\cdot \mathrm{d}\mathbf{r}$ for each of the curves.

(a) We parametrize $\mathcal{C}_1$ by $\mathbf{r}(t) = t\,\hat{\imath} + t\,\hat{\jmath}$ with t running from 0 to 1. Then $x(t) = t$ and $y(t) = t$ so that

$$\mathbf{F}\big(\mathbf{r}(t)\big) = t^2\,\hat{\imath} + (t^2+1)\,\hat{\jmath} \qquad \text{and} \qquad \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t) = \hat{\imath} + \hat{\jmath}$$

so that

$$\begin{aligned}\int_{\mathcal{C}_1} \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \int_0^1 \mathbf{F}\big(\mathbf{r}(t)\big)\cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\,\mathrm{d}t = \int_0^1 \big[t^2\,\hat{\imath} + (t^2+1)\,\hat{\jmath}\big]\cdot[\hat{\imath}+\hat{\jmath}]\,\mathrm{d}t\\ &= \int_0^1 \big[2t^2+1\big]\,\mathrm{d}t\\ &= \frac{5}{3}\end{aligned}$$

(b) We split $\mathcal{C}_2$ into two parts, $\mathcal{C}_{2,x}$ running from P_0 to $(1,0)$ along the x-axis and then $\mathcal{C}_{2,y}$ running from $(1,0)$ to P_1 along the line $x = 1$. We parametrize $\mathcal{C}_{2,x}$ by $\mathbf{r}(x) = x\,\hat{\imath}$ with x running from 0 to 1 and $\mathcal{C}_{2,y}$ by $\mathbf{r}(y) = \hat{\imath} + y\,\hat{\jmath}$ with y running from 0 to 1. Then[26]

$$\begin{aligned}\int_{\mathcal{C}_2} \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \int_{\mathcal{C}_{2,x}} \mathbf{F}\cdot \mathrm{d}\mathbf{r} + \int_{\mathcal{C}_{2,y}} \mathbf{F}\cdot \mathrm{d}\mathbf{r}\\ &= \int_0^1 \big[(x)(0)\,\hat{\imath} + (0^2+1)\,\hat{\jmath}\big]\cdot \overbrace{\frac{\mathrm{d}}{\mathrm{d}x}\big(x\,\hat{\imath}\big)}^{\hat{\imath}}\,\mathrm{d}x + \int_0^1 \big[(1)(y)\,\hat{\imath} + (y^2+1)\,\hat{\jmath}\big]\cdot \overbrace{\frac{\mathrm{d}}{\mathrm{d}y}\big(\hat{\imath} + y\,\hat{\jmath}\big)}^{\hat{\jmath}}\,\mathrm{d}y\\ &= \int_0^1 0\,\mathrm{d}x + \int_0^1 (y^2+1)\,\mathrm{d}y\\ &= \frac{4}{3}\end{aligned}$$

(c) We parametrize $\mathcal{C}_3$ by $\mathbf{r}(t) = t^2\,\hat{\imath} + t\,\hat{\jmath}$ with t running from 0 to 1. Then $x(t) = t^2$ and $y(t) = t$ so that

$$\mathbf{F}\big(\mathbf{r}(t)\big) = t^3\,\hat{\imath} + (t^2+1)\,\hat{\jmath} \qquad \text{and} \qquad \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t) = 2t\,\hat{\imath} + \hat{\jmath}$$

26 You might like to think about why we can split up the integral like this.

so that

$$\int_{\mathcal{C}_3} \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \big[t^3\,\hat{\boldsymbol{\imath}} + (t^2+1)\,\hat{\boldsymbol{\jmath}}\big]\cdot\big[2t\,\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}\big]\,dt = \int_0^1 \big[2t^4 + t^2 + 1\big]\,dt$$
$$= \frac{2}{5} + \frac{1}{3} + 1 = \frac{26}{15}$$

Note that, despite the fact that $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ all start at P_0 and all end at P_1, the three integrals $\int_{\mathcal{C}_1}\mathbf{F}\cdot d\mathbf{r}$, $\int_{\mathcal{C}_2}\mathbf{F}\cdot d\mathbf{r}$ and $\int_{\mathcal{C}_3}\mathbf{F}\cdot d\mathbf{r}$ all have different values.

Example 2.4.3

Example 2.4.4

Set[27]

$$\mathbf{F}(x,y) = 2y\,\hat{\boldsymbol{\imath}} + 3x\,\hat{\boldsymbol{\jmath}}$$

This time we consider two curves.

(a) Let $\mathcal{C}_1$ be circle $x^2+y^2=1$, traversed once counterclockwise, starting at $(1,0)$.

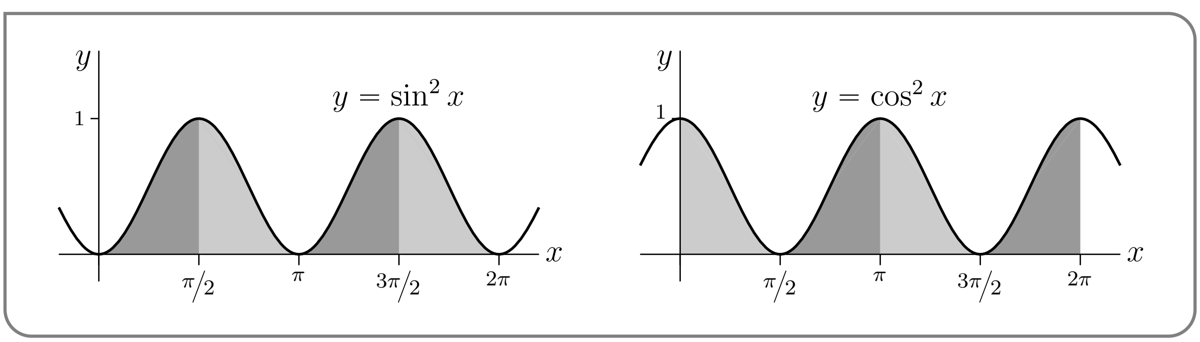
(b) Let $\mathcal{C}_2$ be (trivial) curve which just consists of the single point $(1,0)$.

We shall calculate the work $\int_{\mathcal{C}_i}\mathbf{F}\cdot d\mathbf{r}$ for each curve.

(a) We parametrize $\mathcal{C}_1$ by $\mathbf{r}(t) = \cos t\,\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}}$ with t running from 0 to 2π, just as we did in Example 1.0.1. Then

$$\oint_{\mathcal{C}_1} \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \big[2\sin t\,\hat{\boldsymbol{\imath}} + 3\cos t\,\hat{\boldsymbol{\jmath}}\big]\cdot\big[-\sin t\,\hat{\boldsymbol{\imath}} + \cos t\,\hat{\boldsymbol{\jmath}}\big]\,dt = \int_0^{2\pi}\big[-2\sin^2 t + 3\cos^2 t\big]\,dt$$

You could evaluate these integrals using double angle trig identities like you did in first year calculus. But there is a sneaky, much easier, way. Because $\sin^2 t$ and $\cos^2 t$ are translates of each other, and both are periodic of period π, the two integrals $\int_0^{2\pi}\sin^2 t\,dt$ and $\int_0^{2\pi}\cos^2 t\,dt$ represent the same area and so are equal. See the figure below.



27 Again, the reader should verify that this vector field is not conservative.

Thus

$$\int_0^{2\pi} \sin^2 t \, dt = \int_0^{2\pi} \cos^2 t \, dt = \int_0^{2\pi} \frac{1}{2}\big[\sin^2 t + \cos^2 t\big] \, dt = \frac{1}{2}\int_0^{2\pi} dt = \pi$$

and

$$\oint_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} = -2\int_0^{2\pi} \sin^2 t \, dt + 3\int_0^{2\pi} \cos^2 t \, dt = \pi$$

(b) We parametrize $\mathcal{C}_2$ by $\mathbf{r}(t) = \hat{\imath}$ for all t. Then $\frac{d\mathbf{r}}{dt}(t) = \mathbf{0}$ and $\int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r} = 0$.

Again, despite the fact that $\mathcal{C}_1$ and $\mathcal{C}_2$ both start at $(1, 0)$ and end at $(1, 0)$, the two integrals $\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}$ are different.

Example 2.4.4

Example 2.4.5 (Example 2.3.14, again.)

In Example 2.3.14, we saw that the vector field

$$\mathbf{F}(x, y) = -\frac{y}{x^2 + y^2}\hat{\imath} + \frac{x}{x^2 + y^2}\hat{\jmath} \qquad \text{defined for all } (x, y) \text{ in } \mathbb{R}^2 \text{ except } (x, y) = (0, 0)$$

passed the screening test of Theorem 2.3.9.a, and yet was not conservative. In this example, we will see that this $\mathbf{F}$ violates the conclusion of Theorem 2.4.2, thereby providing a second proof that $\mathbf{F}(x, y)$ is not conservative on $\mathbb{R}^2$ with $(0, 0)$ removed. For the curve $\mathcal{C}$, of Theorem 2.4.2, we use the circle parametrized by $x = a\cos\theta$, $y = a\sin\theta$, $0 \leqslant \theta \leqslant 2\pi$. Then $dx = -a\sin\theta \, d\theta$ and $dy = a\cos\theta \, d\theta$ so that

$$\frac{1}{2\pi}\int_{\mathcal{C}} \frac{x \, dy - y \, dx}{x^2 + y^2} = \frac{1}{2\pi}\int_0^{2\pi} \frac{a^2\cos^2\theta \, d\theta + a^2\sin^2\theta \, d\theta}{a^2\cos^2\theta + a^2\sin^2\theta} = \frac{1}{2\pi}\int_0^{2\pi} d\theta = 1$$

The curve $\mathcal{C}$ has initial point

$$P_0 = (a\cos\theta, \ a\sin\theta)\big|_{\theta=0} = (a, 0)$$

and final point

$$P_1 = (a\cos\theta, \ a\sin\theta)\big|_{\theta=2\pi} = (a, 0) = P_0$$

So, if $\mathbf{F}$ were conservative with potential φ, Theorem 2.4.2 would give that

$$\frac{1}{2\pi}\int_{\mathcal{C}} \frac{x \, dy - y \, dx}{x^2 + y^2} = \varphi(P_1) - \varphi(P_0) = 0$$

Consequently, $\mathbf{F}$ can't be conservative.

Example 2.4.5

2.4.1 ▸▸ Path Independence

This brings us to the following question. Let $\mathbf{F}$ be any fixed vector field. When is it true that, given any two fixed points P_0 and P_1, the integrals

$$\int_{\mathcal{C}} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_{\mathcal{C}'} \mathbf{F}\cdot \mathrm{d}\mathbf{r}$$

for all curves $\mathcal{C}$, $\mathcal{C}'$ that start at P_0 and end at P_1? When can we ignore the path taken? If this is the case we say that "$\int_{\mathcal{C}} \mathbf{F}\cdot \mathrm{d}\mathbf{r}$ is independent of the path chosen" and we write

$$\int_{P_0}^{P_1} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_{\mathcal{C}} \mathbf{F}\cdot \mathrm{d}\mathbf{r}$$

for any path $\mathcal{C}$ from P_0 to P_1. The point of this section is that there is an intimate relation between path independence and conservativeness of vector fields, that we will get to in Theorem 2.4.7.

For simplicity, we will consider only vector fields that are defined and continuous on all of $\mathbb{R}^2$ (i.e. the xy-plane) or $\mathbb{R}^3$ (i.e. the usual three dimensional world). Some discussion about what happens for vector fields that are defined only on part of $\mathbb{R}^2$ or $\mathbb{R}^3$ is given in the optional §4.5.

First we show that if there is one pair of (not necessarily distinct) points P_0, P_1 such that

$$\int_{\mathcal{C}_1} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_{\mathcal{C}_2} \mathbf{F}\cdot \mathrm{d}\mathbf{r}$$

for all curves $\mathcal{C}_1$, $\mathcal{C}_2$ that start at P_0 and end at P_1, then it is also true that, for *any* other pair of points P_0', P_1'

$$\int_{\mathcal{C}_1'} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_{\mathcal{C}_2'} \mathbf{F}\cdot \mathrm{d}\mathbf{r}$$

for all curves $\mathcal{C}_1'$, $\mathcal{C}_2'$ that start at P_0' and end at P_1'. This might seem unlikely at first, but the idea of the proof is really intuitive.

Theorem 2.4.6.

Let $\mathbf{F}$ be a vector field that is defined and continuous on all of $\mathbb{R}^2$ (or $\mathbb{R}^3$). Let P_0, P_1, P_0', P_1' be any four points in $\mathbb{R}^2$ (or $\mathbb{R}^3$). Assume that

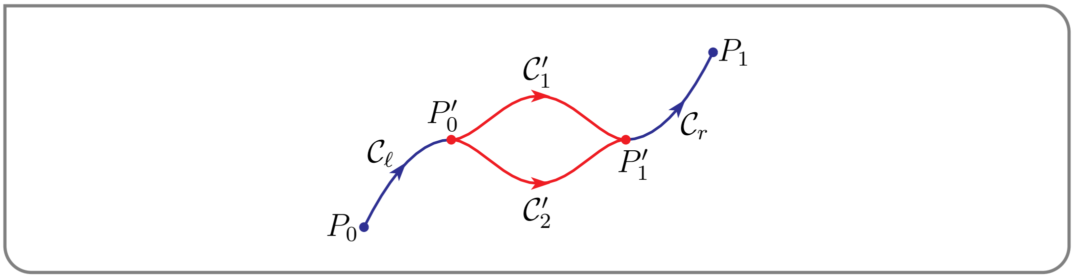
$$\int_{\mathcal{C}_1} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_{\mathcal{C}_2} \mathbf{F}\cdot \mathrm{d}\mathbf{r}$$

for all curves $\mathcal{C}_1$, $\mathcal{C}_2$ that start at P_0 and end at P_1. Then

$$\int_{\mathcal{C}_1'} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_{\mathcal{C}_2'} \mathbf{F}\cdot \mathrm{d}\mathbf{r}$$

for all curves $\mathcal{C}_1'$, $\mathcal{C}_2'$ that start at P_0' and end at P_1'.

Proof. Let $\mathcal{C}_1'$ and $\mathcal{C}_2'$ be any two curves that start at P_0' and end at P_1'. We start by choosing



any two (auxiliary) curves

- $\mathcal{C}_\ell$ that starts at P_0 and ends at P_0' and
- $\mathcal{C}_r$ that starts at P_1' and ends at P_1.

and then we define the curves

- $\mathcal{C}_1$ to be $\mathcal{C}_\ell$, followed by C_1', followed by $\mathcal{C}_r$ and
- $\mathcal{C}_2$ to be $\mathcal{C}_\ell$, followed by C_2', followed by $\mathcal{C}_r$.

Then both $\mathcal{C}_1$ and $\mathcal{C}_2$ start at P_0 and end at P_1, so that, by hypothesis,

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}$$

and, from the construction of $\mathcal{C}_1$ and $\mathcal{C}_2$,

$$\int_{\mathcal{C}_\ell} \mathbf{F} \cdot d\mathbf{r} + \int_{\mathcal{C}_1'} \mathbf{F} \cdot d\mathbf{r} + \int_{\mathcal{C}_r} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}_\ell} \mathbf{F} \cdot d\mathbf{r} + \int_{\mathcal{C}_2'} \mathbf{F} \cdot d\mathbf{r} + \int_{\mathcal{C}_r} \mathbf{F} \cdot d\mathbf{r}$$
$$\implies \int_{\mathcal{C}_1'} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}_2'} \mathbf{F} \cdot d\mathbf{r}$$

as desired. □

We are now ready for our main theorem on conservative fields.

Theorem 2.4.7.

Let $\mathbf{F}$ be a vector field that is defined and continuous on all of $\mathbb{R}^2$ (or $\mathbb{R}^3$). Then the following three statements are equivalent.

(a) $\mathbf{F}$ is conservative. That is, there exists a function φ such that $\mathbf{F} = \boldsymbol{\nabla}\varphi$.

(b) The integral $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve $\mathcal{C}$.

(c) The integral $\int \mathbf{F} \cdot d\mathbf{r}$ is path independent. That is, for any points P_0, P_1 we have $\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}$ for all curves $\mathcal{C}_1$, $\mathcal{C}_2$ that start at P_0 and end at P_1.

That is, if any one of the three statements are true, then all three are true.

Proof. It suffices for us to prove[28] that

- the truth of (a) implies the truth of (b) and
- the truth of (b) implies the truth of (c) and
- the truth of (c) implies the truth of (a).

That's exactly what we will do.
(a) $\Longrightarrow$ (b): Let $\mathcal{C}$ be a closed curve that starts at P_0 and then ends back at P_0. Then, by Theorem 2.4.2 with $P_1 = P_0$,

$$\oint_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \varphi(P_0) - \varphi(P_0) = 0$$

(b) $\Longrightarrow$ (c): Pick any point P_0 and set $P_1 = P_0$. Then we are assuming that $\oint_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$ for all curves that start at P_0 and end at P_1. In particular $\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ takes the same value for all curves that start at P_0 and end at P_1. So Theorem 2.4.6 immediately yields property (c).

(c) $\Longrightarrow$ (a): We are to show that $\mathbf{F}$ is conservative. We'll start by guessing φ and then we'll verify that, for our chosen φ, we really do have $\mathbf{F} = \boldsymbol{\nabla}\varphi$. Our guess for φ is motivated by Theorem 2.4.2. If our $\mathbf{F}$ really is conservative, its potential is going to have to obey $\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \varphi(P_1) - \varphi(P_0)$ for any curve $\mathcal{C}$ that starts at P_0 and ends at P_1. Let's choose $P_0 = \mathbf{0}$. Remembering, from Definition 2.3.1.a, that adding a constant to a potential always yields another potential, we can always choose $\varphi(\mathbf{0}) = 0$. Then $\varphi(P_1) = \int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ for any curve $\mathcal{C}$ that starts at $\mathbf{0}$ and ends at P_1. So define, for each point $\mathbf{x}$, $\varphi(\mathbf{x}) = \int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ for any curve $\mathcal{C}$ that starts at $\mathbf{0}$ and ends at $\mathbf{x}$. Note that, since we we are assuming that (c) is true, the integral $\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ takes the same value for all curves $\mathcal{C}$ that start at $\mathbf{0}$ and end at $\mathbf{x}$.

We now verify that, for this chosen φ, we really do have $\mathbf{F} = \boldsymbol{\nabla}\varphi$. Fix any point $\mathbf{x}$ and any curve $\mathcal{C}_{\mathbf{x}}$ that starts at the origin and ends at $\mathbf{x}$. For any vector $\mathbf{u}$, let $\mathcal{D}_{\mathbf{u}}$ be the curve with parametrization

$$\mathbf{r}_{\mathbf{u}}(t) = \mathbf{x} + t\mathbf{u} \qquad 0 \leqslant t \leqslant 1$$

This curve is a line segment that starts at $\mathbf{x}$ at $t = 0$ and ends at $\mathbf{x} + \mathbf{u}$ at $t = 1$. Observe that $\mathbf{r}'_{\mathbf{u}}(t) = \mathbf{u}$. Recall that, by assumption, $\varphi(\mathbf{x} + s\mathbf{u}) = \int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ for *any* curve $\mathcal{C}$ that starts at $\mathbf{0}$ and ends at $\mathbf{x} + s\mathbf{u}$. So

$$\varphi(\mathbf{x} + s\mathbf{u}) = \int_{\mathcal{C}_{\mathbf{x}} + \mathcal{D}_{s\mathbf{u}}} \mathbf{F} \cdot d\mathbf{r}$$

where $C_{\mathbf{x}} + D_{s\mathbf{u}}$ is the curve which first follows $C_{\mathbf{x}}$ from the origin to $\mathbf{x}$ and then follows $D_{s\mathbf{u}}$ from x to $\mathbf{x} + s\mathbf{u}$. We have

$$\begin{aligned}\int_{C_{\mathbf{x}} + D_{s\mathbf{u}}} \mathbf{F} \cdot d\mathbf{r} &= \int_{C_{\mathbf{x}}} \mathbf{F} \cdot d\mathbf{r} + \int_{D_{s\mathbf{u}}} \mathbf{F} \cdot d\mathbf{r} \\ &= \int_{C_{\mathbf{x}}} \mathbf{F} \cdot d\mathbf{r} + \int_0^1 \mathbf{F}(\mathbf{x} + ts\mathbf{u}) \cdot (s\mathbf{u})\, dt\end{aligned}$$

In the second integral, make the change of variables $\tau = ts$, $\mathrm{d}\tau = s\mathrm{d}t$. This gives

$$\varphi(\mathbf{x} + s\mathbf{u}) = \int_{C_{\mathbf{x}}} \mathbf{F} \cdot d\mathbf{r} + \int_0^s \mathbf{F}(\mathbf{x} + \tau\mathbf{u}) \cdot \mathbf{u}\, d\tau$$

28 This is a pretty efficient, and standard, way to structure the proof of the equivalence of three statements.

By the fundamental theorem of calculus, applied to the second integral,

$$\frac{\mathrm{d}}{\mathrm{d}s}\varphi(\mathbf{x}+s\mathbf{u})\Big|_{s=0} = \mathbf{F}(\mathbf{x}+s\mathbf{u})\cdot\mathbf{u}\Big|_{s=0} = \mathbf{F}(\mathbf{x})\cdot\mathbf{u}$$

Applying this with $\mathbf{u} = \hat{\imath}, \hat{\jmath}, \hat{\mathbf{k}}$ gives us

$$\Big(\frac{\partial\varphi}{\partial x}(\mathbf{x}), \frac{\partial\varphi}{\partial y}(\mathbf{x}), \frac{\partial\varphi}{\partial z}(\mathbf{x})\Big) = \big(\mathbf{F}(\mathbf{x})\cdot\hat{\imath}, \mathbf{F}(\mathbf{x})\cdot\hat{\jmath}, \mathbf{F}(\mathbf{x})\cdot\hat{\mathbf{k}}\big)$$

which is

$$\nabla\varphi(\mathbf{x}) = \mathbf{F}(\mathbf{x})$$

as desired. □

Using this result, we can completely characterize conservative fields on $\mathbb{R}^2$ and $\mathbb{R}^3$.

Theorem 2.4.8.

Let $\mathbf{F}$ be a vector field that is defined and has continuous first order partial derivatives on all of $\mathbb{R}^2$ (or $\mathbb{R}^3$). Then $\mathbf{F}$ is conservative if and only if it passes the screening test $\boldsymbol{\nabla}\times\mathbf{F} = \mathbf{0}$, i.e. is curl free.

Warning 2.4.9.

Note that in Theorem 2.4.8 we are assuming that $\mathbf{F}$ passes the screening test on *all* of $\mathbb{R}^2$ or $\mathbb{R}^3$. We have already seen, in Example 2.3.14, that if the screening test fails at even a single point, for example because the vector field is not defined at that point, then $\mathbf{F}$ need not be conservative. We'll explore what happens in such cases in the (optional) §4.5. We'll see that something can be salvaged.

Proof of Theorem 2.4.8. We'll give the proof for the $\mathbb{R}^2$ case. The proof for the $\mathbb{R}^3$ case is very similar. We have already seen, in Theorem 2.3.9, that if $\mathbf{F}$ is conservative, then it passes the screening test and there is nothing more to do.

So we now have to assume that $\mathbf{F}$ obeys $\frac{\partial F_1}{\partial y}(x,y) = \frac{\partial F_2}{\partial x}(x,y)$ on *all* of $\mathbb{R}^2$ and prove that it is conservative. We'll do so using the strategy of Example 2.3.13 to find a function $\varphi(x,y)$, that obeys

$$\frac{\partial\varphi}{\partial x}(x,y) = F_1(x,y)$$
$$\frac{\partial\varphi}{\partial y}(x,y) = F_2(x,y)$$

The partial derivative $\frac{\partial}{\partial x}$ treats y as a constant. So $\varphi(x,y)$ obeys the first equation if and only if there is a function $\psi(y)$ with

$$\varphi(x,y) = \int_0^x F_1(X,y)\,\mathrm{d}X + \psi(y)$$

This $\varphi(x,y)$ will also obey the second equation if and only if

$$\begin{aligned} F_2(x,y) &= \frac{\partial\varphi}{\partial y}(x,y) \\ &= \frac{\partial}{\partial y}\Big(\int_0^x F_1(X,y)\,\mathrm{d}X \;+\; \psi(y)\Big) \\ &= \int_0^x \frac{\partial F_1}{\partial y}(X,y)\,\mathrm{d}X \;+\; \psi'(y) \end{aligned}$$

So we have to find a $\psi(y)$ that obeys

$$\psi'(y) = F_2(x,y) - \int_0^x \frac{\partial F_1}{\partial y}(X,y)\,\mathrm{d}X$$

This looks bad — no matter what $\psi(y)$ is, the left hand side is independent of x, while it looks like the right hand side depends on x. Fortunately our screening test hypothesis now rides in to the rescue[29]. (We haven't used it yet, and it has to come in somewhere.)

$$\begin{aligned} F_2(x,y) - \int_0^x \frac{\partial F_1}{\partial y}(X,y)\,\mathrm{d}X &= F_2(x,y) - \int_0^x \frac{\partial F_2}{\partial x}(X,y)\,\mathrm{d}X \\ &= F_2(x,y) - F_2(X,y)\Big|_{X=0}^{X=x} \\ &= F_2(0,y) \end{aligned}$$

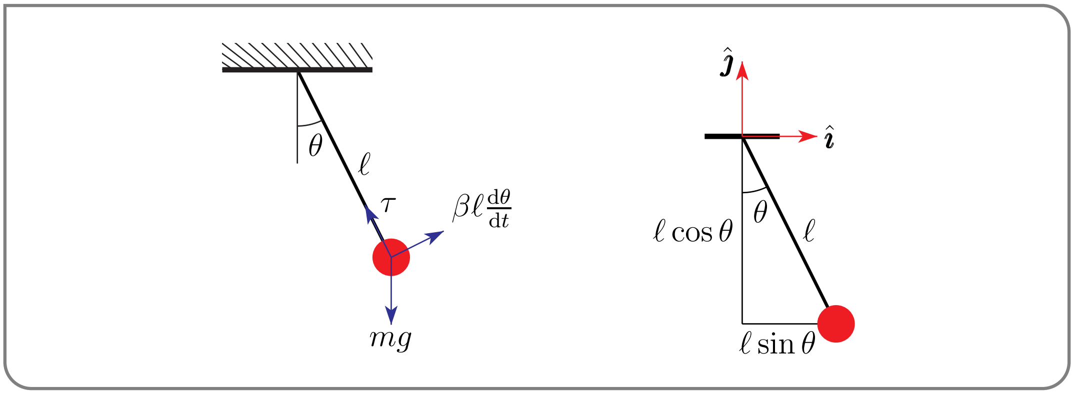
In going from the first line to the second line we used the fundamental theorem of calculus. So choosing

$$\psi(y) = \int_0^y F_2(0,Y)\,\mathrm{d}Y + C$$

for any constant C, does the trick. □

2.5▲ Optional — The Pendulum

Model a pendulum by a mass m that is connected to a hinge by an idealized rod that is massless and of fixed length ℓ. Denote by θ the angle between the rod and vertical. The



29 or bails us out, or saves our bacon, or . . .

forces acting on the mass are

- ◦ gravity, which has magnitude mg and direction $(0, -1)$,
- ◦ tension in the rod, whose magnitude, τ, automatically adjusts itself so that the distance between the mass and the hinge is fixed at ℓ and whose direction, $(-\sin\theta, \cos\theta)$, is always parallel to the rod and
- ◦ possibly some frictional forces, like friction in the hinge and air resistance. We shall assume that the total frictional force has magnitude proportional to the speed[30] of the mass and has direction opposite to the direction of motion of the mass.

If we use a coordinate system centered on the hinge, the (x, y) coordinates of the mass at time t are $\ell\big(\sin\theta(t), -\cos\theta(t)\big)$. Hence its velocity vector is

$$\mathbf{v}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\big[\ell\big(\sin\theta(t), -\cos\theta(t)\big)\big] = \ell\big(\cos\theta(t), \sin\theta(t)\big)\frac{\mathrm{d}\theta}{\mathrm{d}t}(t)$$

and the total frictional force is $-\beta\ell(\cos\theta, \sin\theta)\frac{\mathrm{d}\theta}{\mathrm{d}t}$, for some constant β. The acceleration vector of the mass is

$$\mathbf{a}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(t) = \ell(\cos\theta, \sin\theta)\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \ell(-\sin\theta, \cos\theta)\Big(\frac{\mathrm{d}\theta}{\mathrm{d}t}\Big)^2$$

so that Newton's law of motion, $\mathbf{F} = m\mathbf{a}$, now tells us

$$\begin{aligned} m\mathbf{a}(t) &= m\ell(\cos\theta, \sin\theta)\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + m\ell(-\sin\theta, \cos\theta)\Big(\frac{\mathrm{d}\theta}{\mathrm{d}t}\Big)^2 \\ &= \mathbf{F} = mg(0, -1) + \tau(-\sin\theta, \cos\theta) - \beta\ell(\cos\theta, \sin\theta)\frac{\mathrm{d}\theta}{\mathrm{d}t} \end{aligned}$$

To eliminate the (unknown) coefficient τ we dot this equation with $(\cos\theta, \sin\theta)$, which extracts the component parallel to the direction of motion of the mass. Dotting with $(\cos\theta, \sin\theta)$ gives $m\ell\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -mg\sin\theta - \beta\ell\frac{\mathrm{d}\theta}{\mathrm{d}t}$ or

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{\beta}{m}\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{g}{\ell}\sin\theta = 0$$

which is the equation of motion of the (nonlinear) pendulum. In general, it can be hard to analyse nonlinear differential equations. But if the amplitude of oscillation is small enough that we may approximate $\sin\theta$ by θ we get the equation of motion of the linear pendulum[31] which is

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{\beta}{m}\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{g}{\ell}\theta = 0$$

30 The dependence of air resistance (drag) on the speed v is relatively complex. At low speed drag tends to be approximately proportional to v, while at high speed it tends to be approximately proportional to v^2.

31 When $\beta = 0$, this equation reduces to the equation $\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{\ell}\theta = 0$, which occurs in many different applications, and whose solutions exhibit simple harmonic motion.

These equations may be reformulated as systems of first order ordinary differential equation, that is as equations for the flow lines of a vector field, by the simple expedient of defining (as we did in Example 2.1.4)

$$x(t) = \theta(t) \qquad y(t) = \theta'(t)$$

Then, for the full, nonlinear, equation $\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{\beta}{m}\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{g}{\ell}\sin\theta = 0$

$$\begin{aligned} x'(t) &= \theta'(t) = y(t) \\ y'(t) &= \theta''(t) = -\frac{g}{\ell}\sin x(t) - \frac{\beta}{m}y(t) \end{aligned}$$

The solutions of this first order system of ordinary differential equations are flow lines for the vector field

$$\mathbf{V}\big((x,y)\big) = \Big(y\,,\ -\frac{g}{\ell}\sin x - \frac{\beta}{m}y\Big)$$

When $\beta = 0$, this is exactly the vector field of Example 2.1.4.

Chapter 3

SURFACE INTEGRALS

3.1 ▲ Parametrized Surfaces

For many applications we will need to use integrals over surfaces. One obvious one is just computing surface areas. Another is computing the rate at which fluid traverses a surface. The first step is to simply specify surfaces carefully.

There are three common ways to specify a surface in three dimensions.

(a) *Graph of a function:* Probably the most common way to specify a surface is to give its equation in the form

$$z = f(x,y) \qquad (x,y) \in \mathcal{D} \subset \mathbb{R}^2$$

Here "$(x,y) \in \mathcal{D} \subset \mathbb{R}^2$" just means that (x,y) runs over the subset $\mathcal{D}$ of $\mathbb{R}^2$. For example, if the surface is the top half of the sphere of radius one centred on the origin

$$z = \sqrt{1 - x^2 - y^2} \qquad \text{with } x^2 + y^2 \leqslant 1$$

(b) *Implicitly:* We can also specify that the surface is the set of points (x,y,z) that satisfy the equation $G(x,y,z) = 0$, or, more generally[1], satisfy the equation $G(x,y,z) = K$, with K a constant. For example, the sphere of radius one centred on the origin is the set of points that obey

$$x^2 + y^2 + z^2 = 1$$

We shall explore this surface a little more in Example 3.1.2 below.

(c) *Range of a function:* Probably the most useful way to specify a surface, when one needs to integrate over the surface, is as the range of a function

$$\begin{aligned} &\mathbf{r} : \mathcal{D} \subset \mathbb{R}^2 \to \mathbb{R}^3 \\ &(u,v) \in \mathcal{D} \mapsto \mathbf{r}(u,v) = \big(x(u,v)\,,\, y(u,v)\,,\, z(u,v)\big) \end{aligned}$$

1 Of course we can always convert the equation $G(x,y,z) = K$ into $H(x,y,z) = 0$ with $H(x,y,z) = G(x,y,z) - K$. But it is often more convenient to use $G(x,y,z) = K$.

The upper line means that $\mathbf{r}$ is a function which is defined on the subset $\mathcal{D}$ of $\mathbb{R}^2$ and which assigns to each point on $\mathcal{D}$ a point in $\mathbb{R}^3$. The second line means that the function $\mathbf{r}$ assigns to the element (u,v) of $\mathcal{D}$ the element $\mathbf{r}(u,v) = \big(x(u,v)\,,\,y(u,v)\,,\,z(u,v)\big)$ in $\mathbb{R}^3$. Such a surface is called a parametrized surface — each point of the surface is labelled by the values of the two parameters u and v. Parametrized surfaces are of course the two parameter analog of parametrized curves. Examples of parametrized surfaces come next.

Example 3.1.1

One simple, even trivial, way to parametrize the surface which is the graph

$$z = f(x,y) \qquad (x,y) \in \mathcal{D} \subset \mathbb{R}^2$$

is to choose x and y as the parameters. That is, to choose

$$\mathbf{r}(u,v) = \big(u,v,\,f(u,v)\big), \quad (u,v) \in \mathcal{D} \qquad \text{or} \qquad \mathbf{r}(x,y) = \big(x,y,\,f(x,y)\big), \quad (x,y) \in \mathcal{D}$$

Example 3.1.1

Let's do something a bit more substantial.

Example 3.1.2 (Sphere)

The sphere of radius 1 centred on the origin is the set of points (x,y,z) that obey

$$G(x,y,z) = x^2 + y^2 + z^2 = 1$$

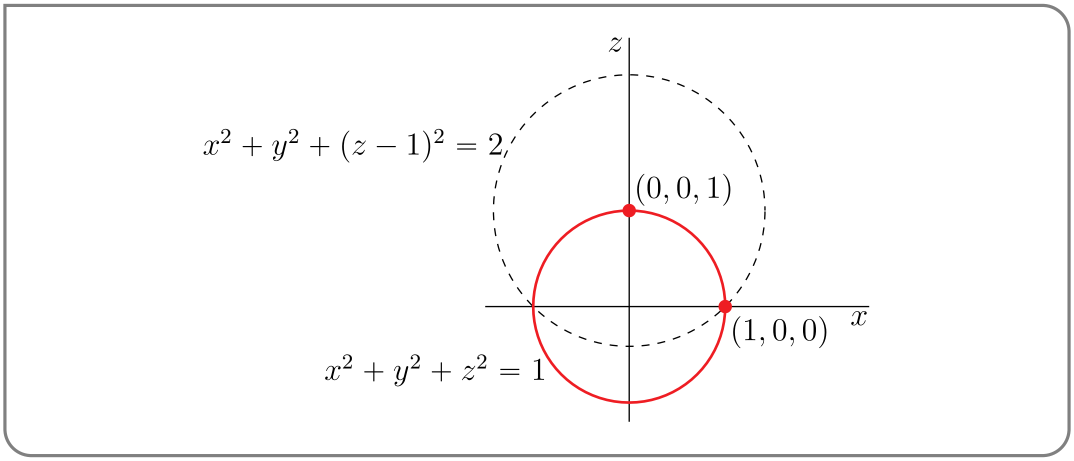
We cannot express this surface as the graph of a function because, for each (x,y) with $x^2 + y^2 < 1$, there are two z's that obey $x^2 + y^2 + z^2 = 1$, namely

$$z = \pm\sqrt{1 - x^2 - y^2}$$

On the other hand, locally, this surface is the graph of a function. This means that, for any point (x_0,y_0,z_0) on the sphere, all points of the surface that are sufficiently near (x_0,y_0,z_0) can be expressed in one of the forms $z = f(x,y)$ or $x = g(y,z)$, or $y = h(x,z)$. For example, the part of the sphere that is within a distance $\sqrt{2}$ of the point $(0,0,1)$ is

$$\begin{aligned}
&\big\{\,(x,y,z)\;\big|\;x^2+y^2+z^2=1,\ |(x,y,z)-(0,0,1)|<\sqrt{2}\,\big\}\\
&=\big\{\,(x,y,z)\;\big|\;x^2+y^2+z^2=1,\ x^2+y^2+(z-1)^2<2\,\big\}\\
&=\big\{\,(x,y,z)\;\big|\;x^2+y^2+z^2=1,\ x^2+y^2+z^2-2z+1<2\,\big\}\\
&=\big\{\,(x,y,z)\;\big|\;x^2+y^2+z^2=1,\ z>0\,\big\}\\
&=\big\{\,(x,y,z)\;\big|\;z=\sqrt{1-x^2-y^2},\ x^2+y^2<1\,\big\}
\end{aligned}$$

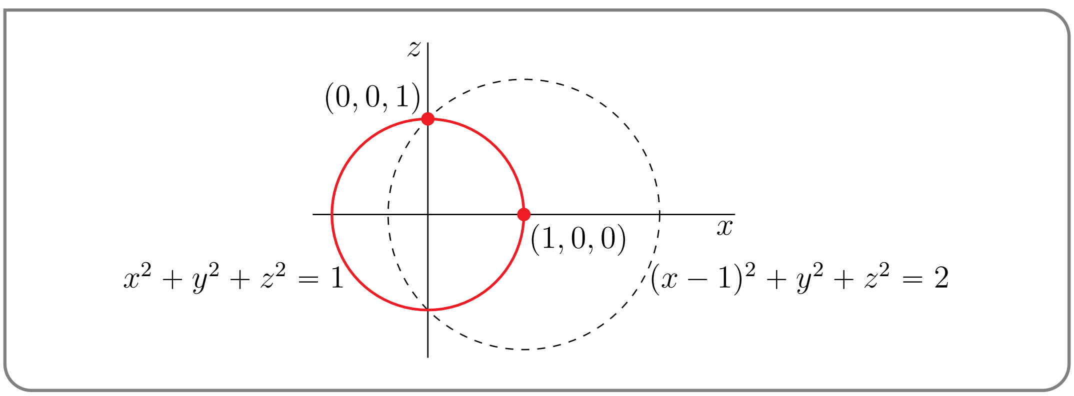
This is illustrated in the figure below which shows the $y = 0$ section of the sphere $x^2 + y^2 + z^2 = 1$ and also the $y = 0$ section of the set of points that are within a distance $\sqrt{2}$ of $(0,0,1)$. (They are the points inside the dashed circle.)



Similarly, as illustrated schematically in the figure below, the part of the sphere that is within a distance $\sqrt{2}$ of the point $(1,0,0)$ is

$$\begin{aligned}
&\{\, (x,y,z) \mid x^2+y^2+z^2=1,\ |(x,y,z)-(1,0,0)| < \sqrt{2}\,\} \\
&= \{\, (x,y,z) \mid x^2+y^2+z^2=1,\ (x-1)^2+y^2+z^2 < 2\,\} \\
&= \{\, (x,y,z) \mid x^2+y^2+z^2=1,\ x^2-2x+1+y^2+z^2 < 2\,\} \\
&= \{\, (x,y,z) \mid x^2+y^2+z^2=1,\ x > 0\,\} \\
&= \{\, (x,y,z) \mid x = \sqrt{1-y^2-z^2},\ y^2+z^2 < 1\,\}
\end{aligned}$$

The figure below shows the $y = 0$ section of the sphere $x^2 + y^2 + z^2 = 1$ and also the $y = 0$ section of the set of points that are within a distance $\sqrt{2}$ of $(1,0,0)$. (Again, they are the points inside the dashed circle.)



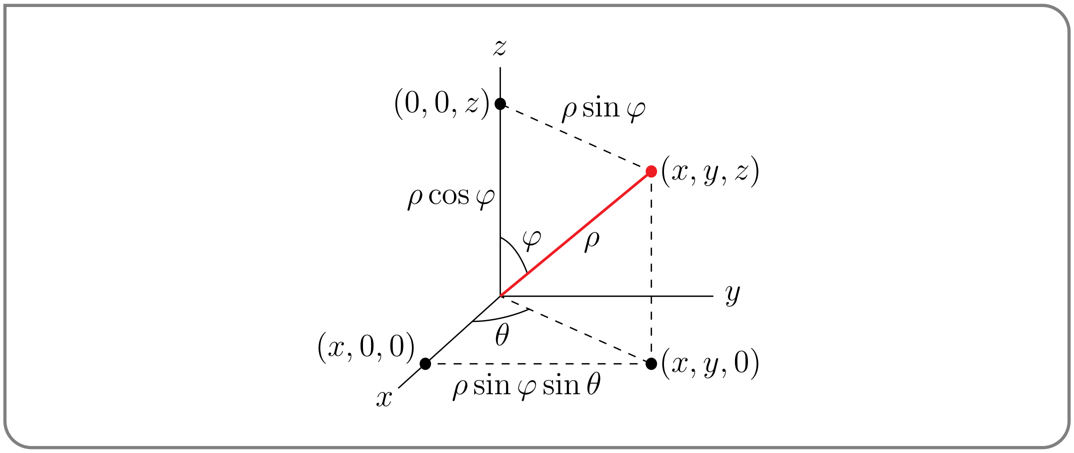
We can parametrize the unit sphere by using spherical coordinates, which you should have seen before. As a reminder, here is a figure showing the definitions of the three spherical coordinates[2]

$\rho =$ distance from (x,y,z) to $(0,0,0)$

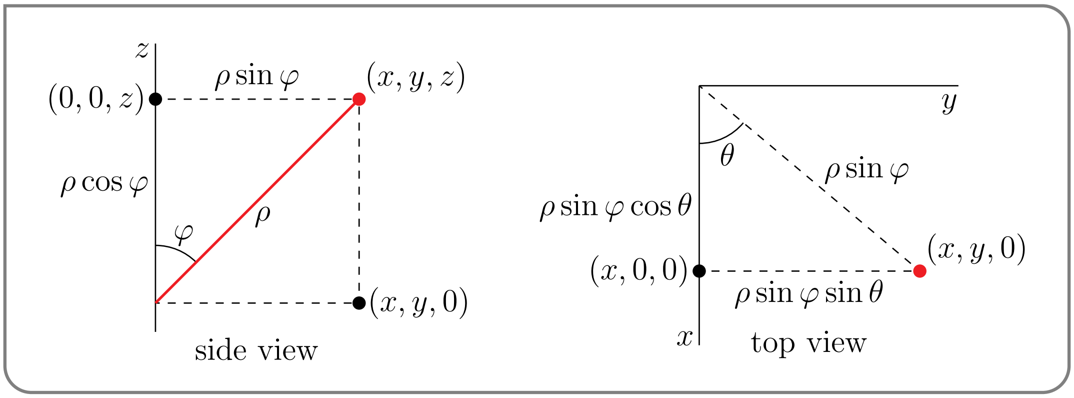
$\varphi =$ angle between the line $\overline{(0,0,0)\,(x,y,z)}$ and the z axis

$\theta =$ angle between the line $\overline{(0,0,0)\,(x,y,0)}$ and the x axis

2 The symbols ρ, φ, θ, are the standard mathematics symbols for the spherical coordinates. Appendix G gives another set of symbols that is commonly used in the physical sciences and engineering.



and here are two more figures giving the side and top views of the previous figure.



From the figure, we see that Cartesian and spherical coordinates are related by

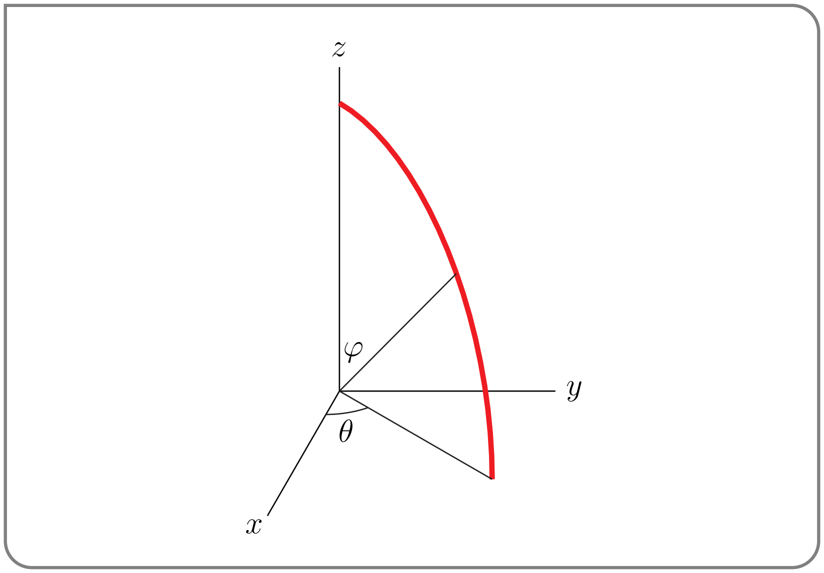
$$\begin{aligned} x &= \rho \sin\varphi \cos\theta \\ y &= \rho \sin\varphi \sin\theta \\ z &= \rho\cos\varphi \end{aligned}$$

The points on the sphere $x^2 + y^2 + z^2 = 1$ are precisely the set of points with $\rho = 1$. So we can use the parametrization

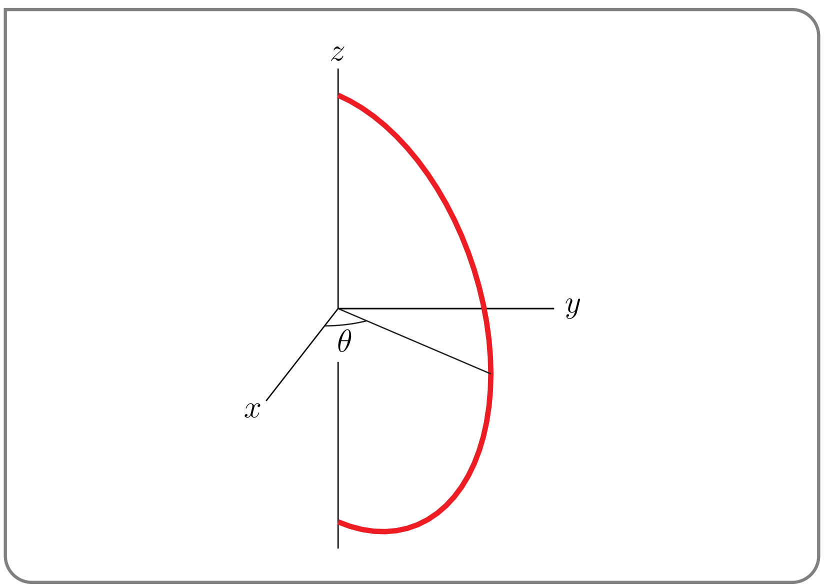
$$\mathbf{r}(\theta,\varphi) = \big(\sin\varphi\cos\theta\,, \sin\varphi\sin\theta\,,\, \cos\varphi\big)$$

Here is how to see that as φ runs over $(0,\pi)$ and θ runs over $[0,2\pi)$, $\mathbf{r}(\theta,\varphi)$ covers the whole sphere $x^2+y^2+z^2 = 1$ except for the north pole ($\varphi = 0$ gives the north pole for all values of θ) and the south pole ($\varphi = \pi$ gives the south pole for all values of θ).

- Fix θ and have φ run over the interval $0 < \varphi \leqslant \pi/2$. Then $\mathbf{r}(\theta,\varphi)$ traces out one quarter of a circle starting at the north pole $\mathbf{r}(\theta,0) = (0,0,1)$ (but excluding the north pole itself) and ending at the point $\mathbf{r}(\theta,\pi/2) = (\cos\theta, \sin\theta, 0)$ in the xy-plane.

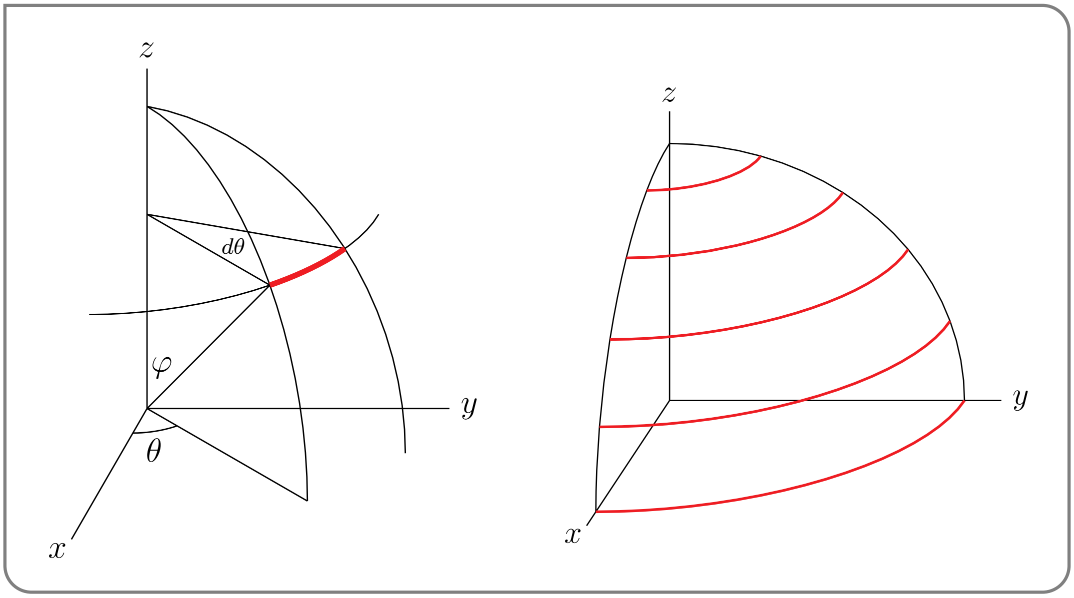


- Keep θ fixed at the same value and extend the interval over which φ runs to $0 < \varphi < \pi$. Now $\mathbf{r}(\theta, \varphi)$ traces out a semi-circle starting at the north pole $\mathbf{r}(\theta, 0) = (0, 0, 1)$, ending at the south pole $\mathbf{r}(\theta, \pi) = (0, 0, -1)$ (but excluding both the north and south poles themselves) and passing through the point $\mathbf{r}(\theta, \pi/2) = (\cos\theta, \sin\theta, 0)$ in the xy-plane.

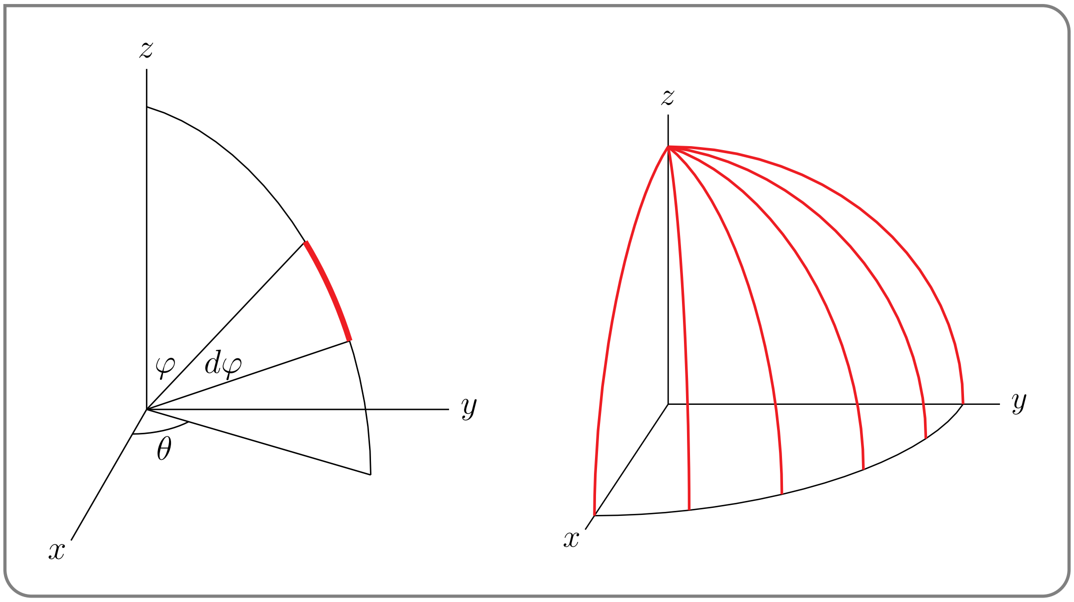


- Finally have θ run over $0 \leqslant \theta < 2\pi$. Then the semicircle rotates about the z-axis, sweeping out the full sphere, except for the north and south poles.

Recall that φ is the angle between the radius vector and the z-axis. If you hold φ fixed and increase θ by a small amount $d\theta$, $\mathbf{r}(\theta, \varphi)$ sweeps out the red circular arc in the figure on the left below. If you hold φ fixed and vary θ from 0 to 2π, $\mathbf{r}(\theta, \varphi)$ sweeps out a line of latitude. The figure on the right below gives the lines of latitude (or at least the parts of those lines in the first octant) for $\varphi = \frac{\pi}{10}, \frac{2\pi}{10}, \frac{3\pi}{10}, \frac{4\pi}{10}$ and $\frac{5\pi}{10} = \frac{\pi}{2}$.



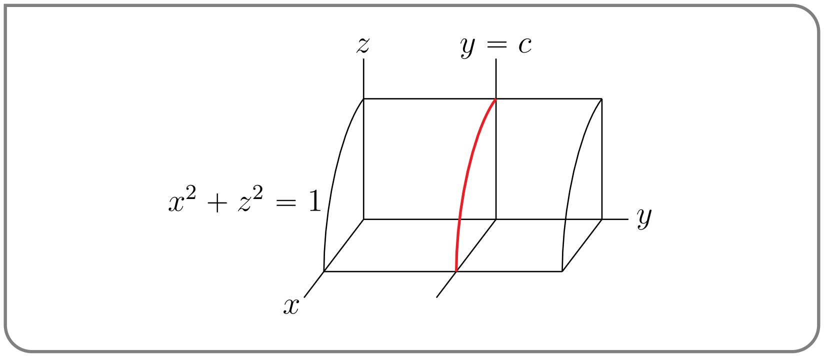
On the other hand, if you hold θ fixed and increase φ by a small amount $\mathrm{d}\varphi$, $\mathbf{r}(\theta,\varphi)$ sweeps out the red circular arc in the figure on the left below. If you hold θ fixed and vary φ from 0 to π, $\mathbf{r}(\theta,\varphi)$ sweeps out a line of longitude. The figure on the right below gives the lines of longitude (or at least the parts of those lines in the first octant) for $\theta = 0$, $\frac{\pi}{10}$, $\frac{2\pi}{10}$, $\frac{3\pi}{10}$, $\frac{4\pi}{10}$ and $\frac{5\pi}{10} = \frac{\pi}{2}$.



Example 3.1.2

Example 3.1.3 (Cylinder)

The surface $x^2 + z^2 = 1$ is an infinite cylinder. Part of this cylinder in the first octant is sketched below.



Note that the section of this cylinder that lies in the xz-plane, and in fact in any plane $y = c$, is the circle $x^2 + z^2 = 1$. We can of course parametrize this circle by $x = \cos\theta$, $z = \sin\theta$. So we can parametrize the whole cylinder by using θ and y as parameters.

$$\mathbf{r}(\theta, y) = \big(\cos\theta\,,\, y\,,\, \sin\theta\big) \qquad 0 \leqslant \theta < 2\pi,\ \ -\infty < y < \infty$$

Example 3.1.3

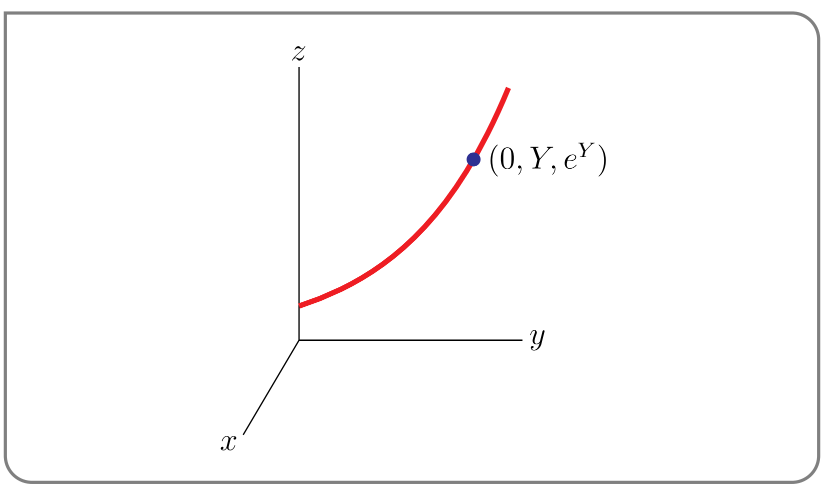
Example 3.1.4 (Surface of Revolution)

In this example, we are going to parametrize a surface of revolution. In your first integral calculus course, you undoubtedly encountered many surfaces created by rotating a curve $y = f(x)$ about the x-axis or the y-axis. In this course, we are used to having the z-axis, rather than the y-axis, run vertically. So in this example, we'll parametrize the surface constructed by rotating the curve

$$z = g(y) = e^y \qquad 0 \leqslant y \leqslant 1$$

about the z-axis. Exactly the same procedure can be used to parametrize surfaces created by rotating about the x-axis or the y-axis too.

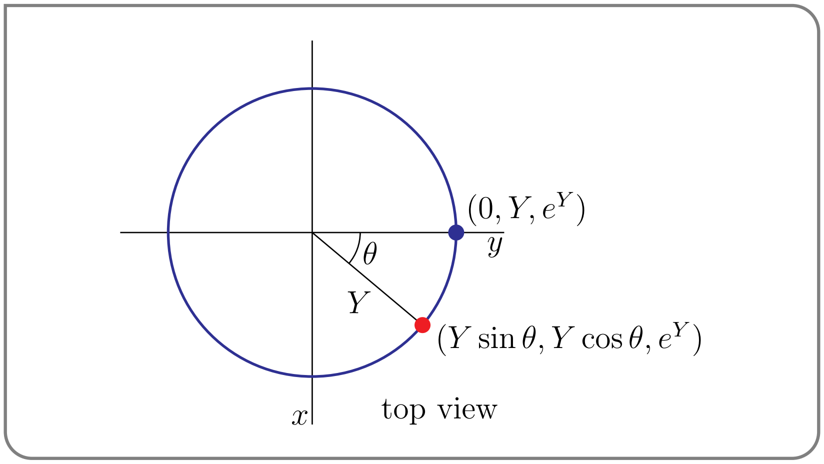
We start by just sketching the curve, considering the yz-plane as the plane $x = 0$ in $\mathbb{R}^3$. The specified curve is the red curve in the figure below. Concentrate on any one point on that curve. It is the blue dot at $(0, Y, e^Y)$ in the figure. When our curve is rotated about



the z-axis, the blue dot sweeps out a circle. The circle that the blue dot sweeps out

- lies in the horizontal plane $z = e^Y$ and
- is centred on the z-axis and
- has radius Y.

We can parametrize the circle swept out in the usual way. Here is a top view of the circle, with the parameter, named θ, indicated.



The coordinates of the red dot are $\big(Y\sin\theta\,,\,Y\cos\theta\,,\,e^Y\big)$. This also gives a parametrization of the surface of revolution

$$\begin{aligned} x(Y,\theta) &= Y\sin\theta \\ y(Y,\theta) &= Y\cos\theta \\ z(Y,\theta) &= e^Y \\ &\quad 0 \leqslant Y \leqslant 1, \qquad 0 \leqslant \theta < 2\pi \end{aligned}$$

Notice, by way of checks, that

- when $\theta = 0$,
$$\big(x(Y,0)\,,\,y(Y,0)\,,\,z(Y,0)\big) = (0, Y, e^Y)$$
runs over the entire desired curve (namely $z = g(y)$, $0 \leqslant y \leqslant 1$), when Y runs over $0 \leqslant Y \leqslant 1$ and
- for any fixed $0 \leqslant Y \leqslant 1$, $\big(x(Y,\theta)\,,\,y(Y,\theta)\,,\,z(Y,\theta)\big)$ runs over the circle $x^2 + y^2 = Y^2$, in the plane $z = e^Y$, when θ runs over $0 \leqslant \theta < 2\pi$.

Also notice that
$$x(Y,\theta)^2 + y(Y,\theta)^2 = Y^2$$
so that
$$Y = \sqrt{x(Y,\theta)^2 + y(Y,\theta)^2}$$
and
$$z(Y,\theta) = e^Y = e^{\sqrt{x(Y,\theta)^2 + y(Y,\theta)^2}}$$

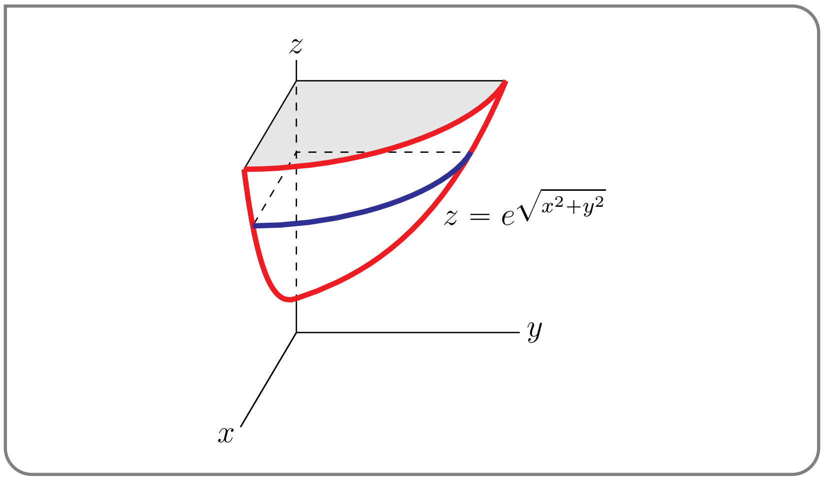
That is, the surface of revolution is contained in the (infinite) surface

$$z = e^{\sqrt{x^2+y^2}}$$

Remembering that $0 \leqslant Y \leqslant 1$, we have that $1 \leqslant z = e^Y \leqslant e$. Thus the surface of revolution is

$$z = e^{\sqrt{x^2+y^2}} \qquad 1 \leqslant z \leqslant e$$

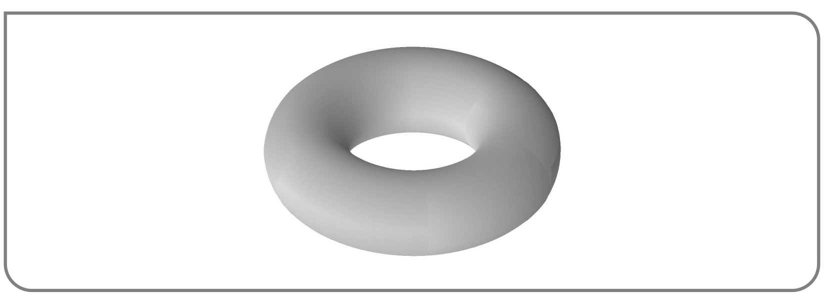
Finally here is a sketch of the part of the surface in the first octant, $x, y, z \geqslant 0$.



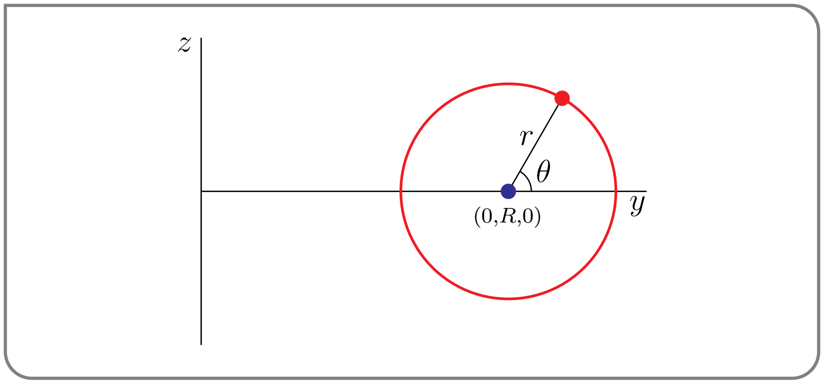
Example 3.1.4

Example 3.1.5 (Torus)

In this example, we are going to parametrize a donut (well, its surface), or an inner tube.



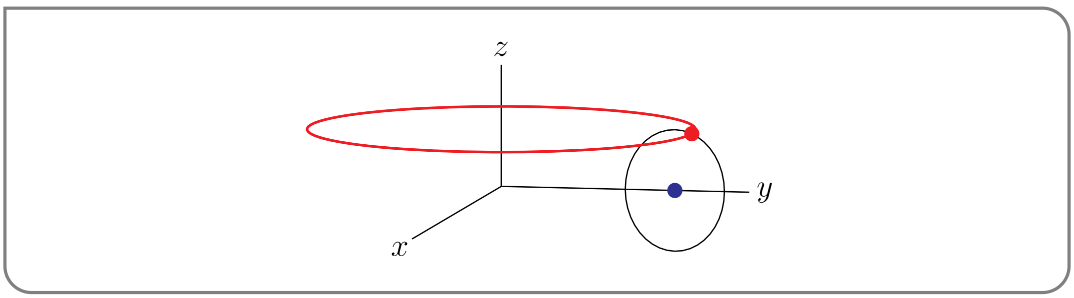
The formal mathematical name for the surface of a donut is a torus. Our strategy will be to first parametrize the section of the torus in the right half of the yz-plane, and then built up the full torus by rotating the circle about the z-axis. The section is a circle, sketched below. We'll assume that the centre of the circle is a distance R from the z-axis, and that



the circle has radius r. Then the red dot on the circle is at

$$x = 0$$
$$y = R + r\cos\theta$$
$$z = r\sin\theta$$

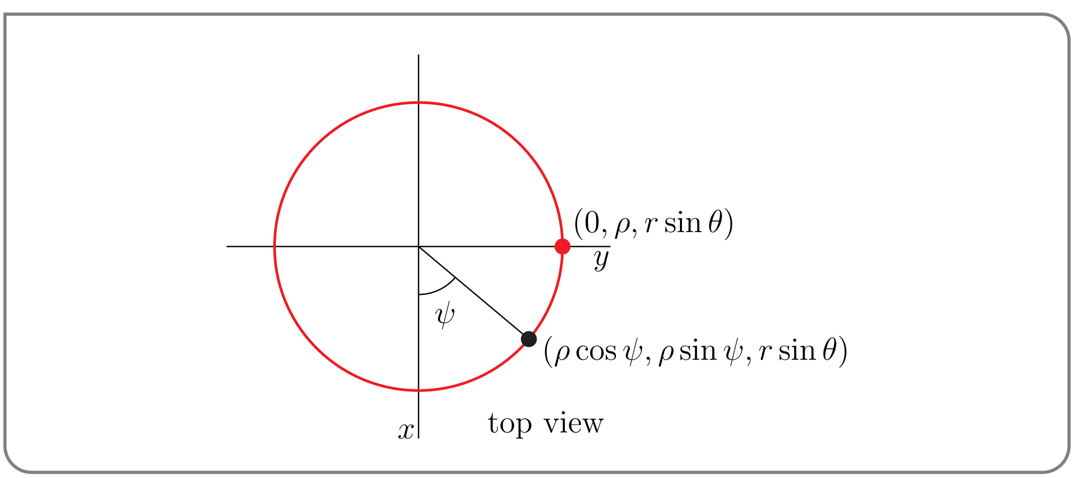
In particular the red dot is a distance $r\sin\theta$ above the xy-plane and is a distance $R + r\cos\theta$ from the z-axis. So when we rotate the section about the z-axis, the red dot sweeps out a circle which is sketched below.



The circle that the red dot sweeps out

- ∘ lies in the plane $z = r\sin\theta$ and
- ∘ is centred on the z-axis and
- ∘ has radius $\rho = R + r\cos\theta$.

We can parametrize the circle swept out in the usual way. Here is a top view of the circle, with the parameter, named ψ, indicated.



So the parametrization of the circle swept out by the red dot, and also the parametrization of the torus, is

$$x = \rho \cos\psi = (R + r\cos\theta)\cos\psi$$
$$y = \rho \sin\psi = (R + r\cos\theta)\sin\psi$$
$$z = r\sin\theta$$

or

$$\mathbf{r}(\theta,\psi) = (R + r\cos\theta)\cos\psi\,\hat{\imath} + (R + r\cos\theta)\sin\psi\,\hat{\jmath} + r\sin\theta\,\hat{\mathbf{k}} \qquad 0 \leqslant \theta,\psi < 2\pi$$

Example 3.1.5

3.2▲ Tangent Planes

If you are confronted with a complicated surface and want to get some idea of what it looks like near a specific point, probably the first thing that you will do is find the plane that best approximates the surface near the point. That is, find the tangent plane to the surface at the point. In general, a good way to specify a plane is to supply

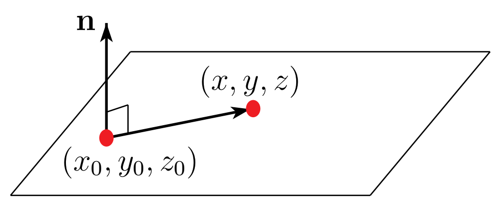
- a nonzero vector **n** (called a normal vector) perpendicular to the plane[3] (to determine the orientation of the plane) and
- one point (x_0, y_0, z_0) on the plane.

If (x, y, z) is any other point on the plane, then the vector

$$(x, y, z) - (x_0, y_0, z_0) = (x - x_0\,,\, y - y_0\,,\, z - z_0)$$

lies entirely in the plane and so is perpendicular to **n**. This gives the following very neat the equation for the plane.

$$\mathbf{n} \cdot (x - x_0\,,\, y - y_0\,,\, z - z_0) = 0$$



The following theorem provides formulae for normal vectors **n** to general surfaces, assuming first that the surface is parametrized, second that the surface is a graph and finally the surface is given by an implicit equation. The formulae are developed in the proof of the theorem.

3 Alternatively, you could find two vectors that are in the plane (and not parallel to each other), and then construct a normal vector by taking their cross product.

Theorem 3.2.1 (Normal vectors to surfaces).

(a) Let

$$\begin{aligned}\mathbf{r} : \mathcal{D} \subset \mathbb{R}^2 &\to \mathbb{R}^3 \\ (u,v) \in \mathcal{D} &\mapsto \mathbf{r}(u,v) = \big(x(u,v)\,,\, y(u,v)\,,\, z(u,v)\big)\end{aligned}$$

be a parametrized surface and let $(x_0, y_0, z_0) = \mathbf{r}(u_0, v_0)$ be a point on the surface. Set

$$\begin{aligned}\mathbf{T}_u &= \frac{\partial}{\partial u}\mathbf{r}(u, v_0)\Big|_{u=u_0} = \Big(\frac{\partial x}{\partial u}(u_0,v_0)\,,\, \frac{\partial y}{\partial u}(u_0,v_0)\,,\, \frac{\partial z}{\partial u}(u_0,v_0)\Big) \\ \mathbf{T}_v &= \frac{\partial}{\partial v}\mathbf{r}(u_0, v)\Big|_{v=v_0} = \Big(\frac{\partial x}{\partial v}(u_0,v_0)\,,\, \frac{\partial y}{\partial v}(u_0,v_0)\,,\, \frac{\partial z}{\partial v}(u_0,v_0)\Big)\end{aligned}$$

Then

$$\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v = \det\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \frac{\partial x}{\partial u}(u_0,v_0) & \frac{\partial y}{\partial u}(u_0,v_0) & \frac{\partial z}{\partial u}(u_0,v_0) \\ \frac{\partial x}{\partial v}(u_0,v_0) & \frac{\partial y}{\partial v}(u_0,v_0) & \frac{\partial z}{\partial v}(u_0,v_0)\end{vmatrix}$$

is normal (i.e. perpendicular) to the surface at (x_0, y_0, z_0).

(b) Let $(x_0, y_0, z_0) = f(x_0, y_0)$ be a point on the surface $z = f(x, y)$. Then,

$$\mathbf{n} = -f_x(x_0,y_0)\,\hat{\imath} - f_y(x_0,y_0)\,\hat{\jmath} + \hat{\mathbf{k}}$$

is normal to the surface at (x_0, y_0, z_0).

(c) Consider the surface given implicitly by the equation $G(x, y, z) = K$, where K is a constant. Let (x_0, y_0, z_0) be a point on the surface and assume that the gradient $\boldsymbol{\nabla} G\big(x_0, y_0, z_0\big) \neq \mathbf{0}$. Then

$$\mathbf{n} = \boldsymbol{\nabla} G\big(x_0, y_0, z_0\big)$$

is normal to the surface at (x_0, y_0, z_0).

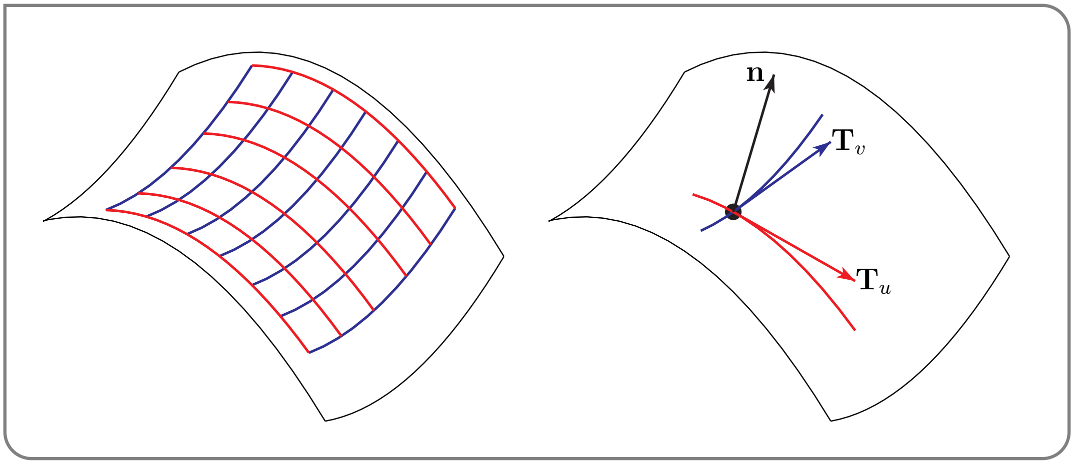
Note that none of the normal vectors $\mathbf{n}$ above need be of unit length.

Note that if we apply part (c) to $G(x, y, z) = z - f(x, y)$ we get the normal vector $\mathbf{n} = \boldsymbol{\nabla} G\big(x_0, y_0, z_0\big) = -f_x(x_0,y_0)\,\hat{\imath} - f_y(x_0,y_0)\,\hat{\jmath} + \hat{\mathbf{k}}$, which is the same as the normal vector provided by part (b). Of course they had to be at least parallel.

Proof. (a) First fix $v = v_0$ and let u vary. Then

$$u \mapsto \mathbf{r}(u, v_0) = \big(x(u,v_0)\,,\, y(u,v_0)\,,\, z(u,v_0)\big)$$

is a curve on the surface (the red curve in the figure on the right below) that passes through (x_0, y_0, z_0) (the black dot in the figure) when $u = u_0$. The tangent vector to this curve at



(x_0, y_0, z_0), which is also a tangent vector to the surface at (x_0, y_0, z_0), is

$$\mathbf{T}_u = \frac{\partial}{\partial u}\mathbf{r}(u, v_0)\Big|_{u=u_0} = \Big(\frac{\partial x}{\partial u}(u_0, v_0)\,,\ \frac{\partial y}{\partial u}(u_0, v_0)\,,\ \frac{\partial z}{\partial u}(u_0, v_0)\Big)$$

It is the red arrow in the figure on the right above.

Next fix $u = u_0$ and let v vary. Then

$$v \mapsto \mathbf{r}(u_0, v) = \big(x(u_0, v)\,,\ y(u_0, v)\,,\ z(u_0, v)\big)$$

is a curve on the surface (the blue curve in the figure on the right above) that passes through (x_0, y_0, z_0) when $v = v_0$. The tangent vector to this curve at (x_0, y_0, z_0), which is also a tangent vector to the surface at (x_0, y_0, z_0), is

$$\mathbf{T}_v = \frac{\partial}{\partial v}\mathbf{r}(u_0, v)\Big|_{v=v_0} = \Big(\frac{\partial x}{\partial v}(u_0, v_0)\,,\ \frac{\partial y}{\partial v}(u_0, v_0)\,,\ \frac{\partial z}{\partial v}(u_0, v_0)\Big)$$

It is the blue arrow in the figure on the right above.

We now have two vectors, namely $\mathbf{T}_u$ and $\mathbf{T}_v$, that are tangent to the surface at (x_0, y_0, z_0). So their cross product

$$\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v = \det\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \frac{\partial x}{\partial u}(u_0, v_0) & \frac{\partial y}{\partial u}(u_0, v_0) & \frac{\partial z}{\partial u}(u_0, v_0) \\ \frac{\partial x}{\partial v}(u_0, v_0) & \frac{\partial y}{\partial v}(u_0, v_0) & \frac{\partial z}{\partial v}(u_0, v_0) \end{vmatrix}$$

is normal (i.e. perpendicular) to the surface at (x_0, y_0, z_0). Note however that this vector need not be normalized. That is, it need not be of unit length.

(b) Next assume that the surface is given by the equation $z = f(x, y)$. Then, renaming u to x and v to y, we may reuse part (a):

$$\mathbf{r}(x, y) = \big(x, y, f(x, y)\big)$$

parametrizes the surface and, at $(x_0, y_0, z_0) = f(x_0, y_0)\big)$,

$$\mathbf{T}_x = \frac{\partial \mathbf{r}}{\partial x}(x_0, y_0) = \big(1\,,\ 0\,,\ f_x(x_0, y_0)\big)$$

$$\mathbf{T}_y = \frac{\partial \mathbf{r}}{\partial y}(x_0, y_0) = \big(0\,,\ 1\,,\ f_y(x_0, y_0)\big)$$

and

$$\mathbf{n} = \mathbf{T}_x \times \mathbf{T}_y = \det \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ 1 & 0 & f_x(x_0, y_0) \\ 0 & 1 & f_y(x_0, y_0) \end{vmatrix} = -f_x(x_0, y_0)\,\hat{\imath} - f_y(x_0, y_0)\,\hat{\jmath} + \hat{\mathbf{k}}$$

(c) Finally assume that the surface is given implicitly by the equation $G(x, y, z) = 0$ or, more generally by the equation, $G(x, y, z) = K$, where K is a constant. If $\mathbf{r}(t) = \big(x(t)\ y(t)\,,\ z(t)\big)$ is any curve with $\mathbf{r}(0) = (x_0, y_0, z_0)$ that lies on the surface, then

$$\begin{aligned} G\big(\mathbf{r}(t)\big) &= K \qquad \text{for all } t \\ \implies \frac{\mathrm{d}}{\mathrm{d}t} G\big(x(t), y(t), z(t)\big) &= 0 \qquad \text{for all } t \end{aligned}$$

Applying the chain rule gives

$$\frac{\partial G}{\partial x}\big(x(t), y(t), z(t)\big)\frac{\mathrm{d}x}{\mathrm{d}t}(t) + \frac{\partial G}{\partial y}\big(x(t), y(t), z(t)\big)\frac{\mathrm{d}y}{\mathrm{d}t}(t) + \frac{\partial G}{\partial z}\big(x(t), y(t), z(t)\big)\frac{\mathrm{d}z}{\mathrm{d}t}(t) = 0$$

The left hand side is exactly the dot product of $\big(\frac{\partial G}{\partial x}\,,\ \frac{\partial G}{\partial y}\,,\ \frac{\partial G}{\partial z}\big) = \boldsymbol{\nabla} G$ with $\big(\frac{\mathrm{d}x}{\mathrm{d}t}\,,\ \frac{\mathrm{d}y}{\mathrm{d}t}\,,\ \frac{\mathrm{d}z}{\mathrm{d}t}\big) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$, so that

$$\begin{aligned} \boldsymbol{\nabla} G\big(\mathbf{r}(t)\big) \cdot \mathbf{r}'(t) &= 0 \qquad \text{for all } t \\ \implies \boldsymbol{\nabla} G\big(x_0, y_0, z_0\big) \cdot \mathbf{r}'(0) &= 0 \end{aligned}$$

This tell us that $\boldsymbol{\nabla} G\big(x_0, y_0, z_0\big)$ is perpendicular to $\mathbf{r}'(0)$, which is a tangent vector to $G = K$ at (x_0, y_0, z_0). This is true for all curves $\mathbf{r}(t)$ on $G = K$ and so is true for all tangent vectors to $G = K$ at (x_0, y_0, z_0). So $\boldsymbol{\nabla} G\big(x_0, y_0, z_0\big)$ is a normal vector to $G(x, y, z) = K$ at (x_0, y_0, z_0). □

Example 3.2.2

Consider the surface

$$\begin{aligned} x &= x(u, v) = u \cos v \\ y &= y(u, v) = u \sin v \\ z &= z(u, v) = u \end{aligned}$$

Observe that

$$x(u, v)^2 + y(u, v)^2 = u^2 = z(u, v)^2$$

So our surface is also

$$G(x, y, z) = x^2 + y^2 - z^2 = 0$$

We shall sketch it shortly. But first, let's find it's tangent plane at $(x_0, y_0, z_0) = \mathbf{r}(u_0, v_0)$. In fact, let's do it twice. Once using the parametrization and once using its implicit equation. First, using the parametrization $\mathbf{r}(u, v) = u \cos v\,\hat{\imath} + u \sin v\,\hat{\jmath} + u\,\hat{\mathbf{k}}$, we have

$$\begin{aligned} \mathbf{T}_u &= \frac{\partial \mathbf{r}}{\partial u}(u_0, v_0) = \cos v_0\,\hat{\imath} + \sin v_0\,\hat{\jmath} + \hat{\mathbf{k}} \\ \mathbf{T}_v &= \frac{\partial \mathbf{r}}{\partial v}(u_0, v_0) = -u_0 \sin v_0\,\hat{\imath} + u_0 \cos v_0\,\hat{\jmath} \end{aligned}$$

so that

$$\begin{aligned}\mathbf{n} &= \big(\cos v_0\,\hat{\imath} + \sin v_0\,\hat{\jmath} + \hat{\mathbf{k}}\big) \times \big(- u_0 \sin v_0\,\hat{\imath} + u_0 \cos v_0\,\hat{\jmath}\big)\\ &= \big(- u_0 \cos v_0\,,\ -u_0 \sin v_0\,,\ u_0\big) = (-x_0, -y_0, z_0)\end{aligned}$$

Next using the implicit equation $G(x, y, z) = x^2 + y^2 - z^2 = 0$, we have the normal vector

$$\nabla G\big(x_0, y_0, z_0\big) = (2x_0, 2y_0, -2z_0) = -2(-x_0, -y_0, z_0)$$

Of course the two vectors $(-x_0, -y_0, z_0)$ and $-2(-x_0, -y_0, z_0)$ are parallel to each other. Either can be used as a normal vector and the tangent plane to $x^2 + y^2 - z^2 = 0$ at (x_0, y_0, z_0) is

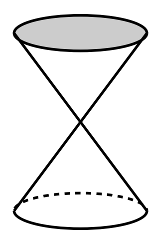
$$0 = \mathbf{n} \cdot (x - x_0, y - y_0, z - z_0) = -x_0(x - x_0) - y_0(y - y_0) + z_0(z - z_0)$$

provided $(x_0, y_0, z_0) \neq \mathbf{0}$. In the event that $(x_0, y_0, z_0) = \mathbf{0}$ the "tangent plane equation" reduces to $0 = 0$ and there is clearly a problem.

More generally, if $\mathbf{T}_u \times \mathbf{T}_v = \mathbf{0}$ (or $\nabla G(x_0, y_0, z_0) = \mathbf{0}$), then either[4]

- the surface fails to have a tangent plane at (x_0, y_0, z_0), or
- our parametrization is screwy[5] there. For example, we can parametrize the xy-plane, $z = 0$, by $\mathbf{r}(u, v) = u \cos v\,\hat{\imath} + u \sin v\,\hat{\jmath}$. (This is just polar coordinates.) Then $\mathbf{T}_u = \cos v_0\,\hat{\imath} + \sin v_0\,\hat{\jmath}$ and $\mathbf{T}_v = -u_0 \sin v_0\,\hat{\imath} + u_0 \cos v_0\,\hat{\jmath}$, so that $\mathbf{T}_u \times \mathbf{T}_v = u_0\hat{\mathbf{k}}$ is $\mathbf{0}$ when $u_0 = 0$. But the plane $z = 0$ is its own tangent plane everywhere.

The surface of current interest is $x^2 + y^2 = z^2$. The intersection of this surface with the horizontal plane $z = z_0$ is $x^2 + y^2 = z_0^2$, which is the circle of radius $|z_0|$ centred on $x = y = 0$. So our surface is a stack of circles. The radius of the circle in the xy-plane is zero. The radius increases linearly as we move away from the xy-plane. Our surface is a cone. It does not have a tangent plane at $(0, 0, 0)$.



Example 3.2.2

Example 3.2.3

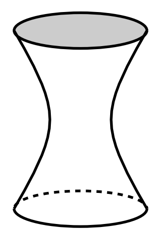
This time we shall find the tangent planes to the surface

$$x^2 + y^2 - z^2 = 1$$

4 We saw the same dichotomy when considering what happened for a curve when $\mathbf{r}'(t) = 0$. See Example 1.1.10.

5 Of course "screwy" is not a mathematically precise word. One way a parametrization $\mathbf{r}(u, v)$ could be "screwy" is if it failed to give a one-to-one correspondence between parameter values (u, v) and points on (part of) the surface. For example, polar coordinates $\mathbf{r}(u, v) = u \cos v\,\hat{\imath} + u \sin v\,\hat{\jmath}$ give $\mathbf{r}(0, v) = (0, 0)$ for all values of v.

As for the cone of the last example, the intersection of this surface with the horizontal plane $z = z_0$ is a circle — the circle of radius $\sqrt{1+z_0^2}$ centred on $x = y = 0$. Our surface is again a stack of circles. The radius of the circle in the xy-plane is 1. The radius increases as we move away from the xy-plane. Here is a sketch of the surface.



It is called a hyperboloid[6] of one sheet.

Using the implicit equation $G(x,y,z) = x^2 + y^2 - z^2 = 1$, we have

$$\nabla G(x_0, y_0, z_0) = (2x_0, 2y_0, -2z_0) = 2(x_0, y_0, -z_0)$$

and we may take $(x_0, y_0, -z_0)$ as a normal vector at (x_0, y_0, z_0). So the tangent plane to $x^2 + y^2 - z^2 = 1$ at (x_0, y_0, z_0) is

$$0 = \mathbf{n} \cdot (x - x_0, y - y_0, z - z_0) = x_0(x - x_0) + y_0(y - y_0) - z_0(z - z_0)$$

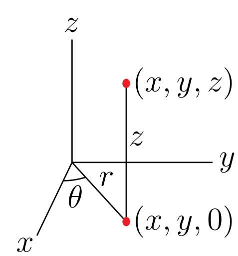
This time $\mathbf{n} = (x_0, y_0, -z_0) \neq \mathbf{0}$, so that we have a tangent plane, at every point of the surface. In particular, the vanishing of $\mathbf{n} = (x_0, y_0, -z_0)$ at $(x_0, y_0, z_0) = (0,0,0)$ is not a problem because $(0,0,0)$ is not on the surface.

Example 3.2.3

Example 3.2.4 (Optional — Parametrizing the Hyperboloid of One Sheet)

The hyperboloid of one sheet, $x^2 + y^2 - z^2 = 1$, has a symmetry. It is invariant under rotation about the z-axis. So it is natural to parametrize the surface using cylindrical coordinates.

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$



In cylindrical coordinates the surface $x^2 + y^2 - z^2 = 1$ is $r^2 - z^2 = 1$, and we could parametrize it by $\mathbf{r}(\theta, z) = \sqrt{1+z^2}\cos\theta\,\hat{\boldsymbol{\imath}} + \sqrt{1+z^2}\sin\theta\,\hat{\boldsymbol{\jmath}} + z\,\hat{\mathbf{k}}$. Alternatively, we can eliminate the square roots in the parametrization by exploiting the hyperbolic trig functions

$$\sinh u = \frac{1}{2}\left(e^u - e^{-u}\right) \qquad \cosh u = \frac{1}{2}\left(e^u + e^{-u}\right)$$

6 There are also hyperboloids of two sheets. See Appendix H.

The functions have properties[7] that are very similar to those of $\sin\theta$ and $\cos\theta$.

$$\frac{d}{du}\cosh u = \sinh u \qquad \frac{d}{du}\sinh u = \cosh u \qquad \cosh^2 u - \sinh^2 u = 1$$

Observe that we can turn $r^2 - z^2 = 1$ into $\cosh^2 u - \sinh^2 u = 1$ simply by setting $r = \cosh u$, $z = \sinh u$. Doing so yields the parametrization

$$\mathbf{r}(\theta, u) = \cosh u\, \cos\theta\, \hat{\imath} + \cosh u\, \sin\theta\, \hat{\jmath} + \sinh u\, \hat{\mathbf{k}}$$

As an exercise in working with hyperbolic trig functions, we'll use this parametrization to find $\hat{\mathbf{n}}$.

$$\begin{aligned} x &= \cosh u\, \cos\theta & x_u &= \sinh u\, \cos\theta & x_\theta &= -\cosh u\, \sin\theta \\ y &= \cosh u\, \sin\theta & y_u &= \sinh u\, \sin\theta & y_\theta &= \cosh u\, \cos\theta \\ z &= \sinh u & z_u &= \cosh u & z_\theta &= 0 \end{aligned}$$

So

$$\begin{aligned} \mathbf{n} = \mathbf{T}_u \times \mathbf{T}_\theta &= \det\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \sinh u\, \cos\theta & \sinh u\, \sin\theta & \cosh u \\ -\cosh u\, \sin\theta & \cosh u\, \cos\theta & 0 \end{vmatrix} \\ &= \big(-\cosh^2 u\, \cos\theta\, ,\, -\cosh^2 u\, \sin\theta\, ,\, \sinh u \cosh u \big) \end{aligned}$$

Example 3.2.4

3.3 Surface Integrals

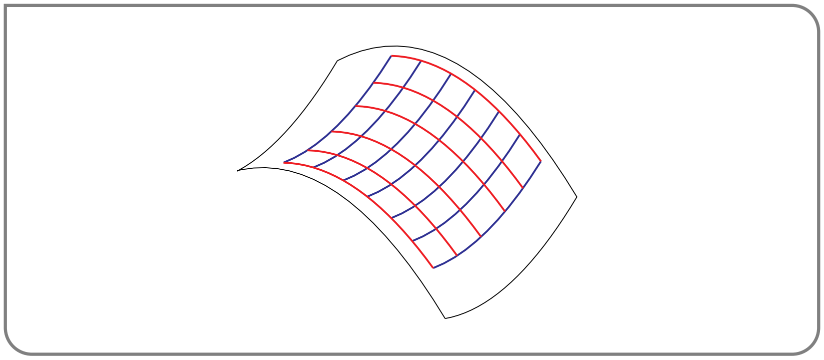
We are now going to define two types of integrals over surfaces.

- Integrals that look like $\iint_S \rho\, dS$ are used to compute the area and, when ρ is, for example, a mass density, the mass of the surface S.
- Integrals that look like $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$, with $\hat{\mathbf{n}}(x, y, z)$ being a unit vector that is perpendicular to S at (x, y, z), are called flux integrals. We shall see in §3.4, that when $\mathbf{v}$ is the velocity field of a moving fluid and ρ is the density of the fluid, then $\iint_S \rho \mathbf{v} \cdot \hat{\mathbf{n}}\, dS$ is the rate at which fluid is crossing the surface S.

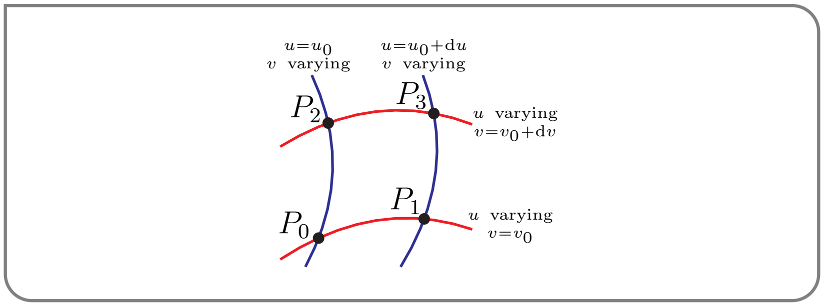
3.3.1 Parametrized Surfaces

Suppose that we wish to integrate over part, S, of a surface that is parametrized by $\mathbf{r}(u, v)$. We start by cutting S up into small pieces by drawing a bunch of curves of constant u (the blue curves in the figure below) and a bunch of curves of constant v (the red curves in the figure below).

7 This is no accident: $\cosh u = \cos(iu)$ and $\sinh u = -i\sin(iu)$, where i is the usual complex number that obeys $i^2 = -1$. You can verify these formulae by just checking that $\cosh u$ and $\cos(iu)$ have the same Taylor expansions and that $\sinh u$ and $-i\sin(iu)$ have the same Taylor expansions.



Concentrate on any one the small pieces. Here is a greatly magnified sketch.



For example, the lower red curve was constructed by holding v fixed at the value v_0, varying u and sketching $\mathbf{r}(u, v_0)$, and the upper red curve was constructed by holding v fixed at the slightly larger value $v_0 + \mathrm{d}v$, varying u and sketching $\mathbf{r}(u, v_0 + \mathrm{d}v)$. So the four intersection points in the figure are

$$\begin{aligned} P_2 &= \mathbf{r}(u_0, v_0 + \mathrm{d}v) & P_3 &= \mathbf{r}(u_0 + \mathrm{d}u, v_0 + \mathrm{d}v) \\ P_0 &= \mathbf{r}(u_0, v_0) & P_1 &= \mathbf{r}(u_0 + \mathrm{d}u, v_0) \end{aligned}$$

Now if

$$\mathbf{R}(t) = \mathbf{r}(u_0 + t\mathrm{d}U\,,\, v_0 + t\mathrm{d}V)$$

(where $\mathrm{d}U$ and $\mathrm{d}V$ are any small constants) then, by Taylor expansion,

$$\begin{aligned} \mathbf{r}\big(u_0 + \mathrm{d}U\,,\, v_0 + \mathrm{d}V\big) &= \mathbf{R}(1) \\ &\approx \big[\mathbf{R}(0) + \mathbf{R}'(0)\,\big(t - 0\big)\big]_{t=1} \\ &= \mathbf{r}(u_0\,,\, v_0) + \frac{\partial \mathbf{r}}{\partial u}(u_0\,,\, v_0)\,\mathrm{d}U + \frac{\partial \mathbf{r}}{\partial v}(u_0\,,\, v_0)\,\mathrm{d}V \end{aligned}$$

Applying this three times, once with $\mathrm{d}U = \mathrm{d}u$, $\mathrm{d}V = 0$, once with $\mathrm{d}U = 0$ $\mathrm{d}V = \mathrm{d}v$, and once with $\mathrm{d}U = \mathrm{d}u$, $\mathrm{d}V = \mathrm{d}v$,

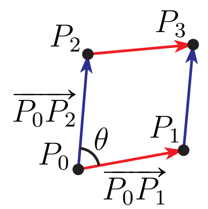
$$\begin{aligned} P_0 &= \mathbf{r}(u_0\,,\, v_0) \\ P_1 &= \mathbf{r}(u_0 + \mathrm{d}u, v_0) &&\approx \mathbf{r}(u_0\,,\, v_0) + \frac{\partial \mathbf{r}}{\partial u}(u_0\,,\, v_0)\,\mathrm{d}u \\ P_2 &= \mathbf{r}(u_0, v_0 + \mathrm{d}v) &&\approx \mathbf{r}(u_0\,,\, v_0) + \frac{\partial \mathbf{r}}{\partial v}(u_0\,,\, v_0)\,\mathrm{d}v \\ P_3 &= \mathbf{r}(u_0 + \mathrm{d}u, v_0 + \mathrm{d}v) &&\approx \mathbf{r}(u_0\,,\, v_0) + \frac{\partial \mathbf{r}}{\partial u}(u_0\,,\, v_0)\,\mathrm{d}u + \frac{\partial \mathbf{r}}{\partial v}(u_0\,,\, v_0)\,\mathrm{d}v \end{aligned}$$

We have dropped all Taylor expansion terms that are of degree two or higher in $\mathrm{d}u$, $\mathrm{d}v$. The reason is that, in defining the integral, we take the limit $\mathrm{d}u, \mathrm{d}v \to 0$. Because of that limit, all of the dropped terms contribute exactly 0 to the integral. We shall not prove this. But we shall show, in the optional §3.3.5, why this is the case.

The small piece of our surface with corners P_0, P_1, P_2, P_3 is approximately a parallelogram with sides

$$\overrightarrow{P_0P_1} \approx \overrightarrow{P_2P_3} \approx \frac{\partial \mathbf{r}}{\partial u}(u_0\,,\, v_0)\,\mathrm{d}u$$

$$\overrightarrow{P_0P_2} \approx \overrightarrow{P_1P_3} \approx \frac{\partial \mathbf{r}}{\partial v}(u_0\,,\, v_0)\,\mathrm{d}v$$



Denote by θ the angle between the vectors $\overrightarrow{P_0P_1}$ and $\overrightarrow{P_0P_2}$. The base of the parallelogram, $\overrightarrow{P_0P_1}$, has length $|\overrightarrow{P_0P_1}|$, and the height of the parallelogram is $|\overrightarrow{P_0P_2}|\ \sin\theta$. So the area of the parallelogram is[8]

$$\begin{aligned} |\overrightarrow{P_0P_1}|\ |\overrightarrow{P_0P_2}|\ \sin\theta &= |\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}| \\ &\approx \left|\frac{\partial \mathbf{r}}{\partial u}(u_0\,,\, v_0) \times \frac{\partial \mathbf{r}}{\partial v}(u_0\,,\, v_0)\right| \mathrm{d}u\mathrm{d}v \end{aligned}$$

Furthermore, $\frac{\partial \mathbf{r}}{\partial u}(u_0\,,\, v_0)$ and $\frac{\partial \mathbf{r}}{\partial v}(u_0\,,\, v_0)$ are tangent vectors to the curves $\mathbf{r}(t\,,\, v_0)$ and $\mathbf{r}(u_0\,,\, t)$ respectively. Both of these curves lie in S. So $\frac{\partial \mathbf{r}}{\partial u}(u_0\,,\, v_0)$ and $\frac{\partial \mathbf{r}}{\partial v}(u_0\,,\, v_0)$ are tangent vectors to S at $\mathbf{r}(u_0, v_0)$ and the cross product $\frac{\partial \mathbf{r}}{\partial u}(u_0\,,\, v_0) \times \frac{\partial \mathbf{r}}{\partial v}(u_0\,,\, v_0)$ is perpendicular to S at $\mathbf{r}(u_0, v_0)$. We have found both $\mathrm{d}S$ and $\hat{\mathbf{n}}\,\mathrm{d}S$, where $\hat{\mathbf{n}}$ is a unit normal vector to the surface.

Equation 3.3.1.

For the parametrized surface $\mathbf{r}(u, v)$,

$$\hat{\mathbf{n}}\,\mathrm{d}S = \pm\frac{\partial \mathbf{r}}{\partial u}(u\,,\, v) \times \frac{\partial \mathbf{r}}{\partial v}(u\,,\, v)\ \mathrm{d}u\mathrm{d}v$$

$$\mathrm{d}S = \left|\frac{\partial \mathbf{r}}{\partial u}(u\,,\, v) \times \frac{\partial \mathbf{r}}{\partial v}(u\,,\, v)\right|\ \mathrm{d}u\mathrm{d}v$$

The $\pm$ sign in (3.3.1) is there because there are two unit normal vectors at each point of a surface, one on each side of the surface. Typically, the application itself tells you which of the two normal vectors should be used. We shall see many examples shortly.

3.3.2 ▸▸ Graphs

The surface which is the graph $z = f(x, y)$ can be parametrized by

$$\mathbf{r}(x, y) = x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + f(x, y)\,\hat{\mathbf{k}}$$

8 As we mentioned above, the approximation below becomes exact when the limit $\mathrm{d}u, \mathrm{d}v \to 0$ is taken in the definition of the integral. See the optional §3.3.5.

As

$$\frac{\partial \mathbf{r}}{\partial x} = \hat{\imath} + \frac{\partial f}{\partial x}\,\hat{\mathbf{k}} \qquad \text{and} \qquad \frac{\partial \mathbf{r}}{\partial y} = \hat{\jmath} + \frac{\partial f}{\partial y}\,\hat{\mathbf{k}}$$

we have

$$\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{bmatrix} = -f_x(x,y)\,\hat{\imath} - f_y(x,y)\,\hat{\jmath} + \hat{\mathbf{k}}$$

So, (3.3.1) gives the following.

Equation 3.3.2.

For the surface $z = f(x,y)$,

$$\hat{\mathbf{n}}\,\mathrm{d}S = \pm\big[-f_x(x,y)\,\hat{\imath} - f_y(x,y)\,\hat{\jmath} + \hat{\mathbf{k}}\big]\,\mathrm{d}x\mathrm{d}y$$

$$\mathrm{d}S = \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\,\mathrm{d}x\mathrm{d}y$$

Similarly, for the surface $x = g(y,z)$,

$$\hat{\mathbf{n}}\,\mathrm{d}S = \pm\big[\hat{\imath} - g_y(y,z)\,\hat{\jmath} - g_z(y,z)\,\hat{\mathbf{k}}\big]\,\mathrm{d}y\mathrm{d}z$$

$$\mathrm{d}S = \sqrt{1 + g_y(y,z)^2 + g_z(y,z)^2}\,\mathrm{d}y\mathrm{d}z$$

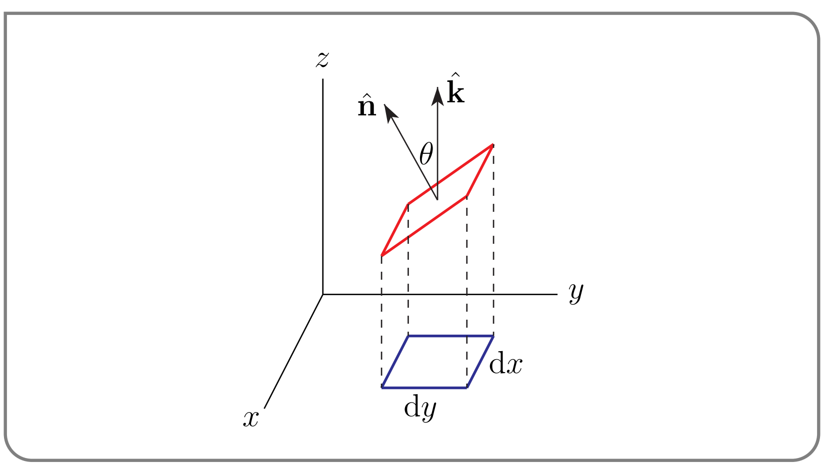
and for the surface $y = h(x,z)$,

$$\hat{\mathbf{n}}\,\mathrm{d}S = \pm\big[-h_x(x,z)\,\hat{\imath} + \hat{\jmath} - h_z(x,z)\,\hat{\mathbf{k}}\big]\,\mathrm{d}x\mathrm{d}z$$

$$\mathrm{d}S = \sqrt{1 + h_x(x,z)^2 + h_z(x,z)^2}\,\mathrm{d}x\mathrm{d}z$$

Again, in any given application, some care must be taken in choosing the sign in (3.3.2), so as to get the appropriate normal vector.

The formulae like $\mathrm{d}S = \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\,\mathrm{d}x\mathrm{d}y$ in (3.3.2) have geometric interpretations. The red parallelogram in the sketch



represents a little piece of our surface. It has area $\mathrm{d}S = \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\,\mathrm{d}x\mathrm{d}y$. The blue parallelogram in the same sketch represents the projection of the red parallelogram onto the xy-plane. It has area $\mathrm{d}x\mathrm{d}y$. The vector $\hat{\mathbf{n}}$ in the sketch is a unit normal for the red parallelogram. We have seen that it is parallel to

$$\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} = -f_x(x,y)\,\hat{\boldsymbol{\imath}} - f_y(x,y)\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}$$

so that the angle θ between $\hat{\mathbf{n}}$ and $\hat{\mathbf{k}}$ obeys

$$\begin{aligned}\cos\theta &= \frac{(-f_x(x,y)\,\hat{\boldsymbol{\imath}} - f_y(x,y)\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}})\cdot\hat{\mathbf{k}}}{|-f_x(x,y)\,\hat{\boldsymbol{\imath}} - f_y(x,y)\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}|\,|\hat{\mathbf{k}}|} \\ &= \frac{1}{\sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}}\end{aligned}$$

The geometric interpretation of $\mathrm{d}S = \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\,\mathrm{d}x\mathrm{d}y$ is that the area $\mathrm{d}S$ of a little piece of surface is the area of its projection on the xy-plane times the factor $\frac{1}{\cos\theta}$ where θ is the angle between $\hat{\mathbf{n}}$ (which is perpendicular to the surface) and $\hat{\mathbf{k}}$ (which is perpendicular to the xy-plane). Notice that

- when θ is close to zero, which corresponds the f being almost constant and our surface being almost parallel to the xy-plane, $\mathrm{d}S$ reduces to almost $\mathrm{d}x\mathrm{d}y$.
- On the other hand, in the limit $\theta \to \frac{\pi}{2}$, which corresponds to f_x and/or f_y becoming infinite and our surface becoming perpendicular to the xy-plane, $\mathrm{d}S$ becomes "infinity times" $\mathrm{d}x\mathrm{d}y$. In this case, we should represent our surface either in the form $x = g(y,z)$ or in the form $y = h(x,z)$, rather than in the form $z = f(x,y)$.

3.3.3 ▸▸ Surfaces Given by Implicit Equations

Finally suppose that the surface is given by the equation $G(x,y,z) = K$, with K a constant. Suppose further that at some point on the surface $\frac{\partial G}{\partial z} \neq 0$. Then near that point we may solve[9] the equation $G(x,y,z) = K$ for z as a function of x and y. That is, the surface also obeys $z = f(x,y)$ for a function $f(x,y)$ that satisfies

$$G\big(x,y,f(x,y)\big) = K$$

near the point. Differentiating this with respect to x and y gives, by the chain rule,

$$\begin{aligned}0 &= \frac{\partial}{\partial x}\Big[G\big(x,y,f(x,y)\big)\Big] = G_x\big(x,y,f(x,y)\big) + G_z\big(x,y,f(x,y)\big)\,f_x(x,y) \\ 0 &= \frac{\partial}{\partial y}\Big[G\big(x,y,f(x,y)\big)\Big] = G_y\big(x,y,f(x,y)\big) + G_z\big(x,y,f(x,y)\big)\,f_y(x,y)\end{aligned}$$

9 This is called the implicit function theorem. We will not prove it. But it is not so hard to understand why it is true, if one thinks in terms of the Taylor expansion of G about the point. For simplicity, let's suppose that the point is $(0,0,0)$ and G happens to be exactly equal to its first order Taylor expansion about $(0,0,0)$. That is, $G(x,y,z) = A + Bx + Cy + Dz$, for some constants A, B, C, D. Since $(0,0,0)$ is on the surface, $A = K$. As $\frac{\partial G}{\partial z} = D \neq 0$ we can easily solve $G(x,y,z) = K$ for z as a function of x and y. Namely $z = \frac{1}{D}(-Bx - Cy)$. The general proof is based on the fact that, under reasonable hypotheses, the first order Taylor expansion is a good approximation to G near $(0,0,0)$.

which implies

$$f_x(x,y) = -\frac{G_x\big(x,y,f(x,y)\big)}{G_z\big(x,y,f(x,y)\big)} \qquad f_y(x,y) = -\frac{G_y\big(x,y,f(x,y)\big)}{G_z\big(x,y,f(x,y)\big)}$$

and

$$\begin{aligned}-f_x(x,y)\,\hat{\imath} - f_y(x,y)\,\hat{\jmath} + \hat{\mathbf{k}} &= \frac{G_x\big(x,y,f(x,y)\big)}{G_z\big(x,y,f(x,y)\big)}\,\hat{\imath} + \frac{G_y\big(x,y,f(x,y)\big)}{G_z\big(x,y,f(x,y)\big)}\,\hat{\jmath} + \hat{\mathbf{k}} \\ &= \frac{\nabla G\big(x,y,f(x,y)\big)}{G_z\big(x,y,f(x,y)\big)}\end{aligned}$$

So, by (3.3.2),

Equation 3.3.3.

For the surface $G(x,y,z) = K$, when $G_z(x,y,z) \neq 0$,

$$\hat{\mathbf{n}}\,\mathrm{d}S = \pm\frac{\nabla G(x,y,z)}{\nabla G(x,y,z)\cdot\hat{\mathbf{k}}}\,\mathrm{d}x\mathrm{d}y$$

$$\mathrm{d}S = \left|\frac{\nabla G(x,y,z)}{\nabla G(x,y,z)\cdot\hat{\mathbf{k}}}\right|\,\mathrm{d}x\mathrm{d}y$$

Similarly, for the surface $G(x,y,z) = K$, when $G_x(x,y,z) \neq 0$,

$$\hat{\mathbf{n}}\,\mathrm{d}S = \pm\frac{\nabla G(x,y,z)}{\nabla G(x,y,z)\cdot\hat{\imath}}\,\mathrm{d}y\mathrm{d}z$$

$$\mathrm{d}S = \left|\frac{\nabla G(x,y,z)}{\nabla G(x,y,z)\cdot\hat{\imath}}\right|\,\mathrm{d}y\mathrm{d}z$$

and for the surface $G(x,y,z) = K$, when $G_y(x,y,z) \neq 0$,

$$\hat{\mathbf{n}}\,\mathrm{d}S = \pm\frac{\nabla G(x,y,z)}{\nabla G(x,y,z)\cdot\hat{\jmath}}\,\mathrm{d}x\mathrm{d}z$$

$$\mathrm{d}S = \left|\frac{\nabla G(x,y,z)}{\nabla G(x,y,z)\cdot\hat{\jmath}}\right|\,\mathrm{d}x\mathrm{d}z$$

If, for some point (x_0,y_0,z_0) we have $G_x(x_0,y_0,z_0) = G_y(x_0,y_0,z_0) = G_z(x_0,y_0,z_0) = 0$, we also have a problem! Often this is a sign that our surface is not smooth at (x_0,y_0,z_0) and in fact does not have a normal vector there. For an example of this, see Example 3.2.2.

3.3.4 ▸▸ Examples of $\iint_S \rho\,\mathrm{d}S$

We'll start by computing, in several different ways, the surface area of the hemisphere

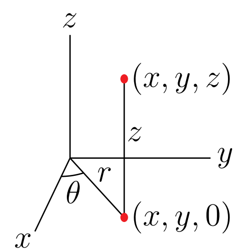
$$x^2 + y^2 + z^2 = a^2 \qquad z \geqslant 0$$

(with $a > 0$). You probably know, from high school, that the answer is $\frac{1}{2} \times 4\pi a^2 = 2\pi a^2$. But you have probably not seen a derivation of this answer. Note that, since $x^2 + y^2 = a^2 - z^2$ on the hemisphere, the set of (x, y)'s for which there is a z with (x, y, z) on the hemisphere is exactly $\{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leqslant a^2 \}$.

Example 3.3.4 (Area of a hemisphere — using cylindrical coordinates)

Let's parametrize the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geqslant 0$, using as parameters the polar coordinates r, θ of the cylindrical coordinates[10]

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$



and then apply (3.3.1). In cylindrical coordinates the equation $x^2 + y^2 + z^2 = a^2$ becomes $r^2 + z^2 = a^2$, and the condition $x^2 + y^2 \leqslant a^2$ is $0 \leqslant r \leqslant a$, $0 \leqslant \theta < 2\pi$.

So the hemisphere can be parametrized by

$$\big(x(r,\theta)\,,\ y(r,\theta)\,,\ z(r,\theta)\big) = \big(r\cos\theta\,,\ r\sin\theta\,,\ \sqrt{a^2 - r^2}\,\big)$$
$$\text{with } \ 0 \leqslant r \leqslant a,\ 0 \leqslant \theta < 2\pi$$

Note that we selected the positive solution $z = \sqrt{a^2 - r^2}$ of $r^2 + z^2 = a^2$ in order to satisfy the condition that $z \geqslant 0$. Since

$$\Big(\frac{\partial x}{\partial r}, \frac{\partial y}{\partial r}, \frac{\partial z}{\partial r}\Big) = \Big(\cos\theta\,,\ \sin\theta\,,\ -\frac{r}{\sqrt{a^2 - r^2}}\Big)$$
$$\Big(\frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}, \frac{\partial z}{\partial \theta}\Big) = (-r\sin\theta\,,\ r\cos\theta\,,\ 0)$$

(3.3.1) yields

$$\begin{aligned}
\hat{\mathbf{n}}\,\mathrm{d}S &= \pm\Big(\frac{\partial x}{\partial r}, \frac{\partial y}{\partial r}, \frac{\partial z}{\partial r}\Big) \times \Big(\frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}, \frac{\partial z}{\partial \theta}\Big)\,\mathrm{d}r\mathrm{d}\theta \\
&= \pm\det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \cos\theta & \sin\theta & -\frac{r}{\sqrt{a^2-r^2}} \\ -r\sin\theta & r\cos\theta & 0 \end{bmatrix} \mathrm{d}r\mathrm{d}\theta \\
&= \pm\Big(\frac{r^2\cos\theta}{\sqrt{a^2 - r^2}}, \frac{r^2\sin\theta}{\sqrt{a^2 - r^2}}, r\Big)\,\mathrm{d}r\mathrm{d}\theta \\
\mathrm{d}S &= \sqrt{\frac{r^4}{a^2 - r^2} + r^2}\,\mathrm{d}r\mathrm{d}\theta = \sqrt{\frac{a^2 r^2}{a^2 - r^2}}\,\mathrm{d}r\mathrm{d}\theta = \frac{ar}{\sqrt{a^2 - r^2}}\,\mathrm{d}r\mathrm{d}\theta
\end{aligned}$$

10 The symbols r, θ, z are the standard mathematics symbols for the cylindrical coordinates. Appendix G gives another set of symbols that is commonly used in the physical sciences and engineering.

So the area of the hemisphere is

$$\begin{aligned}\int_0^a \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta\ \frac{ar}{\sqrt{a^2-r^2}} &= 2\pi a \int_0^a \mathrm{d}r\ \frac{r}{\sqrt{a^2-r^2}} \\ &= 2\pi a \int_{a^2}^0 \frac{-\mathrm{d}u/2}{\sqrt{u}} \qquad \text{with } u = a^2 - r^2,\ \mathrm{d}u = -2r\,\mathrm{d}r \\ &= 2\pi a \Big[-\sqrt{u}\Big]_{a^2}^0 \\ &= 2\pi a^2\end{aligned}$$

as it should be.

Example 3.3.4

Example 3.3.5 (Area of a hemisphere — using an implicit equation)

This time we'll compute the area of the hemisphere by using that, if (x,y,z) is on the hemisphere, then $G(x,y,z) = a^2$ with $G(x,y,z) = x^2 + y^2 + z^2$. Since

$$\nabla G(x,y,z) = \big(2x\,,\,2y\,,\,2z\big)$$

(3.3.3) yields

$$\begin{aligned}\mathrm{d}S &= \left|\frac{\nabla G\big(x,y,z\big)}{\nabla G\big(x,y,z\big)\cdot\hat{\mathbf{k}}}\right| \mathrm{d}x\mathrm{d}y \\ &= \left|\frac{\big(2x\,,\,2y\,,\,2z\big)}{2z}\right| \mathrm{d}x\mathrm{d}y \\ &= \frac{\sqrt{x^2+y^2+z^2}}{|z|}\,\mathrm{d}x\mathrm{d}y \\ &= \frac{a}{\sqrt{a^2-x^2-y^2}}\,\mathrm{d}x\mathrm{d}y \qquad \text{on } x^2+y^2+z^2 = a^2\end{aligned}$$

So the area is $\iint_{x^2+y^2\leqslant a^2} \frac{a}{\sqrt{a^2-x^2-y^2}}\,\mathrm{d}x\mathrm{d}y$. To evaluate this integral, we switch to polar coordinates, substituting $x = r\cos\theta$, $y = r\sin\theta$. This gives

$$\begin{aligned}\text{area} = \iint_{x^2+y^2\leqslant a^2} \frac{a}{\sqrt{a^2-x^2-y^2}}\,\mathrm{d}x\mathrm{d}y &= \int_0^a \mathrm{d}r\ r\int_0^{2\pi} \mathrm{d}\theta\ \frac{a}{\sqrt{a^2-r^2}} \\ &= 2\pi a\int_0^a \mathrm{d}r\ \frac{r}{\sqrt{a^2-r^2}}\end{aligned}$$

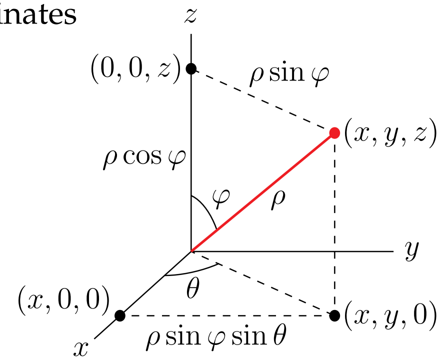
We already showed, in Example 3.3.4, that the value of this integral is $2\pi a^2$.

Example 3.3.5

Example 3.3.6 (Area of a hemisphere — using spherical coordinates)

Of course "integrating over a sphere" cries out for spherical coordinates. So this time we parametrize the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geqslant 0$, using as parameters the angular coordinates θ, φ of the spherical coordinates

$$x = \rho \sin\varphi \cos\theta$$
$$y = \rho \sin\varphi \sin\theta$$
$$z = \rho \cos\varphi$$



and then apply (3.3.1). In spherical coordinates the equation $x^2 + y^2 + z^2 = a^2$ becomes just $\rho^2 = a^2$, and the condition $z \geqslant 0$ is $0 \leqslant \varphi \leqslant \frac{\pi}{2}$, $0 \leqslant \theta < 2\pi$. So the hemisphere can be parametrized[11] by

$$\big(x(\theta,\varphi)\,,\,y(\theta,\varphi)\,,\,z(\theta,\varphi)\big) = \big(a\sin\varphi\cos\theta\,,\,a\sin\varphi\sin\theta\,,\,a\cos\varphi\,\big) \quad 0 \leqslant \varphi \leqslant \frac{\pi}{2},\ 0 \leqslant \theta < 2\pi$$

Since

$$\Big(\frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}, \frac{\partial z}{\partial \theta}\Big) = \big(-a\sin\varphi\sin\theta\,,\,a\sin\varphi\cos\theta\,,\,0\big)$$
$$\Big(\frac{\partial x}{\partial \varphi}, \frac{\partial y}{\partial \varphi}, \frac{\partial z}{\partial \varphi}\Big) = \big(a\cos\varphi\cos\theta\,,\,a\cos\varphi\sin\theta\,,\,-a\sin\varphi\big)$$

(3.3.1) yields

$$\begin{aligned}
\hat{\mathbf{n}}\,\mathrm{d}S &= \pm\Big(\frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}, \frac{\partial z}{\partial \theta}\Big) \times \Big(\frac{\partial x}{\partial \varphi}, \frac{\partial y}{\partial \varphi}, \frac{\partial z}{\partial \varphi}\Big)\,\mathrm{d}\theta\mathrm{d}\varphi \\
&= \pm\big(-a\sin\varphi\sin\theta\,,\,a\sin\varphi\cos\theta\,,\,0\big) \times \big(a\cos\varphi\cos\theta\,,\,a\cos\varphi\sin\theta\,,\,-a\sin\varphi\big)\,\mathrm{d}\theta\mathrm{d}\varphi \\
&= \pm\Big(-a^2\sin^2\varphi\cos\theta\,,\,-a^2\sin^2\varphi\sin\theta\,,\,-a^2\sin\varphi\cos\varphi\Big)\,\mathrm{d}\theta\mathrm{d}\varphi \\
&= \mp a^2\sin\varphi\Big(\sin\varphi\cos\theta\,,\,\sin\varphi\sin\theta\,,\,\cos\varphi\Big)\,\mathrm{d}\theta\mathrm{d}\varphi \\
\mathrm{d}S &= a^2\sin\varphi\sqrt{\sin^2\varphi\cos^2\theta + \sin^2\varphi\sin^2\theta + \cos^2\varphi}\,\mathrm{d}\theta\mathrm{d}\varphi \\
&= a^2\sin\varphi\,\mathrm{d}\theta\mathrm{d}\varphi
\end{aligned}$$

So the area of the hemisphere is

$$\begin{aligned}
a^2\int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\int_0^{2\pi}\mathrm{d}\theta\ \sin\varphi = 2\pi a^2\int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\ \sin\varphi = 2\pi a^2\Big[-\cos\varphi\Big]_0^{\pi/2} \\
= 2\pi a^2
\end{aligned}$$

11 As we have noted before, the spherical coordinate system really breaks down at $\varphi = 0$, because $\rho = 1$, $\varphi = 0$ gives the same point, namely the north pole $(0, 0, 1)$, for all values of θ. We should really treat our integral like an improper integral, first integrating over $\varepsilon < \varphi \leqslant \frac{\pi}{2}$ and then taking the limit $\varepsilon \to 0^+$. However the breakdown of the spherical coordinate system at $\varphi = 0$, just like the breakdown of polar coordinates at $r = 0$, rarely causes problem and it is routine to skip the “improper integral” step.

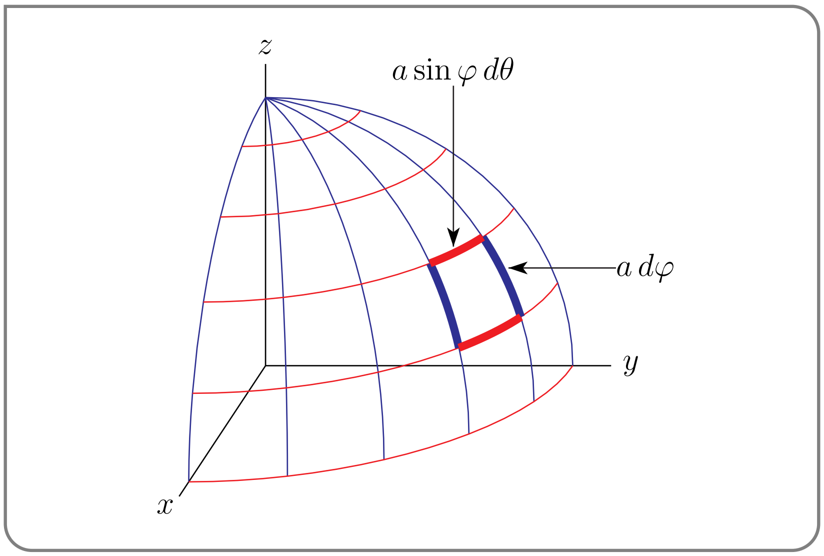
Example 3.3.6

There is an easier way to do this, using a little geometry.

Example 3.3.7 (Area of a hemisphere — using spherical coordinates again)

We are now going to again compute the surface area of the hemisphere using spherical coordinates. But this time instead of determining $\mathrm{d}S$ using the canned formula (3.3.1), we are going to read it off of a sketch.

Sketch the part of the hemisphere that is in the first octant, $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$. Slice it up into small pieces by drawing in curves of constant θ (the blue lines in the figure below) and curves of constant φ (the red lines in the figure below). Each piece is approximately



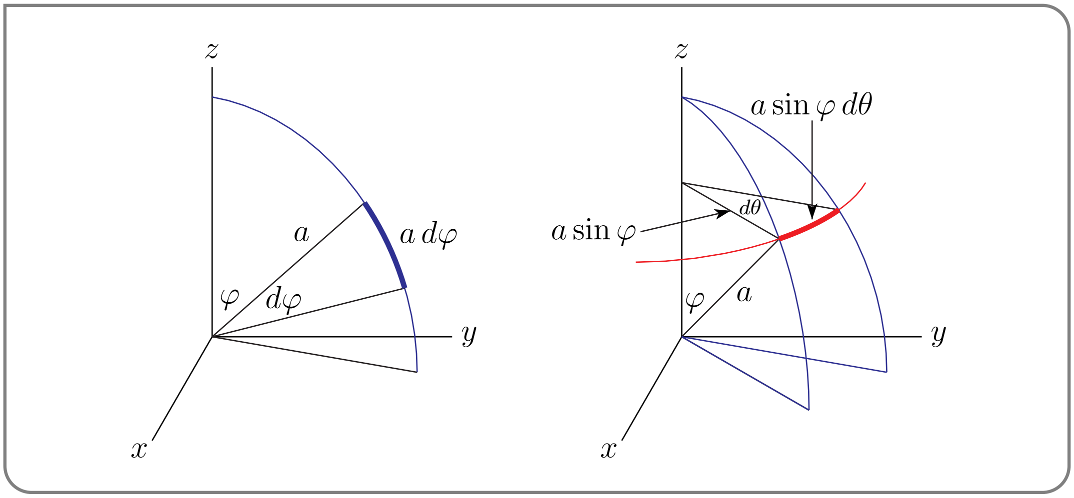
a little rectangle. Concentrate on one of them, like the piece with the thick sides in the figure above. The area, $\mathrm{d}S$, of that piece is (essentially) the product of its height and its width. Each of the two sides of the piece is

- a segment of a circle of radius a (a fat blue line in both the figure above and in the figure on the left below)
- that subtends an angle $\mathrm{d}\varphi$
- and hence is the fraction $\frac{\mathrm{d}\varphi}{2\pi}$ of a full circle of radius a and hence is of length $\frac{\mathrm{d}\varphi}{2\pi}2\pi a = a\mathrm{d}\varphi$.

The top of the piece is

- a segment of a circle of radius $a\sin\varphi$ (a fat red line in both the figure above and in the figure on the right below)
- that subtends an angle $\mathrm{d}\theta$
- and hence is the fraction $\frac{\mathrm{d}\theta}{2\pi}$ of a full circle of radius $a\sin\varphi$ and hence is of length $\frac{\mathrm{d}\theta}{2\pi}2\pi a\sin\varphi = a\sin\varphi\mathrm{d}\theta$.

These are drawn in the figure below.



So the area of our piece is

$$dS = \big(a d\varphi\big)\big(a \sin\varphi d\theta\big) = a^2 \sin\varphi \, d\theta d\varphi$$

This is exactly the same formula that we found for dS in Example 3.3.6 so that we will, yet again, get that the area of a hemisphere of radius a is $2\pi a^2$. (Phew!)

Example 3.3.7

But wait! We can do it again, by yet another method!

Example 3.3.8 (Area of a hemisphere — using $z = f(x,y)$)

We'll compute the area of the hemisphere one last time[12]. This time we'll use that the equation of the hemisphere is

$$z = f(x,y) = \sqrt{a^2 - x^2 - y^2} \qquad \text{with } (x,y) \text{ running over } x^2 + y^2 \leqslant a^2$$

So (3.3.2) yields

$$\begin{aligned} dS &= \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\, dxdy \\ &= \sqrt{1 + \Big(\frac{-x}{\sqrt{a^2-x^2-y^2}}\Big)^2 + \Big(\frac{-y}{\sqrt{a^2-x^2-y^2}}\Big)^2}\, dxdy \\ &= \sqrt{1 + \frac{x^2+y^2}{a^2-x^2-y^2}}\, dxdy \\ &= \sqrt{\frac{a^2}{a^2-x^2-y^2}}\, dxdy \end{aligned}$$

So the area is $\iint_{x^2+y^2\leqslant a^2} \frac{a}{\sqrt{a^2-x^2-y^2}}\, dxdy$. We already found, in Example 3.3.5, that the value of this integral in $2\pi a^2$.

12 We promise!

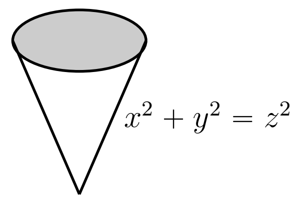
Example 3.3.8

Let's do some more substantial examples, where the integrand is not 1.

Example 3.3.9

Problem: Evaluate $\iint_S x^2y^2z^2\, dS$ where S is the part of the cone $x^2+y^2=z^2$ with $0 \leqslant z \leqslant 1$.
Solution 1. We can express S as

$$z = f(x,y) = \sqrt{x^2+y^2} \qquad x^2+y^2 \leqslant 1$$



Now since

$$f_x(x,y) = \frac{x}{\sqrt{x^2+y^2}} \qquad f_y(x,y) = \frac{y}{\sqrt{x^2+y^2}}$$

(3.3.2) gives[13]

$$\mathrm{d}S = \left[1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}\right]^{1/2} \mathrm{d}x\mathrm{d}y = \sqrt{2}\,\mathrm{d}x\mathrm{d}y$$

Our integral is then

$$\iint_S x^2y^2z^2\, dS = \sqrt{2} \iint_{x^2+y^2\leqslant 1} x^2y^2(x^2+y^2)\,\mathrm{d}x\mathrm{d}y$$

Since we are integrating over a circular domain, let's convert to polar coordinates.

$$\begin{aligned}
\iint_S x^2y^2z^2\, dS &= \sqrt{2}\int_0^{2\pi}\mathrm{d}\theta \int_0^1 \mathrm{d}r\; r(r\cos\theta)^2(r\sin\theta)^2 r^2 \\
&= \sqrt{2}\left[\int_0^{2\pi}\mathrm{d}\theta\; \cos^2\theta\sin^2\theta\right]\left[\int_0^1 \mathrm{d}r\; r^7\right] \\
&= \frac{\sqrt{2}}{8}\int_0^{2\pi}\mathrm{d}\theta\; \cos^2\theta\sin^2\theta = \frac{\sqrt{2}}{32}\int_0^{2\pi}\mathrm{d}\theta\; \sin^2(2\theta) \\
&= \frac{\sqrt{2}}{64}\int_0^{2\pi}\mathrm{d}\theta\; \left[1-\cos(4\theta)\right]
\end{aligned}$$

Remembering[14] that the integral of $\cos(\theta)$, or $\cos(4\theta)$, over a full period is 0, we end up with

$$\iint_S x^2y^2z^2\, dS = \frac{\sqrt{2}}{64}(2\pi) = \frac{\pi\sqrt{2}}{32}$$

13 This answer for $\mathrm{d}S$ is a very clean. Think about why. Hint: review the discussion following (3.3.2).
14 If you have forgotten why, sketch the graph.

Solution 2. We may parametrize[15] the cone by

$$\mathbf{r}(z,\theta) = z\cos\theta\,\hat{\boldsymbol{\imath}} + z\sin\theta\,\hat{\boldsymbol{\jmath}} + z\,\hat{\mathbf{k}} \qquad 0 \leqslant z \leqslant 1,\ 0 \leqslant \theta \leqslant 2\pi$$

Then because

$$\frac{\partial \mathbf{r}}{\partial z} = \cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}} \qquad \text{and} \qquad \frac{\partial \mathbf{r}}{\partial \theta} = -z\sin\theta\,\hat{\boldsymbol{\imath}} + z\cos\theta\,\hat{\boldsymbol{\jmath}}$$

(3.3.1) yields[16]

$$\begin{aligned}
\hat{\mathbf{n}}\,\mathrm{d}S &= \pm\det\begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \cos\theta & \sin\theta & 1 \\ -z\sin\theta & z\cos\theta & 0 \end{bmatrix} \mathrm{d}z\mathrm{d}\theta \\
&= \pm\big[-z\cos\theta\,\hat{\boldsymbol{\imath}} - z\sin\theta\,\hat{\boldsymbol{\jmath}} + z\,\hat{\mathbf{k}}\big]\ \mathrm{d}z\mathrm{d}\theta \\
\mathrm{d}S &= \sqrt{2}z\ \mathrm{d}z\mathrm{d}\theta
\end{aligned}$$

So our integral becomes

$$\begin{aligned}
\iint_S x^2y^2z^2\ dS &= \sqrt{2}\int_0^{2\pi}\mathrm{d}\theta\int_0^1 \mathrm{d}z\ z(z\cos\theta)^2(z\sin\theta)^2z^2 \\
&= \sqrt{2}\int_0^{2\pi}\mathrm{d}\theta\int_0^1\mathrm{d}z\ z^7\cos^2\theta\sin^2\theta \\
&= \frac{\sqrt{2}}{8}\int_0^{2\pi}\mathrm{d}\theta\ \cos^2\theta\sin^2\theta
\end{aligned}$$

We evaluated this integral in Solution 1. So again

$$\iint_S x^2y^2z^2\ dS = \frac{\pi\sqrt{2}}{32}$$

Example 3.3.9

Let's do something more celestial.

Example 3.3.10

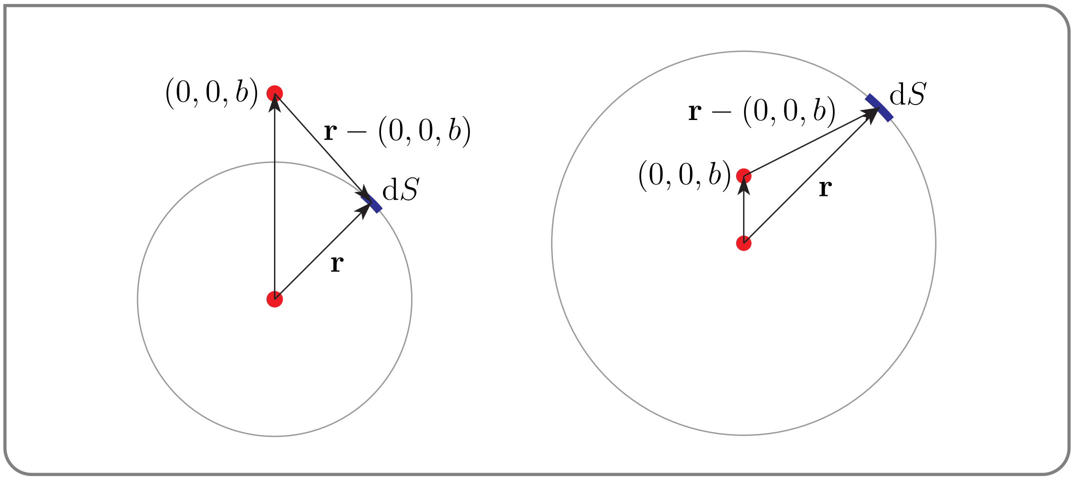
Consider a spherical shell of radius a with mass density μ per unit area. Think of it as a hollow planet[17]. We are going to determine the gravitational force that it exerts on a particle of mass m a distance b away from its centre. This particle can be either outside the shell ($b > a$) or inside the shell ($b < a$). We can choose the coordinate system so that the centre of the shell is at the origin and the particle is at $(0, 0, b)$. By Newton's law of gravitation,

15 We did so previously, with different variable names, in Example 3.2.2.
16 Again the formula for $\mathrm{d}S$ is very neat. Think about why.
17 A favourite of science fiction and fantasy writers. Plug "subterranean fiction" into your favourite search engine. While you're at it, also try "gravity train". We'll look at it in the optional Example 3.3.11.

the force exerted on the particle by a tiny piece of the shell of surface area $\mathrm{d}S$ located at $\mathbf{r}$ is

$$\frac{G\,(\mu \mathrm{d}S)\,m}{|\mathbf{r}-(0,0,b)|^3}(\mathbf{r}-(0,0,b))$$

Here G is the gravitational constant, $\mu\mathrm{d}S$ is the mass of the tiny piece of shell, m is the mass of the particle and $\mathbf{r}-(0,0,b)$ is the vector from the particle to the piece of shell. If



we work in spherical coordinates, as we did in Example 3.3.6,

$$\mathrm{d}S = a^2 \sin\varphi\, \mathrm{d}\varphi \mathrm{d}\theta$$

and

$$\begin{aligned}
\mathbf{r} &= a\sin\varphi\cos\theta\,\hat{\imath} + a\sin\varphi\sin\theta\,\hat{\jmath} + a\cos\varphi\,\hat{\mathbf{k}}\\
\mathbf{r}-(0,0,b) &= a\sin\varphi\cos\theta\,\hat{\imath} + a\sin\varphi\sin\theta\,\hat{\jmath} + (a\cos\varphi - b)\,\hat{\mathbf{k}}\\
|\mathbf{r}-(0,0,b)|^2 &= a^2+b^2-2ab\cos\varphi
\end{aligned}$$

The total force is then

$$\mathbf{F} = G\mu m a^2 \int_0^{\pi} \mathrm{d}\varphi \int_0^{2\pi} \mathrm{d}\theta\ \sin\varphi\ \frac{a\sin\varphi\cos\theta\,\hat{\imath} + a\sin\varphi\sin\theta\,\hat{\jmath} + (a\cos\varphi - b)\,\hat{\mathbf{k}}}{\left[a^2+b^2-2ab\cos\varphi\right]^{3/2}}$$

Note for future reference that the square root in $[a^2+b^2-2ab\cos\varphi]^{3/2}$ is the *positive* square root because $[b^2+a^2-2ab\cos\varphi]^{1/2}$ is the length of $\mathbf{r}-(0,0,b)$, which is positive.

This integral is a little different than other integrals that we have encountered so far in that the integrand is a vector. By definition[18],

$$\iint_S \left[G_1\,\hat{\imath} + G_2\,\hat{\jmath} + G_3\,\hat{\mathbf{k}}\right]\,\mathrm{d}S = \hat{\imath}\iint_S G_1\,\mathrm{d}S + \hat{\jmath}\iint_S G_2\,\mathrm{d}S + \hat{\mathbf{k}}\iint_S G_3\,\mathrm{d}S$$

so we just have to compute the three components separately.

18 Under this definition we still have $\iint(\mathbf{A}+\mathbf{B})\,\mathrm{d}S = \iint \mathbf{A}\,\mathrm{d}S + \iint \mathbf{B}\,\mathrm{d}S$.

In our case, the $\hat{\imath}$ and $\hat{\jmath}$ components

$$\mathbf{F}\cdot\hat{\imath} = G\mu m a^2 \int_0^{\pi} \mathrm{d}\varphi \left[\sin\varphi\ \frac{a\sin\varphi}{\left[a^2+b^2-2ab\cos\varphi\right]^{3/2}} \int_0^{2\pi} \mathrm{d}\theta\ \cos\theta\right]$$

$$\mathbf{F}\cdot\hat{\jmath} = G\mu m a^2 \int_0^{\pi} \mathrm{d}\varphi \left[\sin\varphi\ \frac{a\sin\varphi}{\left[a^2+b^2-2ab\cos\varphi\right]^{3/2}} \int_0^{2\pi} \mathrm{d}\theta\ \sin\theta\right]$$

are both zero[19] because $\int_0^{2\pi}\cos\theta\,\mathrm{d}\theta = \int_0^{2\pi}\sin\theta\,\mathrm{d}\theta = 0$ so that

$$\begin{aligned}\mathbf{F} &= G\mu m a^2\hat{\mathbf{k}} \int_0^{\pi}\mathrm{d}\varphi \int_0^{2\pi}\mathrm{d}\theta\ \sin\varphi\ \frac{a\cos\varphi - b}{\left[a^2+b^2-2ab\cos\varphi\right]^{3/2}} \\ &= 2\pi G\mu m a^2\hat{\mathbf{k}} \int_0^{\pi}\mathrm{d}\varphi\ \sin\varphi\ \frac{a\cos\varphi - b}{\left[a^2+b^2-2ab\cos\varphi\right]^{3/2}}\end{aligned}$$

To evaluate this integral we substitute

$$u = a^2+b^2-2ab\cos\varphi \qquad \mathrm{d}u = 2ab\sin\varphi\,\mathrm{d}\varphi \qquad \cos\varphi = \frac{a^2+b^2-u}{2ab}$$

When $\varphi = 0$, $u = (a-b)^2$ and when $\varphi = \pi$, $u = (a+b)^2$, so

$$\begin{aligned}\mathbf{F} &= \frac{\pi G\mu m a}{b}\hat{\mathbf{k}} \int_{(a-b)^2}^{(a+b)^2} \mathrm{d}u\ \frac{\frac{a^2+b^2-u}{2b} - b}{u^{3/2}} \\ &= \frac{\pi G\mu m a}{b}\hat{\mathbf{k}} \int_{(a-b)^2}^{(a+b)^2} \mathrm{d}u\ \frac{\frac{a^2-b^2-u}{2b}}{u^{3/2}} \\ &= \frac{\pi G\mu m a}{b}\hat{\mathbf{k}} \Big[\Big(\frac{a^2-b^2}{2b}\Big)\frac{u^{-1/2}}{-1/2} - \Big(\frac{1}{2b}\Big)\frac{u^{1/2}}{1/2}\Big]_{(a-b)^2}^{(a+b)^2}\end{aligned}$$

Recalling that $u^{1/2}$ is the *positive* square root,

$$\mathbf{F} = \frac{\pi G\mu m a}{b}\hat{\mathbf{k}}\Big[\Big(\frac{b^2-a^2}{b}\Big)\frac{1}{a+b} - \frac{a+b}{b} - \Big(\frac{b^2-a^2}{b}\Big)\frac{1}{|a-b|} + \frac{|a-b|}{b}\Big]$$

If $b > a$, so that $|a-b| = b-a$

$$\mathbf{F} = \frac{\pi G\mu m a}{b}\hat{\mathbf{k}}\Big[\frac{b-a}{b} - \frac{a+b}{b} - \frac{a+b}{b} + \frac{b-a}{b}\Big] = -\frac{G(4\pi a^2\mu)m}{b^2}\hat{\mathbf{k}}$$

If $b < a$, so that $|a-b| = a-b$

$$\mathbf{F} = \frac{\pi G\mu m a}{b}\hat{\mathbf{k}}\Big[\frac{b-a}{b} - \frac{a+b}{b} + \frac{a+b}{b} + \frac{a-b}{b}\Big] = 0$$

The moral[20] is

19 Think about why the $\hat{\imath}$ and $\hat{\jmath}$ components should both be zero. Think symmetry.
20 These two results appeared in Isaac Newton's Principia Mathematica (1687). They are known as Newton's "superb theorems".

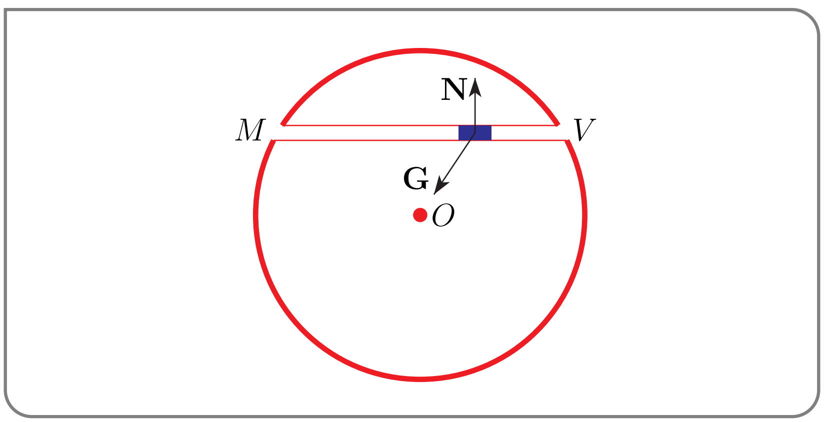
- ○ if the particle is inside the shell, it feels no gravitational force at all, and
- ○ if the particle is outside the shell, it feels the same gravitational force as it would if the entire mass of the shell ($4\pi a^2\mu$) were concentrated at the centre of the shell.

Example 3.3.10

Example 3.3.11 (Optional — Gravity Train)

The "Gravity Train[21]" refers to the following curious, though admittedly not very practical, thought experiment.

- Pretend that the Earth is a perfect sphere of radius R and that it has a constant mass density ρ.
- Pick *any* two distinct points on the surface of the Earth. Call them V and M.
- Bore a tunnel straight through the Earth from V to M.
- Place a train in the tunnel at V. Assume that the only forces acting on the train are gravity, **G**, and a normal force, **N**, that the tunnel imposes on the train to keep it in the tunnel. In particular, there are no frictional forces, like air resistance, and the train does not have an engine. Release the train and assume that it does not melt as it passes through the centre of the Earth.



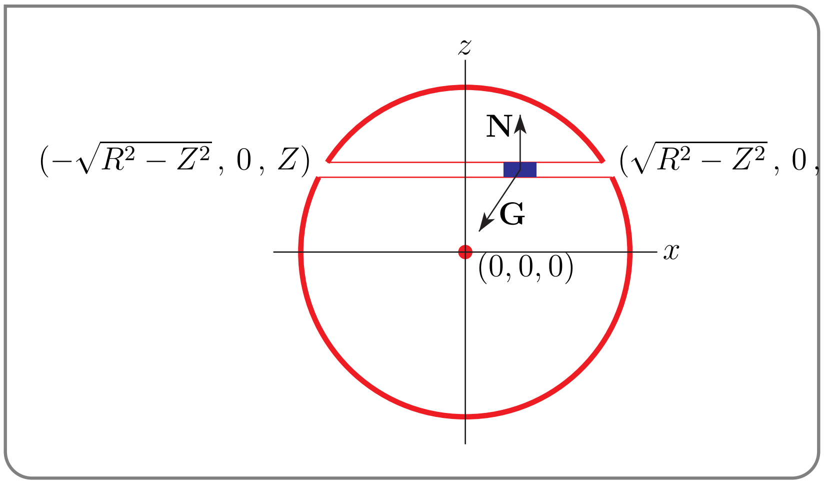
What happens?

We'll simplify our analysis of the motion of the train by picking a convenient coordinate system.

- ○ First translate our coordinate system so that the centre of the Earth, call it O, is at the origin, $(0, 0, 0)$.
- ○ Then rotate our coordinate system about the origin so that the origin, V and M all lie in the xz-plane.

21 The British physicist and architect (he was Surveyor to the City of London and chief assistant to Christopher Wren) Robert Hooke (1635–1703) wrote about the gravity train idea in a letter to Isaac Newton. A gravity train was used in the 2012 movie *Total Recall*.

- Then rotate our coordinate system about the y-axis so that V and M have the same z-coordinate $Z \geqslant 0$. So the coordinates of V and M are $\big(\pm\sqrt{R^2-Z^2}\,,\,0\,,Z\big)$. Let's suppose that V is at $\big(\sqrt{R^2-Z^2}\,,\,0\,,Z\big)$ and M is at $\big(-\sqrt{R^2-Z^2}\,,\,0\,,Z\big)$. It really doesn't matter which is which, but we can always arrange that it is V at $\big(+\sqrt{R^2-Z^2}\,,\,0\,,Z\big)$ by rotating around the z-axis by $180°$ if necessary.



The y- and z-coordinates of the train are always fixed at 0 and Z, respectively. So let's call the x-coordinate at time t $x(t)$, and look at the x-component of Newton's law of motion.

$$m\mathbf{a} = \mathbf{G} + \mathbf{N}$$

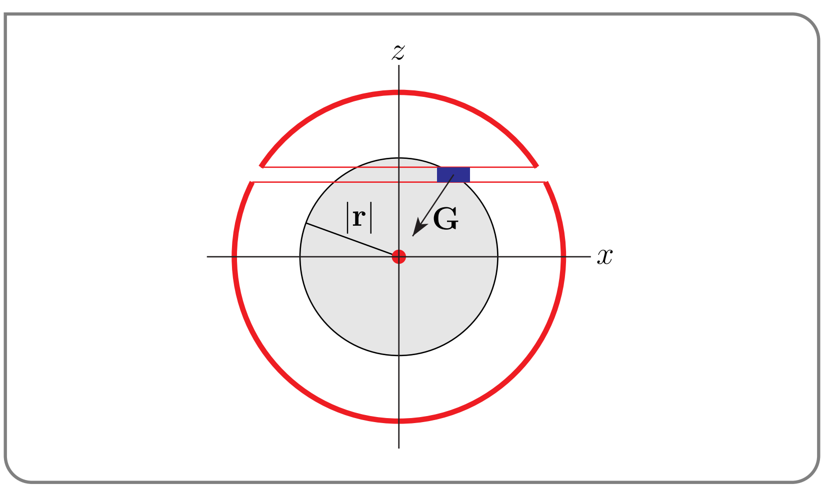
It is

$$mx''(t) = \mathbf{G}\cdot\hat{\boldsymbol{\imath}}$$

because the normal force $\mathbf{N}$ has no $\hat{\boldsymbol{\imath}}$ component. Recall that Newton's law of gravity says that

$$\mathbf{G} = -\frac{GMm}{|\mathbf{r}|^3}\mathbf{r}$$

where G is the gravitational constant, $\mathbf{r}$ is the vector from O to the train, and m is the mass of the train. In this case, because of our computation in Example 3.3.10, the train only feels gravity from shells of the Earth that are inside the train, so that M is the mass of the part



of the Earth whose distance to the centre of the Earth is no more than $|\mathbf{r}|$. So

$$M = \frac{4}{3}\pi|\mathbf{r}|^3\rho$$

and

$$mx''(t) = -\frac{Gm}{|\mathbf{r}|^3}\,\frac{4}{3}\pi|\mathbf{r}|^3\rho\;\mathbf{r}\cdot\hat{\boldsymbol{\imath}}$$

so that

$$x''(t) + \frac{4\pi G\rho}{3}x(t) = 0$$

This is exactly the differential equation of simple harmonic motion. We have seen it before in Example 2.2.7. Except for the constant $\frac{4\pi G\rho}{3}$, it is identical to the equation solved in Example I.4 of the Appendix I, entitled "Review of Linear Ordinary Differential Equations". The general solution is

$$x(t) = C_1\cos\left(\sqrt{\frac{4\pi G\rho}{3}}\,t\right) + C_2\sin\left(\sqrt{\frac{4\pi G\rho}{3}}\,t\right)$$

with C_1 and C_2 being arbitrary constants. If we release the train, from rest, at $t = 0$, then $x(0) = \sqrt{R^2 - Z^2}$ and $x'(0) = 0$ so that $C_1 = \sqrt{R^2 - Z^2}$, $C_2 = 0$ and

$$x(t) = \sqrt{R^2 - Z^2}\cos\left(\sqrt{\frac{4\pi G\rho}{3}}\,t\right)$$

The train reaches M when $x(t) = -\sqrt{R^2 - Z^2}$. That is, when $\cos\left(\sqrt{\frac{4\pi G\rho}{3}}\,t\right) = -1$. So the transit time, T, from V to M obeys

$$\sqrt{\frac{4\pi G\rho}{3}}\,T = \pi \implies T = \pi\sqrt{\frac{3}{4\pi G\rho}} = \sqrt{\frac{3\pi}{4G\rho}}$$

Notice that this transit time depends only on the gravitational constant G and the density of the Earth ρ. In particular it is completely independent of

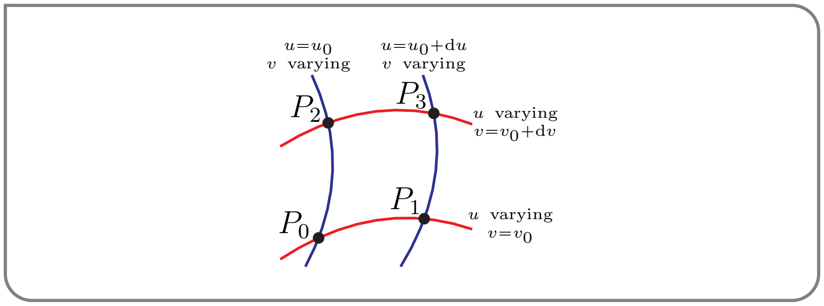
- where V and M are and, in particular,
- how close together V and M are, and also of
- the radius of the Earth.

In the case of the Earth, the transit time is about 42 minutes.

Example 3.3.11

3.3.5 ▸▸ Optional — Dropping Higher Order Terms in $\mathrm{d}u, \mathrm{d}v$

In the course of deriving (3.3.1), that is, $\hat{\mathbf{n}}\,\mathrm{d}S$ and $\mathrm{d}S$ formulae for



we approximated, for example, the vectors

$$\overrightarrow{P_0P_1} = \mathbf{r}(u_0+\mathrm{d}u, v_0) - \mathbf{r}(u_0\,,\,v_0) = \frac{\partial \mathbf{r}}{\partial u}(u_0\,,\,v_0)\,\mathrm{d}u + E_1 \approx \frac{\partial \mathbf{r}}{\partial u}(u_0\,,\,v_0)\,\mathrm{d}u$$
$$\overrightarrow{P_0P_2} = \mathbf{r}(u_0, v_0+\mathrm{d}v) - \mathbf{r}(u_0\,,\,v_0) = \frac{\partial \mathbf{r}}{\partial v}(u_0\,,\,v_0)\,\mathrm{d}v + E_2 \approx \frac{\partial \mathbf{r}}{\partial v}(u_0\,,\,v_0)\,\mathrm{d}v$$

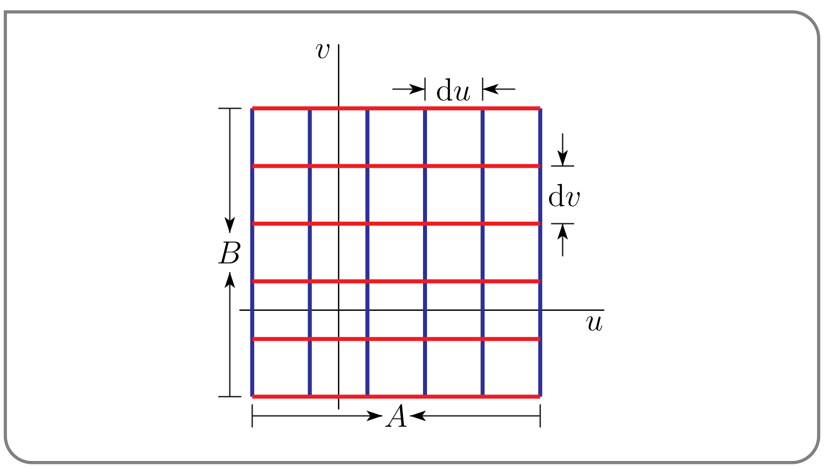
where E_1 is bounded[22] by a constant times $\mathrm{d}u^2$ and E_2 is bounded by a constant times $\mathrm{d}v^2$. That is, we assumed that we could just drop E_1 and E_2.

So we approximated

$$\begin{aligned}\left|\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}\right| &= \left|\left[\frac{\partial \mathbf{r}}{\partial u}(u_0\,,\,v_0)\,\mathrm{d}u + E_1\right] \times \left[\frac{\partial \mathbf{r}}{\partial v}(u_0\,,\,v_0)\,\mathrm{d}v + E_2\right]\right| \\ &= \left|\frac{\partial \mathbf{r}}{\partial u}(u_0\,,\,v_0)\,\mathrm{d}u \times \frac{\partial \mathbf{r}}{\partial v}(u_0\,,\,v_0)\,\mathrm{d}v + E_3\right| \\ &\approx \left|\frac{\partial \mathbf{r}}{\partial u}(u_0\,,\,v_0)\,\mathrm{d}u \times \frac{\partial \mathbf{r}}{\partial v}(u_0\,,\,v_0)\,\mathrm{d}v\right|\end{aligned}$$

where the length of the vector E_3 is bounded by a constant times $\mathrm{d}u^2\,\mathrm{d}v + \mathrm{d}u\,\mathrm{d}v^2$. We'll now see why dropping terms like E_3 does not change the value of the integral at all[23].

Suppose that our domain of integration consists of all (u,v)'s in a rectangle of width A and height B, as in the figure below. Subdivide the rectangle into a grid of $n \times n$



small subrectangles by drawing lines of constant v (the red lines in the figure) and lines of

22 Remember the error in the Taylor polynomial approximations.
23 See the optional §1.1.6 of the CLP-2 text for an analogous argument concerning Riemann sums.

constant v (the blue lines in the figure). Each subrectangle has width $\mathrm{d}u = \frac{A}{n}$ and height $\mathrm{d}v = \frac{B}{n}$. Now suppose that in setting up the integral we make, for each subrectangle, an error that is bounded by some constant times

$$\mathrm{d}u^2\,\mathrm{d}v + \mathrm{d}u\,\mathrm{d}v^2 = \left(\frac{A}{n}\right)^2\frac{B}{n} + \frac{A}{n}\left(\frac{B}{n}\right)^2 = \frac{AB(A+B)}{n^3}$$

Because there are a total of n^2 subrectangles, the total error that we have introduced, for all of these subrectangles, is no larger than a constant times

$$n^2 \times \frac{AB(A+B)}{n^3} = \frac{AB(A+B)}{n}$$

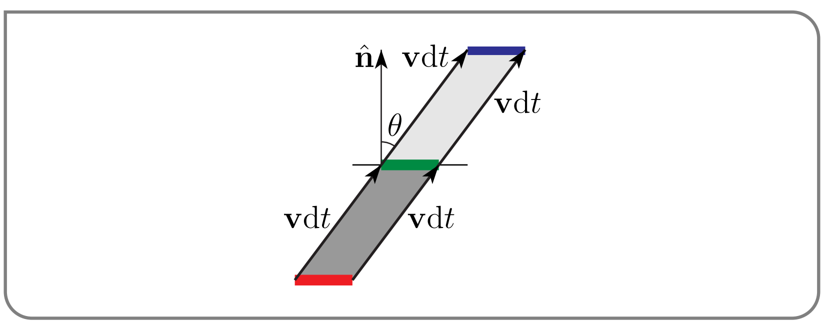
When we define our integral by taking the limit $n \to 0$ of the Riemann sums, this error converges to exactly 0.

3.4▲ Interpretation of Flux Integrals

We defined, in §3.3, two types of integrals over surfaces. We have seen, in §3.3.4, some applications that lead to integrals of the type $\iint_S \rho\,\mathrm{d}S$. We now look at one application that leads to integrals of the type $\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$. Recall that integrals of this type are called flux integrals. Imagine a fluid with

- the density of the fluid (say in kilograms per cubic meter) at position (x, y, z) and time t being $\rho(x, y, z, t)$ and with
- the velocity of the fluid (say in meters per second) at position (x, y, z) and time t being $\mathbf{v}(x, y, z, t)$.

We are going to determine the rate (say in kilograms per second) at which the fluid is flowing through a tiny piece $\mathrm{d}S$ of surface at (x, y, z). During a tiny time interval of length $\mathrm{d}t$ about time t, fluid near $\mathrm{d}S$ moves $\mathbf{v}(x, y, z, t)\mathrm{d}t$. The green line in the figure below is a side view of $\mathrm{d}S$ and $\hat{\mathbf{n}} = \hat{\mathbf{n}}(x, y, z)$ is a unit normal vector to $\mathrm{d}S$. So during that tiny time



interval

- the red line moves to the green line and
- the green line moves to the blue line so that
- the fluid filling the dark grey region below the green line crosses through $\mathrm{d}S$ and moves to light grey region above the green line.

If we denote by θ the angle between $\hat{\mathbf{n}}$ and $\mathbf{v}\mathrm{d}t$,

- the volume of fluid that crosses through $\mathrm{d}S$ during the time interval $\mathrm{d}t$ is the volume whose side view is the dark grey region below the green line. This region has base $\mathrm{d}S$ and height $|\mathbf{v}\mathrm{d}t|\cos\theta$ and so has volume

$$|\mathbf{v}(x,y,z,t)\mathrm{d}t|\cos\theta\,\mathrm{d}S = \mathbf{v}(x,y,z,t)\cdot\hat{\mathbf{n}}(x,y,z)\,\mathrm{d}t\,\mathrm{d}S$$

  because $\hat{\mathbf{n}}(x,y,z)$ has length one.
- The mass of fluid that crosses $\mathrm{d}S$ during the time interval $\mathrm{d}t$ is then

$$\rho(x,y,z,t)\mathbf{v}(x,y,z,t)\cdot\hat{\mathbf{n}}(x,y,z)\,\mathrm{d}t\,\mathrm{d}S$$

- and the rate at which fluid is crossing through $\mathrm{d}S$ is

$$\rho(x,y,z,t)\mathbf{v}(x,y,z,t)\cdot\hat{\mathbf{n}}(x,y,z)\,\mathrm{d}S$$

Integrating $\mathrm{d}S$ over a surface S, we conclude that

Lemma 3.4.1.

The rate at which fluid mass is crossing through a surface S is the flux integral

$$\iint_S \rho(x,y,z,t)\mathbf{v}(x,y,z,t)\cdot\hat{\mathbf{n}}(x,y,z)\,\mathrm{d}S$$

Here ρ is the density of the fluid, $\mathbf{v}$ is the velocity field of the fluid, and $\hat{\mathbf{n}}(x,y,z)$ is a unit normal to S at (x,y,z). If the flux integral is positive the fluid is crossing in the direction $\hat{\mathbf{n}}$. If it is negative the fluid is crossing opposite to the direction of $\hat{\mathbf{n}}$. The rate at which volume of fluid is crossing through a surface S is the flux integral

$$\iint_S \mathbf{v}(x,y,z,t)\cdot\hat{\mathbf{n}}(x,y,z)\,\mathrm{d}S$$

3.4.1 ▸▸ Examples of Flux Integrals

Example 3.4.2 (Point Source)

In Example 2.1.2, we found that the vector field of a point source[24] (in three dimensions) that creates $4\pi m$ liters per second is

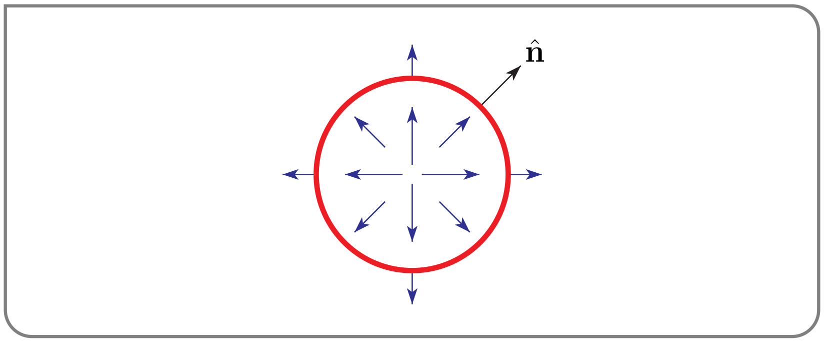
$$\mathbf{v}(x,y,z) = \frac{m}{r(x,y,z)^2}\,\hat{\mathbf{r}}(x,y,z)$$

24 You can imagine that a very small pipe pumps water to the origin.

where

$$r(x,y,z) = \sqrt{x^2+y^2+z^2} \qquad \hat{\mathbf{r}}(x,y,z) = \frac{x\hat{\imath}+y\hat{\jmath}+z\hat{\mathbf{k}}}{\sqrt{x^2+y^2+z^2}}$$

We sketched it in Figure 2.1.1. We'll now compute the flux of this vector field across a sphere centred on the origin. Suppose that the sphere has radius R. Then the outward[25]



pointing normal at a point (x,y,z) on the sphere is

$$\hat{\mathbf{n}}(x,y,z) = \hat{\mathbf{r}}(x,y,z) = \frac{x\hat{\imath}+y\hat{\jmath}+z\hat{\mathbf{k}}}{\sqrt{x^2+y^2+z^2}} = \frac{x\hat{\imath}+y\hat{\jmath}+z\hat{\mathbf{k}}}{R}$$

Note that $\hat{\mathbf{r}}(x,y,z)\cdot\hat{\mathbf{r}}(x,y,z) = 1$ and that, on the sphere, $r(x,y,z) = R$. So the flux of $\mathbf{v}$ outward through the sphere is

$$\begin{aligned}\iint_S \mathbf{v}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \iint_S \frac{m}{r(x,y,z)^2}\hat{\mathbf{r}}(x,y,z)\cdot\hat{\mathbf{r}}(x,y,z)\,\mathrm{d}S \\ &= \iint_S \frac{m}{R^2}\,\mathrm{d}S = \frac{m}{R^2}4\pi R^2 \\ &= 4\pi m\end{aligned}$$

This is the rate at which volume of fluid is exiting the sphere. In our derivation of the vector field we assumed that the fluid is incompressible, so it is also the rate at which the point source is creating fluid.

Example 3.4.2

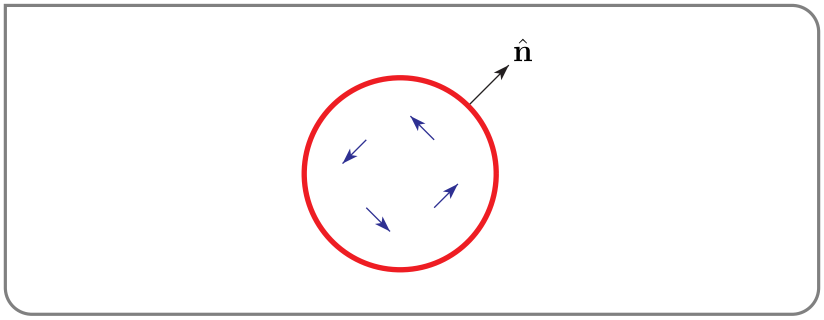
Example 3.4.3 (Vortex)

In Figure 2.1.3, we sketched the vector field (in two dimensions)

$$\mathbf{v}(x,y) = \Omega\big(-y\hat{\imath}+x\hat{\jmath}\big)$$

We'll now compute the flux of this vector field across a circle C centred on the origin. Suppose that the circle has radius R. By definition, in two dimensions, the flux of a

25 It doesn't really matter which unit normal we pick here. We just have to be clear which one we're using. With the outward normal, the flux gives the rate at which fluid crosses the sphere in the outward direction. If we were to use the inward pointing normal, the flux would give the rate at which fluid crosses the sphere in the inward direction.



vector field across a curve C is $\int_C \mathbf{v}\cdot\hat{\mathbf{n}}\ \mathrm{d}s$.

This is the natural analog of the flux in three dimensions — the surface S has been replaced by the curve C, and the surface area $\mathrm{d}S$ of a tiny piece of S has been replaced by the arc length ds of a tiny piece of C.

The outward pointing unit normal at a point (x, y) on our circle C is

$$\hat{\mathbf{n}}(x,y) = \frac{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}}{\sqrt{x^2+y^2}} = \frac{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}}{R}$$

So

$$\mathbf{v}(x,y)\cdot\hat{\mathbf{n}}(x,y) = \frac{\Omega}{R}\big(-y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}}\big)\cdot\big(x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}\big) = 0$$

and the flux across C is

$$\int_C \mathbf{v}\cdot\hat{\mathbf{n}}\ \mathrm{d}s = 0$$

This should not be a surprise — no fluid is crossing C at all. This is exactly what we would expect from looking at the arrows in Figure 2.1.3 or at the stream lines in Example 2.2.6.

Example 3.4.3

Example 3.4.4

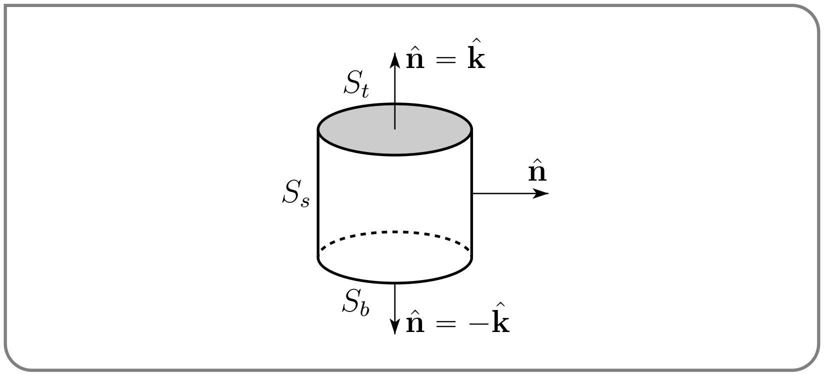
Problem: Evaluate $\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\ dS$ where

$$\mathbf{F}(x,y,z) = (x+y)\,\hat{\boldsymbol{\imath}} + (y+z)\,\hat{\boldsymbol{\jmath}} + (x+z)\,\hat{\mathbf{k}}$$

and S is the boundary of $V = \big\{\ (x,y,z)\ \big|\ 0 \leqslant x^2+y^2 \leqslant 9,\ 0 \leqslant z \leqslant 5\ \big\}$, and $\hat{\mathbf{n}}$ is the outward normal[26] to S.

Solution. The volume V looks like a tin can of radius 3 and height 5. It is natural to

26 It is necessary that the problem specify, one way or another, whether $\hat{\mathbf{n}}$ is the inward pointing normal or the outward pointing normal. Without this, the meaning of $\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\ dS$ is ambiguous. Think about where the orientation of the normal vector gets used in your solution.



decompose its surface S into three parts

$$\begin{aligned} S_t &= \left\{ (x,y,z) \,\middle|\, 0 \leqslant x^2+y^2 \leqslant 9,\ z=5 \right\} = \text{the top} \\ S_b &= \left\{ (x,y,z) \,\middle|\, 0 \leqslant x^2+y^2 \leqslant 9,\ z=0 \right\} = \text{the bottom} \\ S_s &= \left\{ (x,y,z) \,\middle|\, x^2+y^2 = 9,\ 0 \leqslant z \leqslant 5 \right\} = \text{the side} \end{aligned}$$

We'll compute the flux through each of the three parts separately and then add them together.

The Top: On the top, the outward pointing normal to S is $\hat{\mathbf{n}} = \hat{\mathbf{k}}$ and $\mathrm{d}S = \mathrm{d}x\mathrm{d}y$. This is probably intuitively obvious. But if it isn't, you can always derive it by parametrizing the top by $\mathbf{r}(x,y) = x\,\hat{\imath} + y\,\hat{\jmath} + 5\,\hat{\mathbf{k}}$ with $x^2+y^2 \leqslant 9$. So the flux through the top is

$$\iint_{S_t} \mathbf{F}\cdot\hat{\mathbf{n}}\ \mathrm{d}S = \iint_{\substack{x^2+y^2\leqslant 9\\ z=5}} (x+z)\ \mathrm{d}x\mathrm{d}y = \iint_{x^2+y^2\leqslant 9} (x+5)\ \mathrm{d}x\mathrm{d}y$$

The integral $\iint_{x^2+y^2\leqslant 9} x\ \mathrm{d}x\mathrm{d}y = 0$ since x is odd and the domain of integration is symmetric about $x=0$. So

$$\iint_{S_t} \mathbf{F}\cdot\hat{\mathbf{n}}\ \mathrm{d}S = \iint_{x^2+y^2\leqslant 9} 5\ \mathrm{d}x\mathrm{d}y = 5\pi(3)^2 = 45\pi$$

The Bottom: On the bottom, the outward pointing normal to S is $\hat{\mathbf{n}} = -\hat{\mathbf{k}}$ and $\mathrm{d}S = \mathrm{d}x\mathrm{d}y$. So the flux through the bottom is

$$\iint_{S_b} \mathbf{F}\cdot\hat{\mathbf{n}}\ \mathrm{d}S = -\iint_{\substack{x^2+y^2\leqslant 9\\ z=0}} (x+z)\ \mathrm{d}x\mathrm{d}y = -\iint_{x^2+y^2\leqslant 9} x\ \mathrm{d}x\mathrm{d}y = 0$$

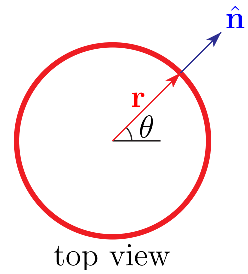
again since x is odd and the domain of integration is symmetric about $x=0$.

The Side: We can parametrize the side by using cylindrical coordinates.

$$\mathbf{r}(\theta,z) = \big(3\cos\theta\,,\, 3\sin\theta\,,\, z\big) \qquad 0 \leqslant \theta < 2\pi,\ 0 \leqslant z \leqslant 5$$

Then, using (3.3.1),

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial \theta} &= (-3\sin\theta\,,\, 3\cos\theta\,,\, 0)\\ \frac{\partial \mathbf{r}}{\partial z} &= (0\,,\,0\,,\,1)\\ \hat{\mathbf{n}}\,\mathrm{d}S &= \frac{\partial \mathbf{r}}{\partial \theta}\times\frac{\partial \mathbf{r}}{\partial z}\,\mathrm{d}\theta\,\mathrm{d}z\\ &= (3\cos\theta\,,\,3\sin\theta\,,\,0)\,\mathrm{d}\theta\,\mathrm{d}z\end{aligned}$$



Note that $\hat{\mathbf{n}} = (\cos\theta\,,\,\sin\theta\,,\,0)$ is outward pointing[27], as desired. Continuing,

$$\begin{aligned}\mathbf{F}\big(x(\theta,z)\,,\,y(\theta,z)\,,\,z(\theta,z)\big) &= 3(\cos\theta+\sin\theta)\,\hat{\boldsymbol{\imath}} + (3\sin\theta+z)\,\hat{\boldsymbol{\jmath}} + (3\cos\theta+z)\,\hat{\mathbf{k}}\\ \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \big\{9\cos^2\theta + 3\sin\theta\cos\theta + 9\sin^2\theta + 3z\sin\theta\big\}\,\mathrm{d}\theta\,\mathrm{d}z\\ &= \big\{9 + 3/2\,\sin(2\theta) + 3z\sin\theta\big\}\,\mathrm{d}\theta\,\mathrm{d}z\end{aligned}$$

So the flux through the side is

$$\begin{aligned}\iint_{S_s} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \int_0^{2\pi}\mathrm{d}\theta\int_0^5\mathrm{d}z\,\big\{9 + 3/2\,\sin(2\theta) + 3z\sin\theta\big\}\\ &= 9\int_0^{2\pi}\mathrm{d}\theta\int_0^5\mathrm{d}z \qquad \text{since } \int_0^{2\pi}\sin\theta\,\mathrm{d}\theta = \int_0^{2\pi}\sin(2\theta)\,\mathrm{d}\theta = 0\\ &= 9\times 2\pi\times 5 = 90\pi\end{aligned}$$

and the total flux is

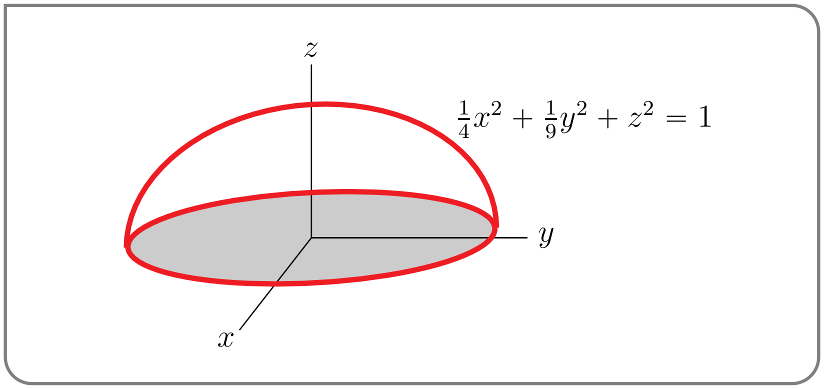
$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iint_{S_t} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S + \iint_{S_b} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S + \iint_{S_s} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = 45\pi + 0 + 90\pi = 135\pi$$

Example 3.4.4

Example 3.4.5

Problem: Evaluate $\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS$ where $\mathbf{F}(x,y,z) = x^4\hat{\boldsymbol{\imath}} + 2y^2\hat{\boldsymbol{\jmath}} + z\hat{\mathbf{k}}$, S is the half of the surface $\frac{1}{4}x^2 + \frac{1}{9}y^2 + z^2 = 1$ with $z \geqslant 0$, and $\hat{\mathbf{n}}$ is the upward pointing unit normal.

27 To check, draw, in your head, a sketch of the top view of the can. "Top view" just means "ignore the z-coordinate". The top view of the can is a circle of radius 3. Then, at a generic point, $\mathbf{r} = (\cos\theta, \sin\theta)$, on the can, draw the unit normal $\hat{\mathbf{n}} = (\cos\theta\,,\,\sin\theta)$ with its tail at $\mathbf{r}$. It is pointing away from the origin, just like $\mathbf{r}$ is. That is, $\hat{\mathbf{n}}$ is pointing outward.



Solution 1. We start by parametrizing the surface, which is half of an ellipsoid. By way of motivation for the parametrization, recall that spherical coordinates, with $\rho = 1$, provide a natural way to parametrize the sphere $x^2 + y^2 + z^2 = 1$. Namely $x = \cos\theta \sin\varphi$, $y = \sin\theta \sin\varphi$, $z = \cos\varphi$. The reason that these spherical coordinates work is that the trig identity $\cos^2\alpha + \sin^2\alpha = 1$ implies

$$x^2 + y^2 = \cos^2\theta \sin^2\varphi + \sin^2\theta \sin^2\varphi = \sin^2\varphi$$

and then

$$(x^2 + y^2) + z^2 = \sin^2\varphi + \cos^2\varphi = 1$$

The equation of our ellipsoid is

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 + z^2 = 1$$

so we can parametrize the ellipsoid by replacing x with $\frac{x}{2}$ and y with $\frac{y}{3}$ in our parametrization of the sphere. That is, we choose the parametrization

$$\begin{aligned} x(\theta, \varphi) &= 2\cos\theta \sin\varphi \\ y(\theta, \varphi) &= 3\sin\theta \sin\varphi \\ z(\theta, \varphi) &= \cos\varphi \end{aligned}$$

with (θ, φ) running over $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant \varphi \leqslant \pi/2$. Note that

$$\frac{1}{4}x(\theta, \varphi)^2 + \frac{1}{9}y(\theta, \varphi)^2 + z(\theta, \varphi)^2 = 1$$

as desired.

Then, using (3.3.1),

$$\begin{aligned} \left(\frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}, \frac{\partial z}{\partial \theta}\right) &= (-2\sin\theta \sin\varphi\,,\, 3\cos\theta \sin\varphi\,,\, 0) \\ \left(\frac{\partial x}{\partial \varphi}, \frac{\partial y}{\partial \varphi}, \frac{\partial z}{\partial \varphi}\right) &= (2\cos\theta \cos\varphi\,,\, 3\sin\theta \cos\varphi\,,\, -\sin\varphi) \\ \hat{\mathbf{n}}\, \mathrm{d}S &= -\left(\frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}, \frac{\partial z}{\partial \theta}\right) \times \left(\frac{\partial x}{\partial \varphi}, \frac{\partial y}{\partial \varphi}, \frac{\partial z}{\partial \varphi}\right) \mathrm{d}\theta \mathrm{d}\varphi \\ &= -(-3\cos\theta \sin^2\varphi, -2\sin\theta \sin^2\varphi, -6\sin\varphi \cos\varphi) \mathrm{d}\theta \mathrm{d}\varphi \end{aligned}$$

The extra minus sign in $\hat{\mathbf{n}}\,\mathrm{d}S$ was put there to make the z component of $\hat{\mathbf{n}}$ positive. (The problem specified that $\hat{\mathbf{n}}$ is to be upward unit normal.) As

$$\mathbf{F}\big(x(\theta,\varphi)\,,\,y(\theta,\varphi)\,,\,z(\theta,\varphi)\big) = 2^4\cos^4\theta\sin^4\varphi\,\hat{\imath} + 2\times 3^2\sin^2\theta\sin^2\varphi\,\hat{\jmath} + \cos\varphi\,\hat{\mathbf{k}}$$

we have

$$\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \Big[3\times 2^4\cos^5\theta\sin^6\varphi + 2\times 2\times 3^2\sin^3\theta\sin^4\varphi + 6\sin\varphi\cos^2\varphi\Big]\,\mathrm{d}\theta\,\mathrm{d}\varphi$$

and the desired integral

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\int_0^{2\pi}\mathrm{d}\theta\,\Big[3\times 2^4\cos^5\theta\sin^6\varphi + 2\times 2\times 3^2\sin^3\theta\sin^4\varphi + 6\sin\varphi\cos^2\varphi\Big]$$

Since $\int_0^{2\pi}\cos^m\theta\,\mathrm{d}\theta = \int_0^{2\pi}\sin^m\theta\,\mathrm{d}\theta = 0$ for all odd[28] natural numbers m,

$$\begin{aligned}\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \int_0^{\pi/2}\mathrm{d}\varphi\int_0^{2\pi}\mathrm{d}\theta\,6\sin\varphi\cos^2\varphi = 12\pi\int_0^{\pi/2}\mathrm{d}\varphi\,\sin\varphi\cos^2\varphi = 12\pi\Big[-\frac{1}{3}\cos^3\varphi\Big]_0^{\pi/2}\\ &= 4\pi\end{aligned}$$

The integral was evaluated by guessing (and checking) that $-\frac{1}{3}\cos^3\varphi$ is an antiderivative of $\sin\varphi\cos^2\varphi$. It can also be done by substituting $u=\cos\varphi$, $\mathrm{d}u = -\sin\varphi\,\mathrm{d}\varphi$.

Solution 2. This time we'll parametrize the half-ellipsoid using a variant of cylindrical coordinates.

$$\begin{aligned}x(r,\theta) &= 2r\cos\theta\\ y(r,\theta) &= 3r\sin\theta\\ z(r,\theta) &= \sqrt{1-r^2}\end{aligned}$$

with (r,θ) running over $0\leqslant\theta\leqslant 2\pi$, $0\leqslant r\leqslant 1$. Because we built the factors of 2 and 3 into $x(r,\theta)$ and $y(r,\theta)$, we have

$$\begin{aligned}&\frac{x(r,\theta)^2}{4}+\frac{y(r,\theta)^2}{9} = r^2\cos^2\theta + r^2\sin^2\theta = r^2\\ \Longrightarrow\ &\frac{x(r,\theta)^2}{4}+\frac{y(r,\theta)^2}{9} + z(r,\theta)^2 = r^2 + \Big(\sqrt{1-r^2}\Big)^2 = 1\end{aligned}$$

as desired. Further $z(r,\theta)\geqslant 0$ by our choice of square root in the definition of $z(r,\theta)$.

So, using (3.3.1),

$$\begin{aligned}\Big(\frac{\partial x}{\partial\theta},\frac{\partial y}{\partial\theta},\frac{\partial z}{\partial\theta}\Big) &= (-2r\sin\theta, 3r\cos\theta, 0)\\ \Big(\frac{\partial x}{\partial r},\frac{\partial y}{\partial r},\frac{\partial z}{\partial r}\Big) &= \Big(2\cos\theta, 3\sin\theta, -\frac{r}{\sqrt{1-r^2}}\Big)\\ \hat{\mathbf{n}}\mathrm{d}S &= -\Big(\frac{\partial x}{\partial\theta},\frac{\partial y}{\partial\theta},\frac{\partial z}{\partial\theta}\Big)\times\Big(\frac{\partial x}{\partial r},\frac{\partial y}{\partial r},\frac{\partial z}{\partial r}\Big)dr\,\mathrm{d}\theta\\ &= -\Big(-\frac{3r^2\cos\theta}{\sqrt{1-r^2}}, -\frac{2r^2\sin\theta}{\sqrt{1-r^2}}, -6r\Big)\mathrm{d}r\,\mathrm{d}\theta\end{aligned}$$

28 Look at the graphs of $\cos^m\varphi$ and $\sin^m\varphi$.

Once again, the extra minus sign in $\hat{\mathbf{n}}\mathrm{d}S$ was put there to make the z component of $\hat{\mathbf{n}}$ positive. Continuing,

$$\mathbf{F}\big(x(r,\theta)\,,\,y(r,\theta)\,,\,z(r,\theta)\big) = 2^4 r^4\cos^4\theta\,\hat{\imath} + 2\times 3^2 r^2\sin^2\theta\,\hat{\jmath} + \sqrt{1-r^2}\,\hat{\mathbf{k}}$$

$$\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \Big[3\times 2^4\frac{r^6}{\sqrt{1-r^2}}\cos^5\theta + 2^2 3^2\frac{r^4}{\sqrt{1-r^2}}\sin^3\theta + 6r\sqrt{1-r^2}\Big]\,\mathrm{d}r\,\mathrm{d}\theta$$

Again using that $\int_0^{2\pi}\cos^m\theta\,\mathrm{d}\theta = \int_0^{2\pi}\sin^m\theta\,\mathrm{d}\theta = 0$ for all odd natural numbers m,

$$\begin{aligned}\int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \int_0^1 dr\int_0^{2\pi}\mathrm{d}\theta\;6r\sqrt{1-r^2}\\ &= 12\pi\int_0^1 dr\;r\sqrt{1-r^2} = 12\pi\Big[-\frac{1}{3}(1-r^2)^{3/2}\Big]_0^1\\ &= 4\pi\end{aligned}$$

The integral was evaluated by guessing (and checking) that $-\frac{1}{3}(1-r^2)^{3/2}$ is an antiderivative of $r\sqrt{1-r^2}$. It can also be done by substituting $u = 1-r^2$, $\mathrm{d}u = -2r\,\mathrm{d}r$.

Solution 3. The surface is of the form $G(x,y,z) = 0$ with $G(x,y,z) = \frac{1}{4}x^2 + \frac{1}{9}y^2 + z^2 - 1$. Hence, using (3.3.3),

$$\hat{\mathbf{n}}\mathrm{d}S = \frac{\boldsymbol{\nabla} G}{\boldsymbol{\nabla} G\cdot\hat{\mathbf{k}}}\mathrm{d}x\,\mathrm{d}y = \frac{\frac{x}{2}\hat{\imath} + \frac{2y}{9}\hat{\jmath} + 2z\hat{\mathbf{k}}}{2z}\mathrm{d}x\,\mathrm{d}y = \Big(\frac{x}{4z}\hat{\imath} + \frac{y}{9z}\hat{\jmath} + \hat{\mathbf{k}}\Big)\mathrm{d}x\,\mathrm{d}y$$

$$\implies \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \Big(\frac{x^5}{4z} + \frac{2y^3}{9z} + z\Big)\mathrm{d}x\,\mathrm{d}y$$

It is true that $\hat{\mathbf{n}}\mathrm{d}S$, and consequently $\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$ become infinite[29] as $z\to 0$. So we should really treat the integral as an improper integral, first integrating over $z\geqslant\varepsilon$ and then taking the limit $\varepsilon\to 0^+$. But, as we shall see, the singularity is harmless. So it is standard to gloss over this point. On S, $z = z(x,y) = \sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}$ and $\frac{x^2}{4}+\frac{y^2}{9}\leqslant 1$, so

$$\int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iint_{\frac{x^2}{4}+\frac{y^2}{9}\leqslant 1}\Big(\frac{x^5}{4z(x,y)} + \frac{2y^3}{9z(x,y)} + z(x,y)\Big)\,\mathrm{d}x\,\mathrm{d}y$$

Both $\frac{x^5}{4z(x,y)}$ and $\frac{2y^3}{9z(x,y)}$ are odd under $x\to -x$, $y\to -y$ and the domain of integration is even under $x\to -x$, $y\to -y$, so their integrals are zero and

$$\begin{aligned}\int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \iint_{\frac{x^2}{4}+\frac{y^2}{9}\leqslant 1} z(x,y)\,\mathrm{d}x\,\mathrm{d}y\\ &= \iint_{\frac{x^2}{4}+\frac{y^2}{9}\leqslant 1}\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

To evaluate this integral, first make the change of variables[30] $x = 2X$, $\mathrm{d}x = 2\mathrm{d}X$, $y = 3Y$,

29 That's because the ellipsoid is becoming vertical as $z\to 0$, so that x and y are not really good parameters there.

30 The reader interested in general changes of variables in multidimensional integrals should look up "Jacobian determinant".

$\mathrm{d}y = 3\mathrm{d}Y$ to give

$$\int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iint_{X^2+Y^2\leqslant 1} \sqrt{1-X^2-Y^2}\,6\,\mathrm{d}X\,\mathrm{d}Y$$

Then switch to polar coordinates, $X = r\cos\theta$, $Y = r\sin\theta$, $\mathrm{d}X\mathrm{d}Y = r\,\mathrm{d}r\mathrm{d}\theta$ to give

$$\int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \int_0^1 dr\int_0^{2\pi}\mathrm{d}\theta\, 6r\sqrt{1-r^2} = 12\pi\int_0^1 dr\, r\sqrt{1-r^2} = 12\pi\Big[-\frac{1}{3}(1-r^2)^{3/2}\Big]_0^1$$
$$= 4\pi$$

Solution 4. The surface is of the form $z = f(x,y)$ with $f(x,y) = \sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}$. Hence, using (3.3.2),

$$\hat{\mathbf{n}}\mathrm{d}S = \Big[-\frac{\partial f}{\partial x}\hat{\boldsymbol{\imath}} - \frac{\partial f}{\partial y}\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}\Big]\,\mathrm{d}x\,\mathrm{d}y = \left[\frac{\frac{x}{4}\hat{\boldsymbol{\imath}} + \frac{y}{9}\hat{\boldsymbol{\jmath}}}{\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}} + \hat{\mathbf{k}}\right]\mathrm{d}x\,\mathrm{d}y$$

$$\implies \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \left[\frac{\frac{x^5}{4}+\frac{2y^3}{9}}{\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}} + \sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}\right]\mathrm{d}x\,\mathrm{d}y$$

Note that our unit normal is upward pointing, as required. As in Solution 3, by the oddness of the x^5 and y^3 terms in the integrand,

$$\int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iint_{\frac{x^2}{4}+\frac{y^2}{9}\leqslant 1}\left[\frac{\frac{x^5}{4}+\frac{2y^3}{9}}{\sqrt{\cdots}} + \sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}\right]\mathrm{d}x\,\mathrm{d}y$$
$$= \iint_{\frac{x^2}{4}+\frac{y^2}{9}\leqslant 1}\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}\,\mathrm{d}x\,\mathrm{d}y$$

Now continue as in Solution 3.

Example 3.4.5

3.5▲ Orientation of Surfaces

One thing that made the flux integrals of the last section possible is that we could choose sensible unit normal vectors $\hat{\mathbf{n}}$. In this section, we explain this more carefully.

Consider the sphere $x^2+y^2+z^2 = 1$. We can think of this surface as having two sides — an inside (the side you see when you are living inside the sphere) and an outside (the side you see when you are living outside the sphere). Concentrate on one point (x_0, y_0, z_0) on the sphere. The surface $x^2+y^2+z^2 = 1$ has precisely two unit normal vectors at (x_0, y_0, z_0), namely

$$\hat{\mathbf{n}}_+ = +(x_0, y_0, z_0) \qquad \text{and} \qquad \hat{\mathbf{n}}_- = -(x_0, y_0, z_0)$$

We can view $\hat{\mathbf{n}}_+$ as being associated to (or attached to) the outside of the sphere and $\hat{\mathbf{n}}_-$ as being associated to (or attached to) the inside of the sphere. Note that, as we move over the sphere, both $\hat{\mathbf{n}}_+$ and $\hat{\mathbf{n}}_-$ change continuously.

Definition 3.5.1.

An oriented surface is a surface together with a *continuous* function

$$\hat{\mathbf{N}} : S \to \mathbb{R}^3$$

such that, for each point p of S, $\hat{\mathbf{N}}(p)$ is a unit normal to S at p.

Example 3.5.2 (Sphere)

One orientation of the sphere $S = \{ (x, y, z) \mid x^2 + y^2 + z^2 = 1 \}$ is

$$\hat{\mathbf{N}}(x, y, z) = (x, y, z)$$

It associates to each point p of S the outward pointing unit normal to S at p. We can think of S with this orientation as being the outer side of S.

The other orientation of the sphere $S = \{ (x, y, z) \mid x^2 + y^2 + z^2 = 1 \}$ is

$$\hat{\mathbf{N}}(x, y, z) = -(x, y, z)$$

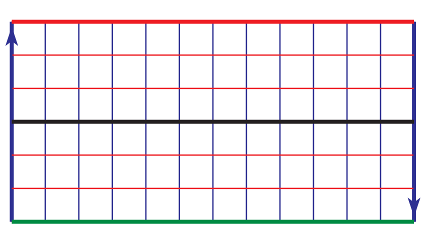
It associates to each point p of S the inward pointing unit normal to S at p. We can think of S with this orientation as being the inner side of S.

While this discussion might seem inordinately picky, it turns out that not all surfaces can be oriented. Our next example exhibits one.

Example 3.5.2

Example 3.5.3 (Optional — The Möbius Strip)

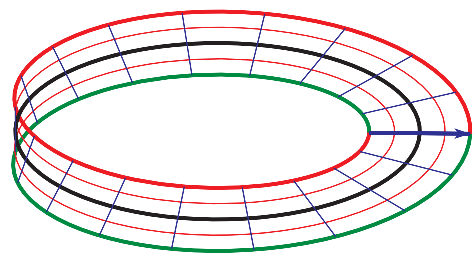
There are some surfaces S for which it is not possible to choose a continuous orientation map $\hat{\mathbf{N}} : S \to \mathbb{R}^3$. Such surfaces are said to be non-orientable. The most famous non-orientable surface is the Möbius[31] strip[32], which you can construct as follows. Take a rectangular strip of paper. Lay it flat and then introduce a half twist so that the arrow on



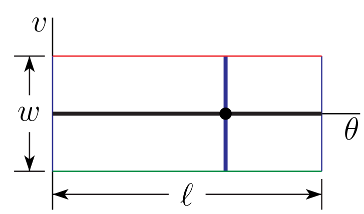
31 August Ferdinand Möbius (1790–1868) was a German mathematician and astronomer. He was a descendant of Martin Luther and a student of Gauss.

32 Another famous non-orientable surface is the Klein bottle. You can easily find discussions of it using your favourite search engine.

the right hand end points upwards, rather than downwards. Then glue the two ends of the strip together, with the two arrows coinciding. That's the Möbius strip.



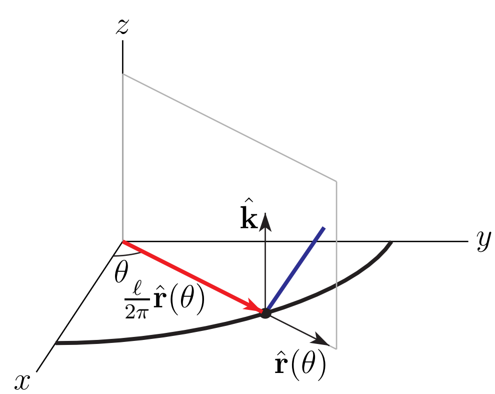
Let's parametrize it. Think of the strip of paper that we used to construct it as consisting of a backbone (the horizontal black line in the figure below) with a bunch of ribs (like the thick blue line in the figure) emanating from it. When we glue the two ends of the



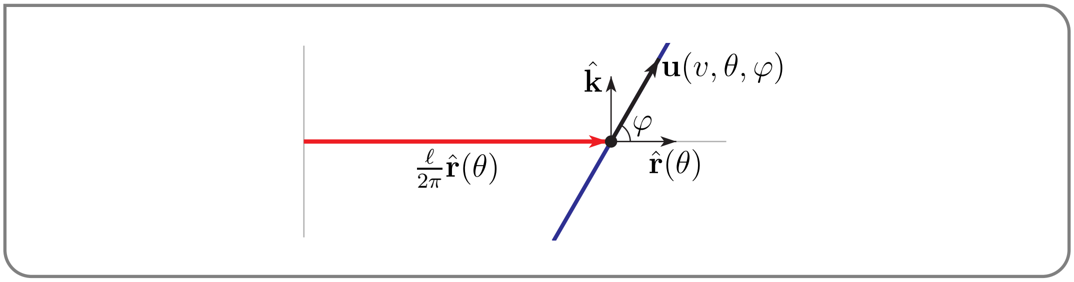
strip together, the black line forms a circle. If the strip has length ℓ, the circle will have circumference ℓ and hence radius $\frac{\ell}{2\pi}$. We'll parametrize it as the circle

$$\frac{\ell}{2\pi}\hat{\mathbf{r}}(\theta) \qquad \text{where } \hat{\mathbf{r}}(\theta) = \cos(\theta)\,\hat{\imath} + \sin(\theta)\,\hat{\jmath}$$

This circle is in the xy-plane. It is the black circle in the figure below. (The figure only shows the part of the circle in the first octant, i.e. with $x, y, z \geqslant 0$.) Now we'll add in the



blue ribs. We'll put the blue rib, that is attached to the backbone at $\frac{\ell}{2\pi}\hat{\mathbf{r}}(\theta)$, in the plane that contains the vectors $\hat{\mathbf{r}}(\theta)$ and $\hat{\mathbf{k}}$. A side view of the plane that contains the vectors $\hat{\mathbf{r}}(\theta)$ and $\hat{\mathbf{k}}$ is sketched in the figure below. To put the half twist into the strip of paper, we



want the blue rib to rotate about the backbone by 180°, i.e. π radians, as θ runs from 0 to 2π. That will be the case if we pick the angle φ in the figure to be $\theta/2$. The vector that is running along the blue rib in the figure is

$$\mathbf{u}(v,\theta,\varphi) = v\cos(\varphi)\,\hat{\mathbf{r}}(\theta) + v\sin(\varphi)\,\hat{\mathbf{k}}$$

where the length, v, of the vector is a parameter. If the width of our original strip of paper is w, then as the parameter v runs from $-w/2$ to $+w/2$, the tip of the vector $\mathbf{u}(v,\theta,\varphi)$ runs over the entire blue rib. So, choosing $\varphi = \theta/2$, our parametrization of the Möbius strip is

$$\begin{aligned}\mathbf{r}(\theta,v) &= \frac{\ell}{2\pi}\hat{\mathbf{r}}(\theta) + \mathbf{u}(v,\theta,\theta/2)\\ &= \frac{\ell}{2\pi}\hat{\mathbf{r}}(\theta) + v\cos\big(\theta/2\big)\hat{\mathbf{r}}(\theta) + v\sin\big(\theta/2\big)\hat{\mathbf{k}} \qquad 0 \leqslant \theta < 2\pi,\ -\frac{w}{2} \leqslant v \leqslant \frac{w}{2}\end{aligned}$$

where $\hat{\mathbf{r}}(\theta) = \cos(\theta)\,\hat{\imath} + \sin(\theta)\,\hat{\jmath}$.

Now that we have parametrized the Möbius strip, let's return to the question of orientability. Recall, from Definition 3.5.1, that, if the Möbius strip were orientable, there would exist a continuous function $\hat{\mathbf{N}}$ which assigns to each point $\mathbf{r}$ of the strip a unit normal vector $\hat{\mathbf{N}}(\mathbf{r})$ at $\mathbf{r}$. First, we'll find the normal vectors to the surface using (3.3.1). The partial derivatives

$$\begin{aligned}\frac{\partial\mathbf{r}}{\partial\theta}(\theta,v) &= \frac{\ell}{2\pi}\hat{\mathbf{r}}'(\theta) + v\cos\big(\theta/2\big)\hat{\mathbf{r}}'(\theta) - \frac{v}{2}\sin\big(\theta/2\big)\hat{\mathbf{r}}(\theta) + \frac{v}{2}\cos\big(\theta/2\big)\hat{\mathbf{k}}\\ \frac{\partial\mathbf{r}}{\partial v}(\theta,v) &= \cos\big(\theta/2\big)\hat{\mathbf{r}}(\theta) + \sin\big(\theta/2\big)\hat{\mathbf{k}}\end{aligned}$$

are relatively messy, so let's just consider the case $v = 0$ (i.e. find the normal vectors on the backbone). Then

$$\begin{aligned}\frac{\partial\mathbf{r}}{\partial\theta}(\theta,0) &= \frac{\ell}{2\pi}\hat{\mathbf{r}}'(\theta)\\ \frac{\partial\mathbf{r}}{\partial v}(\theta,0) &= \cos\big(\theta/2\big)\hat{\mathbf{r}}(\theta) + \sin\big(\theta/2\big)\hat{\mathbf{k}}\end{aligned}$$

Since

$$\begin{aligned}\hat{\mathbf{r}}'(\theta)\times\hat{\mathbf{r}}(\theta) &= \big(-\sin(\theta)\,\hat{\imath} + \cos(\theta)\,\hat{\jmath}\big)\times\big(\cos(\theta)\,\hat{\imath} + \sin(\theta)\,\hat{\jmath}\big) = -\hat{\mathbf{k}}\\ \hat{\mathbf{r}}'(\theta)\times\hat{\mathbf{k}} &= \big(-\sin(\theta)\,\hat{\imath} + \cos(\theta)\,\hat{\jmath}\big)\times\hat{\mathbf{k}} = \hat{\mathbf{r}}(\theta)\end{aligned}$$

we have

$$\frac{\partial\mathbf{r}}{\partial\theta}(\theta,0)\times\frac{\partial\mathbf{r}}{\partial v}(\theta,0) = -\frac{\ell}{2\pi}\Big(\cos\big(\theta/2\big)\,\hat{\mathbf{k}} - \sin\big(\theta/2\big)\,\hat{\mathbf{r}}(\theta)\Big)$$

As $\hat{\mathbf{k}}$ and $\hat{\mathbf{r}}(\theta)$ are mutually perpendicular unit vectors, $\cos\big(\theta/2\big)\,\hat{\mathbf{k}} - \sin\big(\theta/2\big)\,\hat{\mathbf{r}}(\theta)$ has length one, and the two unit normal vectors to the Möbius strip at $\mathbf{r}(\theta,0)$ are

$$\pm\Big(\cos\big(\theta/2\big)\,\hat{\mathbf{k}} - \sin\big(\theta/2\big)\,\hat{\mathbf{r}}(\theta)\Big)$$

So, for each θ, $\hat{\mathbf{N}}\big(\mathbf{r}(\theta,0)\big)$ must be either

$$\cos\big(\theta/2\big)\,\hat{\mathbf{k}} - \sin\big(\theta/2\big)\,\hat{\mathbf{r}}(\theta) \qquad \text{or} \qquad -\big(\cos\big(\theta/2\big)\,\hat{\mathbf{k}} - \sin\big(\theta/2\big)\,\hat{\mathbf{r}}(\theta)\big)$$

Imagine walking along the Möbius strip. The normal vector $\hat{\mathbf{N}}\big(\mathbf{r}(\theta,v)\big)$ is our body when we are at $\mathbf{r}(\theta,v)$ — our feet are at the tail of the vector $\hat{\mathbf{N}}\big(\mathbf{r}(\theta,v)\big)$ and our head is at the arrow of $\hat{\mathbf{N}}\big(\mathbf{r}(\theta,v)\big)$. We start walking at $\mathbf{r}(0,0) = \frac{\ell}{2\pi}\hat{\boldsymbol{\imath}}$. Our body, $\hat{\mathbf{N}}\big(\frac{\ell}{2\pi}\hat{\boldsymbol{\imath}}\big) = \hat{\mathbf{N}}\big(\mathbf{r}(0,0)\big)$ has to be one of $\pm\big(\cos(0)\,\hat{\mathbf{k}} - \sin(0)\,\hat{\mathbf{r}}(0)\big) = \pm\hat{\mathbf{k}}$. Let's suppose that $\hat{\mathbf{N}}\big(\mathbf{r}(0,0)\big) = +\hat{\mathbf{k}}$. (We start upright.) Now we start walking along the backbone of the Möbius strip, increasing θ. Because $\hat{\mathbf{N}}\big(\mathbf{r}(\theta,0)\big)$ has to be continuous, $\hat{\mathbf{N}}\big(\mathbf{r}(\theta,0)\big)$ has to be $+\big(\cos\big(\theta/2\big)\,\hat{\mathbf{k}} - \sin\big(\theta/2\big)\,\hat{\mathbf{r}}(\theta)\big)$. We keep increasing θ. By continuity, $\hat{\mathbf{N}}\big(\mathbf{r}(\theta,0)\big)$ has to be $+\big(\cos\big(\theta/2\big)\,\hat{\mathbf{k}} - \sin\big(\theta/2\big)\,\hat{\mathbf{r}}(\theta)\big)$ for bigger and bigger θ. Eventually we get to $\theta = 2\pi$, i.e. to

$$\mathbf{r}(2\pi,0) = \frac{\ell}{2\pi}\hat{\mathbf{r}}(2\pi) = \frac{\ell}{2\pi}\hat{\boldsymbol{\imath}} = \frac{\ell}{2\pi}\hat{\mathbf{r}}(0) = \mathbf{r}(0,0)$$

We are back to our starting point. Continuity has forced

$$\hat{\mathbf{N}}\big(\mathbf{r}(2\pi,0)\big) = \hat{\mathbf{N}}\big(\mathbf{r}(\theta,0)\big)\Big|_{\theta=2\pi} = +\big(\cos\big(\theta/2\big)\,\hat{\mathbf{k}} - \sin\big(\theta/2\big)\,\hat{\mathbf{r}}(\theta)\big)\Big|_{\theta=2\pi} = -\hat{\mathbf{k}}$$

So we have arrived back upside down. That's a problem — $\hat{\mathbf{N}}\big(\mathbf{r}(2\pi,0)\big) = \hat{\mathbf{N}}\big(\frac{\ell}{2\pi}\hat{\boldsymbol{\imath}}\big)$ and we have already defined $\hat{\mathbf{N}}\big(\frac{\ell}{2\pi}\hat{\boldsymbol{\imath}}\big) = +\hat{\mathbf{k}}$, not $-\hat{\mathbf{k}}$. So the Möbius strip is not orientable. The interested reader should look up M. C. Escher's Möbius Strip II (Red Ants).

Example 3.5.3

Chapter 4

INTEGRAL THEOREMS

4.1▲ Gradient, Divergence and Curl

"Gradient, divergence and curl", commonly called "grad, div and curl", refer to a very widely used family of differential operators and related notations that we'll get to shortly. We will later see that each has a "physical" significance. But even if they were only shorthand[1], they would be worth using.

For example, one of Maxwell's equations (relating the electric field **E** and the magnetic field **B**) written without the use of this notation is

$$\Big(\frac{\partial E_3}{\partial y}-\frac{\partial E_2}{\partial z}\Big)\hat{\boldsymbol{\imath}}-\Big(\frac{\partial E_3}{\partial x}-\frac{\partial E_1}{\partial z}\Big)\hat{\boldsymbol{\jmath}}+\Big(\frac{\partial E_2}{\partial x}-\frac{\partial E_1}{\partial y}\Big)\hat{\mathbf{k}}=-\frac{1}{c}\Big(\frac{\partial B_1}{\partial t}\hat{\boldsymbol{\imath}}+\frac{\partial B_2}{\partial t}\hat{\boldsymbol{\jmath}}+\frac{\partial B_3}{\partial t}\hat{\mathbf{k}}\Big)$$

The same equation written using this notation is

$$\boldsymbol{\nabla}\times\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}$$

The shortest way to write (and easiest way to remember) gradient, divergence and curl uses the symbol "$\boldsymbol{\nabla}$" which is a differential operator like $\frac{\partial}{\partial x}$. It is defined by

$$\boldsymbol{\nabla}=\hat{\boldsymbol{\imath}}\frac{\partial}{\partial x}+\hat{\boldsymbol{\jmath}}\frac{\partial}{\partial y}+\hat{\mathbf{k}}\frac{\partial}{\partial z}$$

and is called "del" or "nabla". Here are the definitions.

1 Good shorthand is not only more brief, but also aids understanding "of the forest by hiding the trees".

Definition 4.1.1.

(a) The gradient of a scalar-valued function $f(x,y,z)$ is the vector field

$$\text{grad}\, f = \nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}$$

Note that the input, f, for the gradient is a scalar-valued function, while the output,∇f, is a vector-valued function.

(b) The divergence of a vector field $\mathbf{F}(x,y,z)$ is the scalar-valued function

$$\text{div}\, \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Note that the input, $\mathbf{F}$, for the divergence is a vector-valued function, while the output, $\nabla \cdot \mathbf{F}$, is a scalar-valued function.

(c) The curl of a vector field $\mathbf{F}(x,y,z)$ is the vector field

$$\text{curl}\, \mathbf{F} = \nabla \times \mathbf{F} = \Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)\hat{\imath} - \Big(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\Big)\hat{\jmath} + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\hat{\mathbf{k}}$$

Note that the input, $\mathbf{F}$, for the curl is a vector-valued function, and the output, $\nabla \times \mathbf{F}$, is a again a vector-valued function.

(d) The Laplacian[2] of a scalar-valued function $f(x,y,z)$ is the scalar-valued function

$$\Delta f = \nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

The Laplacian of a vector field $\mathbf{F}(x,y,z)$ is the vector field

$$\Delta \mathbf{F} = \nabla^2 \mathbf{F} = \nabla \cdot \nabla \mathbf{F} = \frac{\partial^2 \mathbf{F}}{\partial x^2} + \frac{\partial^2 \mathbf{F}}{\partial y^2} + \frac{\partial^2 \mathbf{F}}{\partial z^2}$$

Note that the Laplacian maps either a scalar-valued function to a scalar-valued function, or a vector-valued function to a vector-valued function.

The gradient, divergence and Laplacian all have obvious generalizations to dimensions other than three. That is not the case for the curl. It does have a, far from obvious, generalization, which uses differential forms. Differential forms are well beyond our scope, but are introduced in the optional §4.7.

Example 4.1.2

2 Pierre-Simon Laplace (1749–1827) was a French mathematician and astronomer. He is also the Laplace of Laplace's equation, the Laplace transform, and the Laplace-Bayes estimator. He was Napoleon's examiner when Napoleon attended the Ecole Militaire in Paris.

As an example of an application in which both the divergence and curl appear, we have Maxwell's equations[3] [4] [5], which form the foundation of classical electromagnetism.

$$\begin{aligned}\nabla \cdot \mathbf{E} &= 4\pi\rho \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} &= 0 \\ \nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} &= \frac{4\pi}{c}\mathbf{J}\end{aligned}$$

Here **E** is the electric field, **B** is the magnetic field, ρ is the charge density, **J** is the current density and c is the speed of light.

Example 4.1.2

4.1.1 ▸▸ Vector Identities

Two computationally extremely important properties of the derivative $\frac{\mathrm{d}}{\mathrm{d}x}$ are linearity and the product rule.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\big(af(x) + bg(x)\big) &= a\frac{\mathrm{d}f}{\mathrm{d}x}(x) + b\frac{\mathrm{d}g}{\mathrm{d}x}(x) \\ \frac{\mathrm{d}}{\mathrm{d}x}\big(f(x)\,g(x)\big) &= \frac{\mathrm{d}f}{\mathrm{d}x}(x)\,g(x) + f(x)\,\frac{\mathrm{d}g}{\mathrm{d}x}(x)\end{aligned}$$

Gradient, divergence and curl also have properties like these, which indeed stem (often easily) from them. First, here are the statements of a bunch of them. (A memory aid and proofs will come later.) In fact, here are a very large number of them. Many are included just for completeness. Only a relatively small number are used a lot. They are in red.

Theorem 4.1.3 (Gradient Identities).

(a) $\nabla(f+g) = \nabla f + \nabla g$

(b) $\nabla(cf) = c\,\nabla f$, for any *constant* c

(c) $\nabla(fg) = (\nabla f)g + f(\nabla g)$

(d) $\nabla(f/g) = \big(g\,\nabla f - f\,\nabla g\big)/g^2$ at points $\mathbf{x}$ where $g(\mathbf{x}) \neq 0$.

(e) $\nabla(\mathbf{F}\cdot\mathbf{G}) = \mathbf{F}\times(\nabla\times\mathbf{G}) - (\nabla\times\mathbf{F})\times\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F} + (\mathbf{F}\cdot\nabla)\mathbf{G}$

Here[6]

$$(\mathbf{G}\cdot\nabla)\mathbf{F} = \mathbf{G}_1\frac{\partial \mathbf{F}}{\partial x} + \mathbf{G}_2\frac{\partial \mathbf{F}}{\partial y} + \mathbf{G}_3\frac{\partial \mathbf{F}}{\partial z}$$

3 To be picky, these are Maxwell's equations in the absence of a material medium and in Gaussian units.
4 One important consequence of Maxwell's equations is that electromagnetic radiation, like light, propagate at the speed of light.
5 James Clerk Maxwell (1831–1879) was a Scottish mathematical physicist. In a poll of prominent physicists, Maxwell was voted the third greatest physicist of all time. Only Newton and Einstein beat him.

Theorem 4.1.4 (Divergence Identities).

(a) $\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$

(b) $\nabla \cdot (c\mathbf{F}) = c\,\nabla \cdot \mathbf{F}$, for any *constant* c

(c) $\nabla \cdot (f\mathbf{F}) = (\nabla f) \cdot \mathbf{F} + f\,\nabla \cdot \mathbf{F}$

(d) $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G})$

Theorem 4.1.5 (Curl Identities).

(a) $\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}$

(b) $\nabla \times (c\mathbf{F}) = c\,\nabla \times \mathbf{F}$, for any *constant* c

(c) $\nabla \times (f\mathbf{F}) = (\nabla f) \times \mathbf{F} + f\,\nabla \times \mathbf{F}$

(d) $\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \cdot \mathbf{G}) - (\nabla \cdot \mathbf{F})\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$

Here

$$(\mathbf{G} \cdot \nabla)\mathbf{F} = \mathbf{G}_1 \frac{\partial \mathbf{F}}{\partial x} + \mathbf{G}_2 \frac{\partial \mathbf{F}}{\partial y} + \mathbf{G}_3 \frac{\partial \mathbf{F}}{\partial z}$$

Theorem 4.1.6 (Laplacian Identities).

(a) $\nabla^2 (f + g) = \nabla^2 f + \nabla^2 g$

(b) $\nabla^2 (cf) = c\,\nabla^2 f$, for any *constant* c

(c) $\nabla^2 (fg) = f\,\nabla^2 g + 2\nabla f \cdot \nabla g + g\,\nabla^2 f$

Theorem 4.1.7 (Degree Two Identities).

(a) $\nabla \cdot (\nabla \times \mathbf{F}) = 0$ (divergence of curl)

(b) $\nabla \times (\nabla f) = 0$ (curl of gradient)

(c) $\nabla \cdot \big(f\{\nabla g \times \nabla h\}\big) = \nabla f \cdot (\nabla g \times \nabla h)$

(d) $\nabla \cdot (f\nabla g - g\nabla f) = f\,\nabla^2 g - g\,\nabla^2 f$

(e) $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$ (curl of curl)

6 This is really the only definition that makes sense. For example $\mathbf{G} \cdot (\nabla \mathbf{F})$ does not make sense because you can't take the gradient of a vector-valued function.

Memory Aid. Most of the vector identities (in fact all of them except Theorem 4.1.3.e, Theorem 4.1.5.d and Theorem 4.1.7) are really easy to guess. Just combine the conventional linearity and product rules with the facts that

- if the left hand side is a vector (scalar), then the right hand side must also be a vector (scalar) and
- the only valid products of two vectors are the dot and cross products and
- the product of a scalar with either a scalar or a vector cannot be either a dot or cross product and
- $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$. (The cross product is antisymmetric.)

For example, consider Theorem 4.1.4.c, which says $\boldsymbol{\nabla} \cdot (f\mathbf{F}) = (\boldsymbol{\nabla} f) \cdot \mathbf{F} + f\, \boldsymbol{\nabla} \cdot \mathbf{F}$.

- The left hand side, $\boldsymbol{\nabla} \cdot (f\mathbf{F})$, is a scalar, so the right hand side must also be a scalar.
- The left hand side, $\boldsymbol{\nabla} \cdot (f\mathbf{F})$, is a derivative of the product of f and $\mathbf{F}$, so, mimicking the product rule, the right hand side will be a sum of two terms, one with $\mathbf{F}$ multiplying a derivative of f, and one with f multiplying a derivative of $\mathbf{F}$.
- The derivative acting on f must be $\boldsymbol{\nabla} f$, because $\boldsymbol{\nabla} \cdot f$ and $\boldsymbol{\nabla} \times f$ are not well-defined. To end up with a scalar, rather than a vector, we must take the dot product of $\boldsymbol{\nabla} f$ and $\mathbf{F}$. So that term is $(\boldsymbol{\nabla} f) \cdot \mathbf{F}$.
- The derivative acting on $\mathbf{F}$ must be either $\boldsymbol{\nabla} \cdot \mathbf{F}$ or $\boldsymbol{\nabla} \times \mathbf{F}$. We also need to multiply by the scalar f and end up with a scalar. So the derivative must be a scalar, i.e. $\boldsymbol{\nabla} \cdot \mathbf{F}$ and that term is $f\{\boldsymbol{\nabla} \cdot \mathbf{F}\}$.
- Our final guess is $\boldsymbol{\nabla} \cdot (f\mathbf{F}) = (\boldsymbol{\nabla} f) \cdot \mathbf{F} + f\, \boldsymbol{\nabla} \cdot \mathbf{F}$, which, thankfully, is correct.

Proof of Theorems 4.1.3, 4.1.4, 4.1.5, 4.1.6 and 4.1.7. All of the proofs (except for those of Theorem 4.1.7.c,d, which we will return to later) consist of

- writing out the definition of the left hand side and
- writing out the definition of the right hand side and
- observing (possibly after a little manipulation) that they are the same.

For Theorem 4.1.3.a,b, Theorem 4.1.4.a,b, Theorem 4.1.5.a,b and Theorem 4.1.6.a,b, the computation is trivial — one line per identity, if one uses some efficient notation. Rename the coordinates x, y, z to x_1, x_2, x_3 and the standard unit basis vectors $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\mathbf{k}}$ to $\hat{\boldsymbol{\imath}}_1, \hat{\boldsymbol{\imath}}_2, \hat{\boldsymbol{\imath}}_3$. Then $\boldsymbol{\nabla} = \sum_{n=1}^{3} \hat{\boldsymbol{\imath}}_n \frac{\partial}{\partial x_n}$ and the proof of, for example, Theorem 4.1.4.a is

$$\begin{aligned} \boldsymbol{\nabla} \cdot (\mathbf{F} + \mathbf{G}) &= \sum_{n=1}^{3} \frac{\partial}{\partial x_n} \hat{\boldsymbol{\imath}}_n \cdot (\mathbf{F} + \mathbf{G}) \\ &= \sum_{n=1}^{3} \frac{\partial}{\partial x_n} \hat{\boldsymbol{\imath}}_n \cdot \mathbf{F} + \sum_{n=1}^{3} \frac{\partial}{\partial x_n} \hat{\boldsymbol{\imath}}_n \cdot \mathbf{G} = \boldsymbol{\nabla} \cdot \mathbf{F} + \boldsymbol{\nabla} \cdot \mathbf{G} \end{aligned}$$

For Theorem 4.1.3.c,d, Theorem 4.1.4.c, Theorem 4.1.5.c and Theorem 4.1.6.c, the computation is easy — a few lines per identity. For example, the proof of Theorem 4.1.5.c is

$$\begin{aligned}
\boldsymbol{\nabla}\times(f\mathbf{F}) &= \sum_{n=1}^{3}\frac{\partial}{\partial x_n}\hat{\boldsymbol{\imath}}_n\times(f\mathbf{F}) = \sum_{n=1}^{3}\frac{\partial}{\partial x_n}\big(f\,\{\hat{\boldsymbol{\imath}}_n\times\mathbf{F}\}\big)\\
&= \sum_{n=1}^{3}\frac{\partial f}{\partial x_n}\hat{\boldsymbol{\imath}}_n\times\mathbf{F} + f\sum_{n=1}^{3}\frac{\partial}{\partial x_n}\hat{\boldsymbol{\imath}}_n\times\mathbf{F} \qquad \text{(by Theorem 1.1.3.b)}\\
&= (\boldsymbol{\nabla}f)\times\mathbf{F} + f\,\boldsymbol{\nabla}\times\mathbf{F}
\end{aligned}$$

The similar verification of Theorems 4.1.3.c,d, 4.1.4.c and 4.1.6.c are left as exercises. The latter two are parts (a) and (c) of Question 3 in Section 4.1 of the CLP-4 problem book.

For Theorem 4.1.4.d, the computation is also easy if one uses the fact that

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$$

which is Lemma 4.1.8.a below. The verification of Theorem 4.1.4.d is part (b) of Question 3 in Section 4.1 of the CLP-4 problem book.

That leaves the proofs of Theorem 4.1.3.e, Theorem 4.1.5.d, Theorem 4.1.7.a,b,c,d,e, which we write out explicitly.

Theorem 4.1.3.e:

First write out the left hand side as

$$\boldsymbol{\nabla}(\mathbf{F}\cdot\mathbf{G}) = \sum_{n=1}^{3}\hat{\boldsymbol{\imath}}_n\frac{\partial}{\partial x_n}(\mathbf{F}\cdot\mathbf{G}) = \sum_{n=1}^{3}\hat{\boldsymbol{\imath}}_n\Big(\frac{\partial\mathbf{F}}{\partial x_n}\cdot\mathbf{G}\Big) + \sum_{n=1}^{3}\hat{\boldsymbol{\imath}}_n\Big(\mathbf{F}\cdot\frac{\partial\mathbf{G}}{\partial x_n}\Big)$$

Then rewrite $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\cdot\mathbf{a})\mathbf{b} - (\mathbf{b}\cdot\mathbf{a})\mathbf{c}$, which is Lemma 4.1.8.b below, as

$$(\mathbf{c}\cdot\mathbf{a})\mathbf{b} = \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) + (\mathbf{b}\cdot\mathbf{a})\mathbf{c}$$

Applying it once with $\mathbf{b} = \hat{\boldsymbol{\imath}}_n$, $\mathbf{c} = \frac{\partial\mathbf{F}}{\partial x_n}$, $\mathbf{a} = \mathbf{G}$ and once with $\mathbf{b} = \hat{\boldsymbol{\imath}}_n$, $\mathbf{c} = \frac{\partial\mathbf{G}}{\partial x_n}$, $\mathbf{a} = \mathbf{F}$ gives

$$\begin{aligned}
\boldsymbol{\nabla}(\mathbf{F}\cdot\mathbf{G}) &= \sum_{n=1}^{3}\Big[\mathbf{G}\times\Big(\hat{\boldsymbol{\imath}}_n\times\frac{\partial\mathbf{F}}{\partial x_n}\Big) + (\mathbf{G}\cdot\hat{\boldsymbol{\imath}}_n)\frac{\partial\mathbf{F}}{\partial x_n}\Big] + \sum_{n=1}^{3}\Big[\mathbf{F}\times\Big(\hat{\boldsymbol{\imath}}_n\times\frac{\partial\mathbf{G}}{\partial x_n}\Big) + (\mathbf{F}\cdot\hat{\boldsymbol{\imath}}_n)\frac{\partial\mathbf{G}}{\partial x_n}\Big]\\
&= \mathbf{G}\times(\boldsymbol{\nabla}\times\mathbf{F}) + (\mathbf{G}\cdot\boldsymbol{\nabla})\mathbf{F} + \mathbf{F}\times(\boldsymbol{\nabla}\times\mathbf{G}) + (\mathbf{F}\cdot\boldsymbol{\nabla})\mathbf{G}
\end{aligned}$$

Theorem 4.1.5.d:

We use the same trick. Write out the left hand side as

$$\boldsymbol{\nabla}\times(\mathbf{F}\times\mathbf{G}) = \sum_{n=1}^{3}\hat{\boldsymbol{\imath}}_n\times\frac{\partial}{\partial x_n}(\mathbf{F}\times\mathbf{G}) = \sum_{n=1}^{3}\hat{\boldsymbol{\imath}}_n\times\Big(\frac{\partial\mathbf{F}}{\partial x_n}\times\mathbf{G}\Big) + \sum_{n=1}^{3}\hat{\boldsymbol{\imath}}_n\times\Big(\mathbf{F}\times\frac{\partial\mathbf{G}}{\partial x_n}\Big)$$

Applying $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\cdot\mathbf{a})\mathbf{b} - (\mathbf{b}\cdot\mathbf{a})\mathbf{c}$, which is Lemma 4.1.8.b below,

$$\begin{aligned}
\boldsymbol{\nabla}\times(\mathbf{F}\times\mathbf{G}) &= \sum_{n=1}^{3}\Big[G_n\frac{\partial\mathbf{F}}{\partial x_n} - \frac{\partial F_n}{\partial x_n}\mathbf{G}\Big] + \sum_{n=1}^{3}\Big[\frac{\partial G_n}{\partial x_n}\mathbf{F} - F_n\frac{\partial\mathbf{G}}{\partial x_n}\Big]\\
&= (\mathbf{G}\cdot\boldsymbol{\nabla})\mathbf{F} - (\boldsymbol{\nabla}\cdot\mathbf{F})\mathbf{G} + (\boldsymbol{\nabla}\cdot\mathbf{G})\mathbf{F} - (\mathbf{F}\cdot\boldsymbol{\nabla})\mathbf{G}
\end{aligned}$$

Theorem 4.1.7.a:
Substituting in

$$\nabla \times \mathbf{F} = \Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)\hat{\boldsymbol{\imath}} - \Big(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\Big)\hat{\boldsymbol{\jmath}} + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\hat{\mathbf{k}}$$

gives

$$\begin{aligned}\nabla \cdot (\nabla \times F) &= \frac{\partial}{\partial x}\Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big) - \frac{\partial}{\partial y}\Big(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\Big) + \frac{\partial}{\partial z}\Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\\ &= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_1}{\partial y \partial z} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y}\\ &= 0\end{aligned}$$

because the two red terms have cancelled, the two blue terms have cancelled and the two black terms have cancelled.

Theorem 4.1.7.b:
Substituting in

$$\nabla f = \frac{\partial f}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}$$

gives

$$\nabla \times (\nabla f) = \Big(\frac{\partial}{\partial y}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial y}\Big)\hat{\boldsymbol{\imath}} - \Big(\frac{\partial}{\partial x}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial x}\Big)\hat{\boldsymbol{\jmath}} + \Big(\frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial y}\frac{\partial f}{\partial x}\Big)\hat{\mathbf{k}} = 0$$

Theorem 4.1.7.c:
By Theorem 4.1.4.c, followed by Theorem 4.1.4.d,

$$\begin{aligned}\nabla \cdot \big[f(\nabla g \times \nabla h)\big] &= \nabla f \cdot (\nabla g \times \nabla h) + f \nabla \cdot (\nabla g \times \nabla h)\\ &= \nabla f \cdot (\nabla g \times \nabla h) + f\big[(\nabla \times \nabla g) \cdot \nabla h - \nabla g \cdot (\nabla \times \nabla h)\big]\end{aligned}$$

By Theorem 4.1.7.b, $\nabla \times \nabla g = \nabla \times \nabla h = 0$, so

$$\nabla \cdot \big[f(\nabla g \times \nabla f)\big] = \nabla f \cdot (\nabla g \times \nabla h)$$

Theorem 4.1.7.d:
By Theorem 4.1.4.c,

$$\begin{aligned}\nabla \cdot (f \nabla g - g \nabla f) &= (\nabla f) \cdot (\nabla g) + f\, \nabla \cdot (\nabla g) - (\nabla g) \cdot (\nabla f) + g\, \nabla \cdot (\nabla f)\\ &= f\, \nabla^2 g - g\, \nabla^2 f\end{aligned}$$

Theorem 4.1.7.e:

$$\nabla \times (\nabla \times \mathbf{F}) = \sum_{\ell=1}^{3} \hat{\boldsymbol{\imath}}_\ell \frac{\partial}{\partial x_\ell} \times \left(\sum_{m=1}^{3} \hat{\boldsymbol{\imath}}_m \frac{\partial}{\partial x_m} \times \sum_{n=1}^{3} \hat{\boldsymbol{\imath}}_n F_n \right) = \sum_{\ell,m,n=1}^{3} \hat{\boldsymbol{\imath}}_\ell \times \big(\hat{\boldsymbol{\imath}}_m \times \hat{\boldsymbol{\imath}}_n\big)\, \frac{\partial^2 \ F_n}{\partial x_\ell \partial x_m}$$

Using $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{b} \cdot \mathbf{a})\mathbf{c}$, we have

$$\hat{\boldsymbol{\imath}}_\ell \times \big(\hat{\boldsymbol{\imath}}_m \times \hat{\boldsymbol{\imath}}_n\big) = (\hat{\boldsymbol{\imath}}_\ell \cdot \hat{\boldsymbol{\imath}}_n)\hat{\boldsymbol{\imath}}_m - (\hat{\boldsymbol{\imath}}_\ell \cdot \hat{\boldsymbol{\imath}}_m)\hat{\boldsymbol{\imath}}_n = \delta_{\ell,n}\hat{\boldsymbol{\imath}}_m - \delta_{\ell,m}\hat{\boldsymbol{\imath}}_n$$

where[7]

$$\delta_{m,n} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

Hence

$$\begin{aligned} \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{F}) &= \sum_{\ell,m,n=1}^{3} \delta_{\ell,n}\hat{\boldsymbol{\imath}}_m \frac{\partial^2 \ F_n}{\partial x_\ell \partial x_m} - \sum_{\ell,m,n=1}^{3} \delta_{\ell,m}\hat{\boldsymbol{\imath}}_n \frac{\partial^2 \ F_n}{\partial x_\ell \partial x_m} \\ &= \sum_{m,n=1}^{3} \hat{\boldsymbol{\imath}}_m \frac{\partial}{\partial x_m}\frac{\partial F_n}{\partial x_n} - \sum_{m,n=1}^{3} \hat{\boldsymbol{\imath}}_n \frac{\partial^2 F_n}{\partial x_m^2} \\ &= \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{F}) - \boldsymbol{\nabla}^2\mathbf{F} \end{aligned}$$

□

Lemma 4.1.8.

(a) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$

(b) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{b} \cdot \mathbf{a})\mathbf{c}$

Proof. (a) Here are two proofs. For the first, just write out both sides

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= (a_1, a_2, a_3) \cdot (b_2c_3 - b_3c_2\,,\, b_3c_1 - b_1c_3\,,\, b_1c_2 - b_2c_1) \\ &= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1 \\ (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} &= (a_2b_3 - a_3b_2\,,\, a_3b_1 - a_1b_3\,,\, a_1b_2 - a_2b_1) \cdot (c_1, c_2, c_3) \\ &= a_2b_3c_1 - a_3b_2c_1 + a_3b_1c_2 - a_1b_3c_2 + a_1b_2c_3 - a_2b_1c_3 \end{aligned}$$

and observe that they are the same.

For the second proof, we again write out both sides, but this time we express them in

7 $\delta_{m,n}$ is called the Kronecker delta function. It is named after the German number theorist and logician Leopold Kronecker (1823–1891). He is reputed to have said "God made the integers. All else is the work of man."

terms of determinants.

$$\begin{aligned}
\mathbf{a}\cdot\mathbf{b}\times\mathbf{c} &= (a_1,a_2,a_3)\cdot\det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}}\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{bmatrix}\\
&= a_1\det\begin{bmatrix}b_2 & b_3\\ c_2 & c_3\end{bmatrix} - a_2\det\begin{bmatrix}b_1 & b_3\\ c_1 & c_3\end{bmatrix} + a_3\det\begin{bmatrix}b_1 & b_2\\ c_1 & c_2\end{bmatrix}\\
&= \det\begin{bmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathbf{a}\times\mathbf{b}\cdot\mathbf{c} &= \det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{bmatrix}\cdot(c_1,c_2,c_3)\\
&= c_1\det\begin{bmatrix}a_2 & a_3\\ b_2 & b_3\end{bmatrix} - c_2\det\begin{bmatrix}a_1 & a_3\\ b_1 & b_3\end{bmatrix} + c_3\det\begin{bmatrix}a_1 & a_2\\ b_1 & b_2\end{bmatrix}\\
&= \det\begin{bmatrix}c_1 & c_2 & c_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{bmatrix}
\end{aligned}$$

Exchanging two rows in a determinant changes the sign of the determinant. Moving the top row of a 3×3 determinant to the bottom row requires two exchanges of rows. So the two 3×3 determinants are equal.

(b) The proof is not exceptionally difficult — just write out both sides and grind. Substituting in

$$\mathbf{b}\times\mathbf{c} \;=\; (b_2c_3-b_3c_2)\hat{\boldsymbol{\imath}} - (b_1c_3-b_3c_1)\hat{\boldsymbol{\jmath}} + (b_1c_2-b_2c_1)\hat{\mathbf{k}}$$

gives, for the left hand side,

$$\begin{aligned}
\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) =&\ \det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}}\\ a_1 & a_2 & a_3\\ b_2c_3-b_3c_2 & -b_1c_3+b_3c_1 & b_1c_2-b_2c_1\end{bmatrix}\\
=&\ \hat{\boldsymbol{\imath}}\big[a_2(b_1c_2-b_2c_1) - a_3(-b_1c_3+b_3c_1)\big]\\
&-\hat{\boldsymbol{\jmath}}\big[a_1(b_1c_2-b_2c_1) - a_3(b_2c_3-b_3c_2)\big]\\
&+\hat{\mathbf{k}}\big[a_1(-b_1c_3+b_3c_1) - a_2(b_2c_3-b_3c_2)\big]
\end{aligned}$$

On the other hand, the right hand side

$$\begin{aligned}
(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c} &= (a_1c_1+a_2c_2+a_3c_3)(b_1\hat{\boldsymbol{\imath}}+b_2\hat{\boldsymbol{\jmath}}+b_3\hat{\mathbf{k}}) - (a_1b_1+a_2b_2+a_3b_3)(c_1\hat{\boldsymbol{\imath}}+c_2\hat{\boldsymbol{\jmath}}+c_3\hat{\mathbf{k}})\\
&= \hat{\boldsymbol{\imath}}\ \big[a_1b_1c_1 + a_2b_1c_2 + a_3b_1c_3 - a_1b_1c_1 - a_2b_2c_1 - a_3b_3c_1\big]\\
&\quad +\hat{\boldsymbol{\jmath}}\ \big[a_1b_2c_1 + a_2b_2c_2 + a_3b_2c_3 - a_1b_1c_2 - a_2b_2c_2 - a_3b_3c_2\big]\\
&\quad +\hat{\mathbf{k}}\ \big[a_1b_3c_1 + a_2b_3c_2 + a_3b_3c_3 - a_1b_1c_3 - a_2b_2c_3 - a_3b_3c_3\big]\\
&= \hat{\boldsymbol{\imath}}\ [a_2b_1c_2 + a_3b_1c_3 - a_2b_2c_1 - a_3b_3c_1]\\
&\quad +\hat{\boldsymbol{\jmath}}\ [a_1b_2c_1 + a_3b_2c_3 - a_1b_1c_2 - a_3b_3c_2]\\
&\quad +\hat{\mathbf{k}}\ [a_1b_3c_1 + a_2b_3c_2 - a_1b_1c_3 - a_2b_2c_3]
\end{aligned}$$

The last formula that we had for the left hand side is the same as the last formula we had for the right hand side. □

Example 4.1.9 (Screening tests)

We have seen the vector identity Theorem 4.1.7.b before. It says that if a vector field $\mathbf{F}$ is of the form $\mathbf{F} = \nabla\varphi$ for some some function φ (that is, if $\mathbf{F}$ is conservative), then

$$\nabla \times \mathbf{F} = \nabla \times (\nabla\varphi) = 0$$

Conversely, we have also seen, in Theorem 2.4.8, that, if $\mathbf{F}$ is defined and has continuous first order partial derivatives on all of $\mathbb{R}^3$, and if $\nabla \times \mathbf{F} = 0$, then $\mathbf{F}$ is conservative. The vector identity Theorem 4.1.7.b is our screening test for conservativeness.

Because its right hand side is zero, the vector identity Theorem 4.1.7.a is suggestive. It says that if a vector field $\mathbf{F}$ is of the form $\mathbf{F} = \nabla \times \mathbf{A}$ for some some vector field $\mathbf{A}$, then

$$\nabla \cdot \mathbf{F} = \nabla \cdot (\nabla \times \mathbf{A}) = 0$$

When $\mathbf{F} = \nabla \times \mathbf{A}$, $\mathbf{A}$ is called a *vector potential* for $\mathbf{F}$. We shall see in Theorem 4.1.16, below, that, conversely, if $\mathbf{F}(\mathbf{x})$ is defined and has continuous first order partial derivatives on all of $\mathbb{R}^3$, and if $\nabla \cdot \mathbf{F} = 0$, then $\mathbf{F}$ has a vector potential[8]. The vector identity Theorem 4.1.7.a is indeed another screening test.

As an example, consider the Maxwell's equations

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} &= 0 \end{aligned}$$

that we saw in Example 4.1.2. The first equation implies that (assuming $\mathbf{B}$ is sufficiently smooth) there is a vector field $\mathbf{A}$, called the magnetic potential, with $\mathbf{B} = \nabla \times \mathbf{A}$. Substituting this into the second equation gives

$$0 = \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial}{\partial t}\nabla \times \mathbf{A} = \nabla \times \Big(\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\Big)$$

So $\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$ passes the screening test of Theorem 4.1.7.b and there is a function φ (called the electric potential) with

$$\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = -\nabla\varphi$$

We have put in the minus sign just to provide compatibility with the usual physics terminology.

Example 4.1.9

Example 4.1.10

Problem: Let $\mathbf{r}(x,y,z) = x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + z\,\hat{\mathbf{k}}$ and let $\psi(x,y,z)$ be an arbitrary function. Verify that

$$\nabla \cdot \big(\mathbf{r} \times \nabla\psi\big) = 0$$

8 Does this remind you of Theorem 2.4.8? It should.

Solution. By the vector identity Theorem 4.1.4.d,

$$\nabla \cdot \big(\mathbf{r} \times \nabla\psi\big) = (\nabla \times \mathbf{r}) \cdot \nabla\psi - \mathbf{r} \cdot \big(\nabla \times (\nabla\psi)\big)$$

By the vector identity Theorem 4.1.7.b, the second term is zero. Now since

$$\nabla \times \mathbf{r} = \Big(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\Big)\hat{\imath} - \Big(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\Big)\hat{\jmath} + \Big(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\Big)\hat{\mathbf{k}} = \mathbf{0}$$

the first term is also zero. Indeed $\nabla \cdot \big(\mathbf{r} \times \nabla\psi\big) = 0$ holds for any curl free $\mathbf{r}(x, y, z)$.

Example 4.1.10

4.1.2 ▸▸ Vector Potentials

We'll now further explore the vector potentials that were introduced in Example 4.1.9. First, here is the formal definition.

Definition 4.1.11.

The vector field $\mathbf{A}$ is said to be a *vector potential* for the vector field $\mathbf{B}$ if

$$\mathbf{B} = \nabla \times \mathbf{A}$$

As we saw in Example 4.1.9, if a vector field $\mathbf{B}$ has a vector potential, then the vector identity Theorem 4.1.7.a implies that $\nabla \cdot \mathbf{B} = 0$. This fact deserves to be called a theorem.

Theorem 4.1.12 (Screening test for vector potentials)**.**

If there exists a vector potential for the vector field $\mathbf{B}$, then

$$\nabla \cdot \mathbf{B} = 0$$

Of course, we'll consider the converse soon. Also note that the vector potential, when it exists, is far from unique. Two vector fields $\mathbf{A}$ and $\tilde{\mathbf{A}}$ are both vector potentials for the same vector field if and only if

$$\nabla \times \mathbf{A} = \nabla \times \tilde{\mathbf{A}} \iff \nabla \times (\mathbf{A} - \tilde{\mathbf{A}}) = \mathbf{0}$$

That is, if and only if the difference $\mathbf{A} - \tilde{\mathbf{A}}$ passes the conservative field screening test of Theorems 2.3.9 and 2.4.8. In particular, if $\mathbf{A}$ is one vector potential for a vector field $\mathbf{B}$ (i.e. if $\mathbf{B} = \nabla \times \mathbf{A}$), and if ψ is any function, then

$$\nabla \times (\mathbf{A} + \nabla\psi) = \nabla \times \mathbf{A} + \nabla \times \nabla\psi = \mathbf{B}$$

by the vector identity Theorem 4.1.7.b. That is, $\mathbf{A} + \nabla\psi$ is another vector potential for $\mathbf{B}$.

To simplify computations, we can always choose ψ so that, for example, the third component of $\mathbf{A} + \nabla\psi$, namely $\big(\mathbf{A} + \nabla\psi\big) \cdot \hat{\mathbf{k}} = \mathbf{A}_3 + \frac{\partial\psi}{\partial z}$, is zero — just choose $\psi = -\int \mathbf{A}_3\, \mathrm{d}z$. We have just proven

Lemma 4.1.13.

If the vector field $\mathbf{B}$ has a vector potential, then, in particular, there is a vector potential $\mathbf{A}$ for $\mathbf{B}$ with[9] $\mathbf{A}_3 = 0$.

Here is an example which exploits this choice to simplify the computations used to find a vector potential.

Example 4.1.14

Let

$$\mathbf{B} = yz\,\hat{\imath} + zx\,\hat{\jmath} + xy\,\hat{\mathbf{k}}$$

This vector field has been set up carefully to obey

$$\nabla \cdot \mathbf{B} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(zx) + \frac{\partial}{\partial z}(xy) = 0$$

and so passes the screening test of Theorem 4.1.12.

Let's try and find a vector potential for $\mathbf{B}$. That is, let's try and find a vector field $\mathbf{A} = A_1\,\hat{\imath} + A_2\,\hat{\jmath} + A_3\,\hat{\mathbf{k}}$ that obeys $\nabla \times \mathbf{A} = \mathbf{B}$, or equivalently,

$$\begin{aligned}
\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} &= B_1 = yz \\
-\frac{\partial A_3}{\partial x} + \frac{\partial A_1}{\partial z} &= B_2 = zx \\
\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} &= B_3 = xy
\end{aligned}$$

This system is nasty to solve because every equation contains more than one of the three unknowns, A_1, A_2, A_3. Let us take advantage of our observation above that, if any vector potential exists, then, in particular, a vector potential $\mathbf{A}$ exists that also obeys $A_3 = 0$. So let's also require that $A_3 = 0$. Then the equations above simplify to

$$\begin{aligned}
-\frac{\partial A_2}{\partial z} &= yz \\
\frac{\partial A_1}{\partial z} &= zx \\
\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} &= xy
\end{aligned}$$

This system is much easier because, now that we have chosen $A_3 = 0$, the first equation contains only a single unknown, namely A_2 and we can find all A_2's that obey the first equation simply by integrating with respect to z:

$$A_2 = -\frac{yz^2}{2} + N(x, y)$$

9 There is nothing special about the subscript 3 here. By precisely the same argument, we could come up with another vector potential whose second component is zero, and with a third vector potential whose first component is zero.

Note that, because $\frac{\partial}{\partial z}$ treats x and y as constants, the constant of integration N is allowed to depend on x and y.

Similarly, the second equation contains only a single unknown, A_1, and is easily solved by integrating with respect to z. The second equation is satisfied if and only if

$$A_1 = \frac{xz^2}{2} + M(x,y)$$

for some function M.

Finally, the third equation is also satisfied if and only if $M(x,y)$ and $N(x,y)$ obey

$$\frac{\partial}{\partial x}\Big(-\frac{yz^2}{2} + N(x,y)\Big) - \frac{\partial}{\partial y}\Big(\frac{xz^2}{2} + M(x,y)\Big) = xy$$

which simplifies to

$$\frac{\partial N}{\partial x}(x,y) - \frac{\partial M}{\partial y}(x,y) = xy$$

This is one linear equation in two unknowns, M and N. Typically, we can easily solve one linear equation in one unknown. So we are free to eliminate one of the unknowns by setting, for example, $M = 0$, and then choose any N that obeys

$$\frac{\partial N}{\partial x}(x,y) = xy$$

Integrating with respect to x gives, as one possible choice, $N(x,y) = \frac{x^2 y}{2}$. So we have found a vector potential. Namely

$$\mathbf{A} = \frac{xz^2}{2}\hat{\boldsymbol{\imath}} + \Big(-\frac{yz^2}{2} + \frac{x^2y}{2}\Big)\hat{\boldsymbol{\jmath}}$$

One can, and indeed should, quickly check that $\nabla \times \mathbf{A} = \mathbf{B}$.

Example 4.1.14

Let's do another.

Example 4.1.15

Let

$$\mathbf{B} = (2x)\,\hat{\boldsymbol{\imath}} + (2z - 2x)\,\hat{\boldsymbol{\jmath}} + (2x - 2z)\,\hat{\mathbf{k}}$$

This vector field obeys

$$\nabla \cdot \mathbf{B} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(2z - 2x) + \frac{\partial}{\partial z}(2x - 2z) = 0$$

and so passes the screening test of Theorem 4.1.12. We'll now find a vector potential $\mathbf{A} = A_1\,\hat{\boldsymbol{\imath}} + A_2\,\hat{\boldsymbol{\jmath}} + A_3\,\hat{\mathbf{k}}$ for $\mathbf{B}$. As in the last example, we'll simplify the computations by further requiring[10] that $A_3 = 0$.

10 Of course, we could equally well pick $A_1 = 0$ or $A_2 = 0$.

The requirements that $\boldsymbol{\nabla} \times \mathbf{A} = \mathbf{B}$ and $A_3 = 0$ come down to

$$\begin{aligned} -\frac{\partial A_2}{\partial z} &= 2x \\ \frac{\partial A_1}{\partial z} &= 2z - 2x \\ \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} &= 2x - 2z \end{aligned}$$

Because $\frac{\partial}{\partial z}$ treats x and y as constants, the first equation is satisfied if and only if there is a function $N(x, y)$

$$A_2 = -2xz + N(x, y)$$

and second equation is satisfied if and only if there is a function $M(x, y)$

$$A_1 = z^2 - 2xz + M(x, y)$$

Finally, the third equation is also satisfied if and only if $M(x, y)$ and $N(x, y)$ obey

$$\begin{aligned} \frac{\partial}{\partial x}\Big(-2xz + N(x,y)\Big) - \frac{\partial}{\partial y}\Big(z^2 - 2xz + M(x,y))\Big) &= 2x - 2z \\ \iff \quad -2z + \frac{\partial N}{\partial x}(x,y) - \frac{\partial M}{\partial y}(x,y) &= 2x - 2z \\ \iff \quad \frac{\partial N}{\partial x}(x,y) - \frac{\partial M}{\partial y}(x,y) &= 2x \end{aligned}$$

All of the z's in this equation have cancelled out[11], and we can choose, for example, $M(x, y) = 0$ and $N(x, y) = x^2$. So we have found a vector potential. Namely

$$\mathbf{A} = (z^2 - 2xz)\hat{\boldsymbol{\imath}} + (x^2 - 2xz)\hat{\boldsymbol{\jmath}}$$

Again it is a good idea to check that $\boldsymbol{\nabla} \times \mathbf{A} = \mathbf{B}$.

Example 4.1.15

We can use exactly the strategy of the last examples to prove

Theorem 4.1.16.

Let $\mathbf{B}$ be a vector field that is defined and has all of its first order partial derivatives continuous on all of $\mathbb{R}^3$. Then there exists a vector potential for $\mathbf{B}$ if and only if it passes the screening test $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$.

Proof. We already know that the existence of a vector potential implies that $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$. So we just have to assume that $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ and prove that this implies the existence of a vector

11 If the z's had not cancelled out, no $N(x, y)$ and $M(x, y)$, which after all are independent of z, could satisfy the equation. That would have been a sure sign of a user error.

field $\mathbf{A}$ that obeys $\nabla \times \mathbf{A} = \mathbf{B}$. Hence we need to solve

$$\begin{aligned} \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} &= B_1(x,y,z) \\ -\frac{\partial A_3}{\partial x} + \frac{\partial A_1}{\partial z} &= B_2(x,y,z) \\ \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} &= B_3(x,y,z) \end{aligned}$$

We'll explicitly find such an $\mathbf{A}$ using exactly the strategy of Example 4.1.14. In particular, we'll look for an $\mathbf{A}$ that also has $A_3 = 0$. Then the equations simplify to

$$\begin{aligned} -\frac{\partial A_2}{\partial z} &= B_1(x,y,z) \\ \frac{\partial A_1}{\partial z} &= B_2(x,y,z) \\ \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} &= B_3(x,y,z) \end{aligned}$$

The first equation is satisfied if and only if

$$A_2(x,y,z) = -\int_0^z B_1(x,y,\tilde{z})\,\mathrm{d}\tilde{z} + N(x,y)$$

for some function $N(x,y)$. And the second equation is satisfied if and only if

$$A_1(x,y,z) = \int_0^z B_2(x,y,\tilde{z})\,\mathrm{d}\tilde{z} + M(x,y)$$

So all three equations are satisfied if and only only if we can find $M(x,y)$ and $N(x,y)$ that obey

$$\frac{\partial}{\partial x}\Big(\overbrace{-\int_0^z B_1(x,y,\tilde{z})\,\mathrm{d}\tilde{z} + N(x,y)}^{A_2(x,y,z)}\Big) - \frac{\partial}{\partial y}\Big(\overbrace{\int_0^z B_2(x,y,\tilde{z})\,\mathrm{d}\tilde{z} + M(x,y)}^{A_1(x,y,z)}\Big) = B_3(x,y,z)$$

which is the case if and only if

$$\frac{\partial N}{\partial x}(x,y) - \frac{\partial M}{\partial y}(x,y) = B_3(x,y,z) + \int_0^z \Big(\frac{\partial B_1}{\partial x}(x,y,\tilde{z}) + \frac{\partial B_2}{\partial y}(x,y,\tilde{z})\Big)\,\mathrm{d}\tilde{z}$$

Oof! At first sight, it looks like we have a very big problem here. No matter what N and M we pick the left hand side will depend on x and y only — not on z. But it appears like the right hand side depends on z too. Fortunately the screening test (which we have not used to this point in the proof) rides to the rescue and ensures that the right hand actually does does not depend on z. By the screening test,

$$\nabla \cdot \mathbf{B} = \frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z} = 0$$

and we have

$$\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} = -\frac{\partial B_3}{\partial z}$$

so that the right hand side is

$$B_3(x,y,z) + \int_0^z \Big(-\frac{\partial B_3}{\partial z}(x,y,\tilde{z}) \Big)\, \mathrm{d}\tilde{z} = B_3(x,y,z) + \Big[-B_3(x,y,\tilde{z}) \Big]_{\tilde{z}=0}^{\tilde{z}=z} = B_3(x,y,0)$$

by the fundamental theorem of calculus. So we just have to choose M and N to obey

$$\frac{\partial N}{\partial x}(x,y) - \frac{\partial M}{\partial y}(x,y) = B_3(x,y,0)$$

For example, $M = 0$, $N(x,y) = \int_0^x B_3(\tilde{x},y,0)\, \mathrm{d}\tilde{x}$ work. So not only have we proven that a vector potential exists, but we have found a formula for it. □

Warning 4.1.17.

Note that in Theorem 4.1.16 we are assuming that $\mathbf{B}$ passes the screening test on *all* of $\mathbb{R}^3$. If that is not the case, for example because the vector field is not defined on all of $\mathbb{R}^3$, then $\mathbf{B}$ can fail to have a vector potential. An example (the point source) is provided in Example 4.4.8.

4.1.3 ▸▸ Interpretation of the Gradient

In this section we'll develop an interpretation of the gradient $\nabla f(\mathbf{r}_0)$. This should just be a review of material that you have seen before.

Suppose that you are moving through space and that your position at time t is $\mathbf{r}(t) = \big(x(t), y(t), z(t)\big)$. As you move along, you measure, for example, the temperature. If the temperature at position (x,y,z) is $f(x,y,z)$, then the temperature that you measure at time t is $f\big(x(t), y(t), z(t)\big)$. So the rate of change of temperature that you feel is

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} f\big(x(t), y(t), z(t)\big) \\
&\quad = \frac{\partial f}{\partial x}\big(x(t),y(t),z(t)\big)\frac{\mathrm{d}x}{\mathrm{d}t}(t) + \frac{\partial f}{\partial y}\big(x(t),y(t),z(t)\big)\frac{\mathrm{d}y}{\mathrm{d}t}(t) + \frac{\partial f}{\partial z}\big(x(t),y(t),z(t)\big)\frac{\mathrm{d}z}{\mathrm{d}t}(t) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by the chain rule)} \\
&\quad = \nabla f\big(\mathbf{r}(t)\big) \cdot \mathbf{r}'(t) \\
&\quad = \big|\nabla f\big(\mathbf{r}(t)\big)\big|\, \big|\mathbf{r}'(t)\big| \cos\theta
\end{aligned}$$

where θ is the angle between the gradient vector $\nabla f\big(\mathbf{r}(t)\big)$ and the velocity vector $\mathbf{r}'(t)$. This is the rate of change per unit time. We can get the rate of change per unit distance travelled by moving with speed one, so that $\big|\mathbf{r}'(t)\big| = 1$ and then

$$\frac{\mathrm{d}}{\mathrm{d}t} f\big(\mathbf{r}(t)\big) = \big|\nabla f\big(\mathbf{r}(t)\big)\big| \cos\theta$$

If, at a given moment $t = t_0$, you are at $\mathbf{r}(t_0) = \mathbf{r}_0$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} f\big(\mathbf{r}(t)\big)\Big|_{t=t_0} = \big|\nabla f(\mathbf{r}_0)\big| \cos\theta$$

Recall that θ is the angle between our direction of motion and the gradient vector $\nabla f(\mathbf{r}_0)$. So to maximize the rate of change of temperature that we feel, as we pass through $\mathbf{r}_0$, we should choose our direction of motion to be the direction of the gradient vector $\nabla f(\mathbf{r}_0)$. In conclusion

Equation 4.1.18.

$$\nabla f(\mathbf{r}_0) \quad \text{has direction} = \begin{cases}\text{direction of maximum rate of}\\ \text{change of } f \text{ at } \mathbf{r}_0\end{cases}$$

$$\text{has magnitude} = \begin{cases}\text{magnitude of maximum rate of}\\ \text{change (per unit distance) of } f \text{ at } \mathbf{r}_0\end{cases}$$

4.1.4 ▸▸ Interpretation of the Divergence

In this section we'll develop an interpretation of the divergence $\nabla \cdot \mathbf{v}(\mathbf{r}_0)$ of the vector field $\mathbf{v}(\mathbf{r})$ at the point $\mathbf{r}_0$. We shall do so in two steps.

- First we'll express $\nabla \cdot \mathbf{v}(\mathbf{r}_0)$ in terms of flux integrals.
- Then we'll use the interpretation of flux integrals given in Lemma 3.4.1 to get an interpretation of $\nabla \cdot \mathbf{v}(\mathbf{r}_0)$.

Think of $\mathbf{v}(x,y,z)$ as the velocity of a fluid at (x,y,z) and fix any point $\mathbf{r}_0 = (x_0,y_0,z_0)$. Let, for any $\varepsilon > 0$, S_ε be the sphere

- centered at $\mathbf{r}_0$
- of radius ε.
- Denote by $\hat{\mathbf{n}}(x,y,z)$ the outward normal to S_ε at (x,y,z).

We shall prove, in Lemma 4.1.20, below, that we can write $\nabla \cdot \mathbf{v}(\mathbf{r}_0)$ as the limit

$$\nabla \cdot \mathbf{v}(x_0,y_0,z_0) = \lim_{\varepsilon\to 0} \frac{1}{\frac{4}{3}\pi\varepsilon^3} \iint_{S_\varepsilon} \mathbf{v}(x,y,z) \cdot \hat{\mathbf{n}}(x,y,z)\,\mathrm{d}S$$

Once we have that lemma we can use that

- $\frac{4}{3}\pi\varepsilon^3$ is the volume of the interior of the sphere S_ε and
- by Lemma 3.4.1, $\iint_{S_\varepsilon} \mathbf{v}(x,y,z) \cdot \hat{\mathbf{n}}(x,y,z)\,\mathrm{d}S$ is the rate[12] at which fluid is exiting S_ε

to conclude that

12 Lemma 3.4.1 is being applied with the density ρ set equal to one, so, more precisely, the rate is the number of units of volume of fluid exiting S_ε per unit time

Equation 4.1.19.

$$\nabla\cdot\mathbf{v}(\mathbf{r}_0) = \begin{cases}\text{rate at which fluid is exiting an infinitesimal sphere}\\ \text{centred at } \mathbf{r}_0\text{, per unit time, per unit volume}\end{cases}$$
$$= \text{strength of the source at } \mathbf{r}_0$$

Here is the critical computation.

Lemma 4.1.20.

$$\nabla\cdot\mathbf{v}(x_0,y_0,z_0) = \lim_{\varepsilon\to 0}\frac{1}{\frac{4}{3}\pi\varepsilon^3}\iint_{S_\varepsilon}\mathbf{v}(x,y,z)\cdot\hat{\mathbf{n}}(x,y,z)\,\mathrm{d}S$$

Proof. (Optional). [13]

By translating our coordinate system, it suffices to consider $\mathbf{r}_0 = (x_0, y_0, z_0) = (0,0,0)$. Then

$$S_\varepsilon = \{\,(x,y,z)\;\big|\;|(x,y,z)| = \varepsilon\,\} \qquad \hat{\mathbf{n}}(x,y,z) = \frac{1}{\varepsilon}(x,y,z)$$

We expand $\mathbf{v}(x,y,z)$ in a Taylor expansion in powers of x, y, and z, to first order, with second order error term.

$$\mathbf{v}(x,y,z) = \mathbf{A} + \mathbf{B}\,x + \mathbf{C}\,y + \mathbf{D}\,z + \mathbf{R}(x,y,z)$$

where

$$\mathbf{A} = \mathbf{v}(0,0,0) \qquad \mathbf{B} = \frac{\partial\mathbf{v}}{\partial x}(0,0,0) \qquad \mathbf{C} = \frac{\partial\mathbf{v}}{\partial y}(0,0,0) \qquad \mathbf{D} = \frac{\partial\mathbf{v}}{\partial z}(0,0,0)$$

and the error term $\mathbf{R}(x,y,z)$ is bounded by a constant times[14] $x^2+y^2+z^2$. In particular there is a constant K so that, on S_ε,

$$|\mathbf{R}(x,y,z)| \leqslant K\varepsilon^2$$

So

$$\iint_{S_\varepsilon}\mathbf{v}(x,y,z)\cdot\hat{\mathbf{n}}(x,y,z)\,\mathrm{d}S = \frac{1}{\varepsilon}\iint_{S_\varepsilon}\big(\mathbf{A}+\mathbf{B}\,x+\mathbf{C}\,y+\mathbf{D}\,z+\mathbf{R}(x,y,z)\big)\cdot(x,y,z)\,\mathrm{d}S$$

13 There is another, easier to understand, proof of this result given in §4.4.1. We cannot give that proof here because it uses the divergence theorem, which we will get to later in the chapter.

14 Terms like xy, xz and yz are not needed because, for example, $|xy| \leqslant \frac{1}{2}(x^2+y^2)$. This inequality is equivalent to $\big(|x|-|y|\big)^2 \geqslant 0$.

Multiply out the dot product so that the integrand becomes

$$\begin{aligned}
&\mathbf{A}\cdot\hat{\boldsymbol{\imath}}\,x &&+\ \mathbf{A}\cdot\hat{\boldsymbol{\jmath}}\,y &&+\mathbf{A}\cdot\hat{\mathbf{k}}\,z\\
+&\mathbf{B}\cdot\hat{\boldsymbol{\imath}}\,x^2 &&+\ \mathbf{B}\cdot\hat{\boldsymbol{\jmath}}\,xy &&+\mathbf{B}\cdot\hat{\mathbf{k}}\,xz\\
+&\mathbf{C}\cdot\hat{\boldsymbol{\imath}}\,xy &&+\ \mathbf{C}\cdot\hat{\boldsymbol{\jmath}}\,y^2 &&+\mathbf{C}\cdot\hat{\mathbf{k}}\,yz\\
+&\mathbf{D}\cdot\hat{\boldsymbol{\imath}}\,xz &&+\ \mathbf{D}\cdot\hat{\boldsymbol{\jmath}}\,yz &&+\mathbf{D}\cdot\hat{\mathbf{k}}\,z^2\\
+&\mathbf{R}(x,y,z)\cdot(x,y,z)
\end{aligned}$$

That's a lot of terms. But most of them integrate to zero, simply because the integral of an odd function over an even domain is zero. Because S_ε is invariant under $x\to -x$ and under $y\to -y$ and under $z\to -z$ we have

$$\iint_{S_\varepsilon} x\,\mathrm{d}S=\iint_{S_\varepsilon} y\,\mathrm{d}S=\iint_{S_\varepsilon} z\,\mathrm{d}S=\iint_{S_\varepsilon} xy\,\mathrm{d}S=\iint_{S_\varepsilon} xz\,\mathrm{d}S=\iint_{S_\varepsilon} yz\,\mathrm{d}S=0$$

which is a relief. We are now left with

$$\iint_{S_\varepsilon}\mathbf{v}(x,y,z)\cdot\hat{\mathbf{n}}(x,y,z)\,\mathrm{d}S=\frac{1}{\varepsilon}\iint_{S_\varepsilon}\big(\mathbf{B}\cdot\hat{\boldsymbol{\imath}}\,x^2+\mathbf{C}\cdot\hat{\boldsymbol{\jmath}}\,y^2+\mathbf{D}\cdot\hat{\mathbf{k}}\,z^2\big)\,\mathrm{d}S$$
$$+\frac{1}{\varepsilon}\iint_{S_\varepsilon}\mathbf{R}(x,y,z)\cdot(x,y,z)\,\mathrm{d}S$$

As well S_ε is invariant[15] under the interchange of x and y and also under the interchange of x and z. Consequently

$$\begin{aligned}
\iint_{S_\varepsilon} x^2\,\mathrm{d}S=\iint_{S_\varepsilon} y^2\,\mathrm{d}S=\iint_{S_\varepsilon} z^2\,\mathrm{d}S&=\frac{1}{3}\iint_{S_\varepsilon}\big[x^2+y^2+z^2\big]\,\mathrm{d}S\\
&=\frac{1}{3}\iint_{S_\varepsilon}\varepsilon^2\,\mathrm{d}S\qquad\text{since } x^2+y^2+z^2=\varepsilon^2\text{ on }S_\varepsilon\\
&=\frac{4}{3}\pi\varepsilon^4
\end{aligned}$$

since the surface area of the sphere S_ε is $4\pi\varepsilon^2$. So far, we have

$$\begin{aligned}
\iint_{S_\varepsilon}\mathbf{v}(x,y,z)\cdot\hat{\mathbf{n}}(x,y,z)\,\mathrm{d}S&=\frac{4}{3}\pi\varepsilon^3\big(\mathbf{B}\cdot\hat{\boldsymbol{\imath}}+\mathbf{C}\cdot\hat{\boldsymbol{\jmath}}+\mathbf{D}\cdot\hat{\mathbf{k}}\big)+\frac{1}{\varepsilon}\iint_{S_\varepsilon}\mathbf{R}(x,y,z)\cdot(x,y,z)\,\mathrm{d}S\\
&=\frac{4}{3}\pi\varepsilon^3\boldsymbol{\nabla}\cdot\mathbf{v}(\mathbf{0})+\frac{1}{\varepsilon}\iint_{S_\varepsilon}\mathbf{R}(x,y,z)\cdot(x,y,z)\,\mathrm{d}S\quad\text{(review the definitions of }\mathbf{B},\mathbf{C},\mathbf{D})
\end{aligned}$$

which implies

$$\lim_{\varepsilon\to0}\frac{1}{\frac{4}{3}\pi\varepsilon^3}\iint_{S_\varepsilon}\mathbf{v}(x,y,z)\cdot\hat{\mathbf{n}}(x,y,z)\,\mathrm{d}S=\boldsymbol{\nabla}\cdot\mathbf{v}(\mathbf{0})+\lim_{\varepsilon\to0}\frac{3}{4\pi\varepsilon^4}\iint_{S_\varepsilon}\mathbf{R}(x,y,z)\cdot(x,y,z)\,\mathrm{d}S$$

15 Spheres have lots of symmetry!

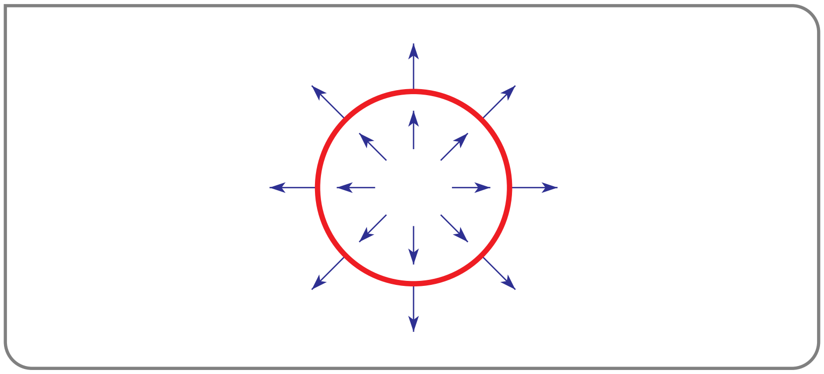
Finally, it suffices to recall that $|\mathbf{R}(x,y,z)| \leqslant K\varepsilon^2$ and, on S_ε, $|(x,y,z)| = \varepsilon$, so that

$$\begin{aligned}\frac{3}{4\pi\varepsilon^4}\left|\iint\limits_{S_\varepsilon} \mathbf{R}(x,y,z)\cdot(x,y,z)\,\mathrm{d}S\right| &\leqslant \frac{3}{4\pi\varepsilon^4}\iint\limits_{S_\varepsilon} |\mathbf{R}(x,y,z)|\,|(x,y,z)|\,\mathrm{d}S \\ &\leqslant \frac{3}{4\pi\varepsilon^4}\iint\limits_{S_\varepsilon} K\varepsilon^3\,\mathrm{d}S = \frac{3}{4\pi\varepsilon^4}\,K\varepsilon^3\left(4\pi\varepsilon^2\right) \\ &= 3K\varepsilon\end{aligned}$$

converges to zero as $\varepsilon \to 0$. So we are left with the desired result. □

Example 4.1.21

Here is a sketch of the vector field $\mathbf{v}(x,y,z) = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{\mathbf{k}}$ and a sphere centered on the origin, like S_ε.



This velocity field has fluid being created and pushed out through the sphere. We have

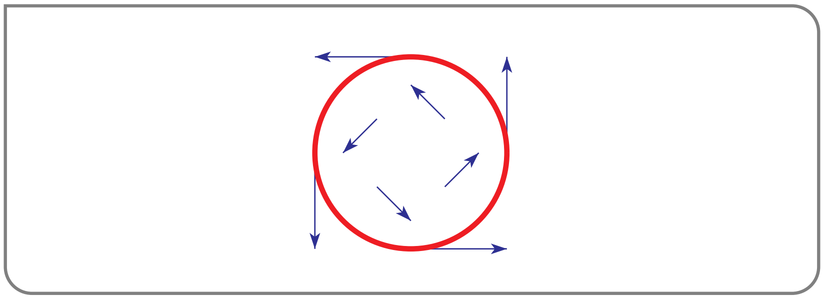
$$\nabla\cdot\mathbf{v}(\mathbf{0}) = 3$$

consistent with our interpretation (4.1.19).

Example 4.1.21

Example 4.1.22

Here is a sketch of the vector field $\mathbf{v}(x,y,z) = -y\,\hat{\imath} + x\,\hat{\jmath}$ and a sphere centered on the origin, like S_ε.



This velocity field just has fluid going around in circles. No fluid actually crosses the sphere. The divergence

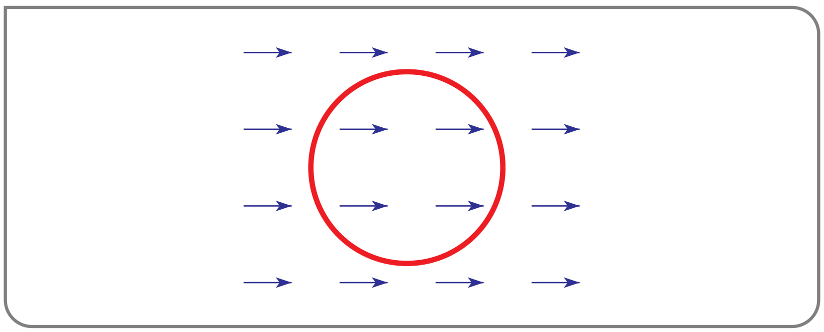
$$\nabla \cdot \mathbf{v}(\mathbf{0}) = 0$$

consistent with our interpretation (4.1.19).

Example 4.1.22

Example 4.1.23

Here is a sketch of the vector field $\mathbf{v}(x,y,z) = \hat{\imath}$ and a sphere centered on the origin, like S_ε.



This velocity field just has fluid moving uniformly to the right. Fluid enters the sphere from the left and leaves through the right at precisely the same rate, so that the net rate at fluid crosses the sphere is zero. The divergence

$$\nabla \cdot \mathbf{v}(\mathbf{0}) = 0$$

again consistent with our interpretation (4.1.19).

Example 4.1.23

4.1.5 ▸▸ Interpretation of the Curl

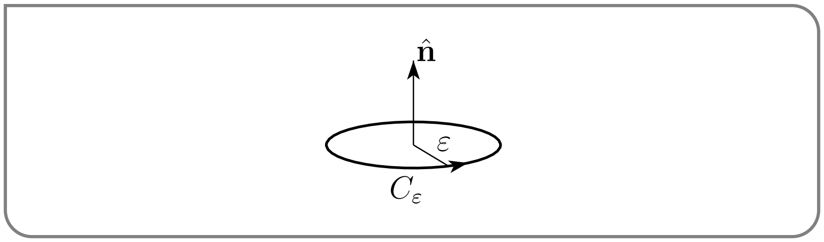
We'll now develop the interpretation of the curl, or more precisely, of $\nabla \times \mathbf{v}(\mathbf{r}_0) \cdot \hat{\mathbf{n}}$ for any unit vector $\hat{\mathbf{n}}$. As we did in developing the interpretation of divergence, we'll

- first express $\nabla \times \mathbf{v}(\mathbf{r}_0) \cdot \hat{\mathbf{n}}$ as a limit of integrals, and
- then we'll interpret the integrals.

To specify the integrals involved, let C_ε be the circle which

- is centered at $\mathbf{r}_0$
- has radius ε
- lies in the plane through $\mathbf{r}_0$ perpendicular to $\hat{\mathbf{n}}$

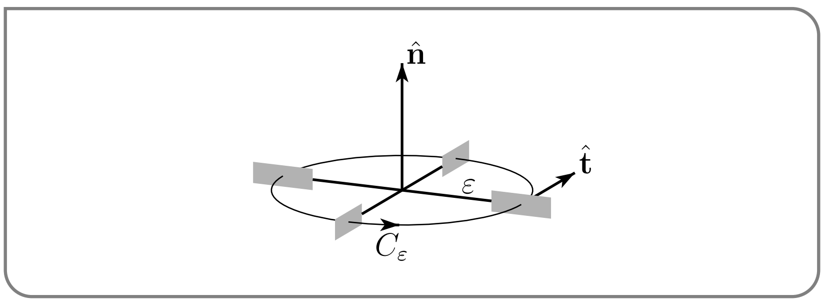
- is oriented in the standard way with respect to $\hat{\mathbf{n}}$. Imagine standing on the circle with your feet on the plane through $\mathbf{r}_0$ perpendicular to $\hat{\mathbf{n}}$, with the vector from your feet to your head in the same direction as $\hat{\mathbf{n}}$ and with your left arm pointing towards $\mathbf{r}_0$. Then you are facing in the positive direction for C_ε.



We shall show in Lemma 4.1.25, below, that

$$\nabla \times v(\mathbf{r}_0) \cdot \hat{\mathbf{n}} = \lim_{\varepsilon \to 0} \frac{1}{\pi \varepsilon^2} \oint_{C_\varepsilon} \mathbf{v}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r}$$

Now let's work on interpreting the right hand side, and in particular on interpreting the integral $\oint_{C_\varepsilon} \mathbf{v}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r}$, which is called the circulation of $\mathbf{v}$ around C_ε. Place a tiny paddle-wheel in the fluid with its axle running along $\hat{\mathbf{n}}$ and its paddles along C_ε, as in the figure below, except that the paddlewheel is really expensive and has a lot more than just four



paddles. Pretend[16] that you are one of the paddles.

- If the paddlewheel is rotating at Ω radians per unit time, then in one unit of time you sweep out an arc of a circle of radius ε that subtends an angle Ω. That arc has length $\Omega\varepsilon$. So you are moving at speed $\Omega\varepsilon$.
- If you are at $\mathbf{r}$, the component of the fluid velocity in your direction of motion, i.e. tangential to C_ε, is $\mathbf{v}(\mathbf{r}) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}$, because $\hat{\mathbf{t}} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}$, with s denoting arc length along the circle, is a unit vector tangential to C_ε.
- All paddles have to move at the same speed. So the speed of the paddles, $\Omega\varepsilon$, should be the average value of $\mathbf{v}(\mathbf{r}) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}$ around the circle.

Thus the rate of rotation, Ω, of the paddlewheel should be determined by

$$\Omega\varepsilon = \frac{\oint_{C_\varepsilon} \mathbf{v}(\mathbf{r}) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} \mathrm{d}s}{\oint_{C_\varepsilon} \mathrm{d}s} = \frac{\oint_{C_\varepsilon} \mathbf{v}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r}}{2\pi\varepsilon}$$

16 Method acting might help you here.

Consequently, $\nabla \times v(\mathbf{r}_0) \cdot \hat{\mathbf{n}}$ is the limit as ε (the radius of the paddlewheel) tends to zero of

$$\frac{1}{\pi\varepsilon^2} \oint_{C_\varepsilon} \mathbf{v}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = 2\Omega$$

That's our interpretation.

Equation 4.1.24.

If a fluid has velocity field $\mathbf{v}$ and you place an infinitesimal paddlewheel at $\mathbf{r}_0$ with its axle in direction $\mathbf{n}$, then it rotates at $\frac{1}{2}\nabla \times \mathbf{v}(\mathbf{r}_0) \cdot \hat{\mathbf{n}}$ radians per unit time. In particular, to maximize the rate of rotation, orient the paddlewheel so that $\hat{\mathbf{n}} \parallel \nabla \times \mathbf{v}(\mathbf{r}_0)$.

There will be some examples at the end of this section. First, we show

Lemma 4.1.25.

$$\nabla \times \mathbf{v}(\mathbf{r}_0) \cdot \hat{\mathbf{n}} = \lim_{\varepsilon \to 0} \frac{1}{\pi\varepsilon^2} \oint_{C_\varepsilon} \mathbf{v}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r}$$

Proof. (Optional). [17]

Just as we did in the proof of Lemma 4.1.20, we can always translate our coordinate system so that $\mathbf{r}_0 = (x_0, y_0, z_0) = (0, 0, 0)$. We can also rotate our coordinate system so that $\hat{\mathbf{n}} = \hat{\mathbf{k}}$. Because $\mathbf{r}_0 = (0, 0, 0)$ and $\hat{\mathbf{n}} = \hat{\mathbf{k}}$, so that C_ε lies in the xy-plane, we can parametrize C_ε by

$$\mathbf{r}(t) = \varepsilon \cos t\, \hat{\boldsymbol{\imath}} + \varepsilon \sin t\, \hat{\boldsymbol{\jmath}}$$

Again as we did in the proof of Lemma 4.1.20, expand $\mathbf{v}(x, y, z)$ in a Taylor expansion in powers of x, y, and z, to first order, with second order error term.

$$\mathbf{v}(x, y, z) = \mathbf{A} + \mathbf{B}\, x + \mathbf{C}\, y + \mathbf{D}\, z + \mathbf{R}(x, y, z)$$

where

$$\mathbf{A} = \mathbf{v}(0, 0, 0) \qquad \mathbf{B} = \frac{\partial \mathbf{v}}{\partial x}(0, 0, 0) \qquad \mathbf{C} = \frac{\partial \mathbf{v}}{\partial y}(0, 0, 0) \qquad \mathbf{D} = \frac{\partial \mathbf{v}}{\partial z}(0, 0, 0)$$

and the error term $\mathbf{R}(x, y, z)$ is bounded by a constant times $x^2 + y^2 + z^2$. In particular there is a constant K so that, on C_ε,

$$|\mathbf{R}(x, y, z)| \leqslant K\varepsilon^2$$

17 There is another, easier to understand, proof of this result given in §4.4.1. We cannot give that proof here because it uses Stokes' theorem, which we will get to later in the chapter.

So

$$\oint_{C_\varepsilon} \mathbf{v}(\mathbf{r})\cdot \mathrm{d}\mathbf{r} = \int_0^{2\pi} \big(\mathbf{A} + \mathbf{B}\,\varepsilon\,\cos t + \mathbf{C}\,\varepsilon\,\sin t + \mathbf{R}(\mathbf{r}(t))\big)\cdot\big(-\varepsilon\sin t\,\hat{\boldsymbol{\imath}} + \varepsilon\cos t\,\hat{\boldsymbol{\jmath}}\big)\,\mathrm{d}t$$

Again, multiply out the dot product so that the integrand becomes

$$\begin{aligned}
&-\varepsilon\mathbf{A}\cdot\hat{\boldsymbol{\imath}}\,\sin t &&+\ \varepsilon\mathbf{A}\cdot\hat{\boldsymbol{\jmath}}\,\cos t\\
&-\varepsilon^2\mathbf{B}\cdot\hat{\boldsymbol{\imath}}\,\sin t\cos t &&+\ \varepsilon^2\mathbf{B}\cdot\hat{\boldsymbol{\jmath}}\,\cos^2 t\\
&-\varepsilon^2\mathbf{C}\cdot\hat{\boldsymbol{\imath}}\,\sin^2 t &&+\ \varepsilon^2\mathbf{C}\cdot\hat{\boldsymbol{\jmath}}\,\sin t\cos t\\
&+\mathbf{R}(\mathbf{r}(t))\cdot\big(-\varepsilon\sin t\,\hat{\boldsymbol{\imath}} + \varepsilon\cos t\,\hat{\boldsymbol{\jmath}}\big)
\end{aligned}$$

Again most of these terms integrate to zero, because

$$\begin{aligned}
\int_0^{2\pi}\sin t\,\mathrm{d}t &= \int_0^{2\pi}\cos t\,\mathrm{d}t = 0\\
\int_0^{2\pi}\sin t\cos t\,\mathrm{d}t &= \frac{1}{2}\int_0^{2\pi}\sin(2t)\,\mathrm{d}t = 0
\end{aligned}$$

and the $\sin^2 t$ and $\cos^2 t$ terms are easily integrated using (see Example 2.4.4)

$$\int_0^{2\pi}\sin^2 t\,\mathrm{d}t = \int_0^{2\pi}\cos^2 t\,\mathrm{d}t = \frac{1}{2}\int_0^{2\pi}\big[\sin^2 t + \cos^2 t\big]\,\mathrm{d}t = \pi$$

So we are left with

$$\oint_{C_\varepsilon}\mathbf{v}(\mathbf{r})\cdot\mathrm{d}\mathbf{r} = \pi\varepsilon^2\mathbf{B}\cdot\hat{\boldsymbol{\jmath}} - \pi\varepsilon^2\mathbf{C}\cdot\hat{\boldsymbol{\imath}} + \int_0^{2\pi}\mathbf{R}(\mathbf{r}(t))\cdot\big(-\varepsilon\sin t\,\hat{\boldsymbol{\imath}} + \varepsilon\cos t\,\hat{\boldsymbol{\jmath}}\big)\,\mathrm{d}t$$

which implies that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{1}{\pi\varepsilon^2}\oint_{C_\varepsilon}\mathbf{v}(\mathbf{r})\cdot\mathrm{d}\mathbf{r}\\
&\quad= \frac{\partial \mathbf{v}_2}{\partial x}(0,0,0) - \frac{\partial \mathbf{v}_1}{\partial y}(0,0,0) + \lim_{\varepsilon\to 0}\frac{1}{\pi\varepsilon^2}\int_0^{2\pi}\mathbf{R}(\mathbf{r}(t))\cdot\big(-\varepsilon\sin t\,\hat{\boldsymbol{\imath}} + \varepsilon\cos t\,\hat{\boldsymbol{\jmath}}\big)\,\mathrm{d}t\\
&\quad= \big(\boldsymbol{\nabla}\times\mathbf{v}(0,0,0)\big)\cdot\hat{\mathbf{k}} + \lim_{\varepsilon\to 0}\frac{1}{\pi\varepsilon^2}\int_0^{2\pi}\mathbf{R}(\mathbf{r}(t))\cdot\big(-\varepsilon\sin t\,\hat{\boldsymbol{\imath}} + \varepsilon\cos t\,\hat{\boldsymbol{\jmath}}\big)\,\mathrm{d}t
\end{aligned}$$

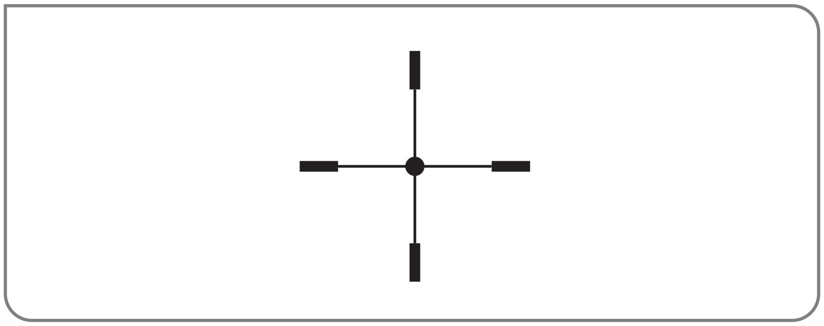
Finally, it suffices to recall that $|\mathbf{R}(x,y,z)| \leqslant K\varepsilon^2$, so that

$$\begin{aligned}
\frac{1}{\pi\varepsilon^2}\bigg|\int_0^{2\pi}\mathbf{R}(\mathbf{r}(t))\cdot\big(-\varepsilon\sin t\,\hat{\boldsymbol{\imath}} + \varepsilon\cos t\,\hat{\boldsymbol{\jmath}}\big)\,\mathrm{d}t\bigg| &\leqslant \frac{1}{\pi\varepsilon}\int_0^{2\pi}|\mathbf{R}(\mathbf{r}(t))|\,\mathrm{d}t\\
&\leqslant \frac{1}{\pi\varepsilon}\int_0^{2\pi}K\varepsilon^2\,\mathrm{d}t = \frac{1}{\pi\varepsilon}K\varepsilon^2\,(2\pi)\\
&= 2K\varepsilon
\end{aligned}$$

converges to zero as $\varepsilon\to 0$.

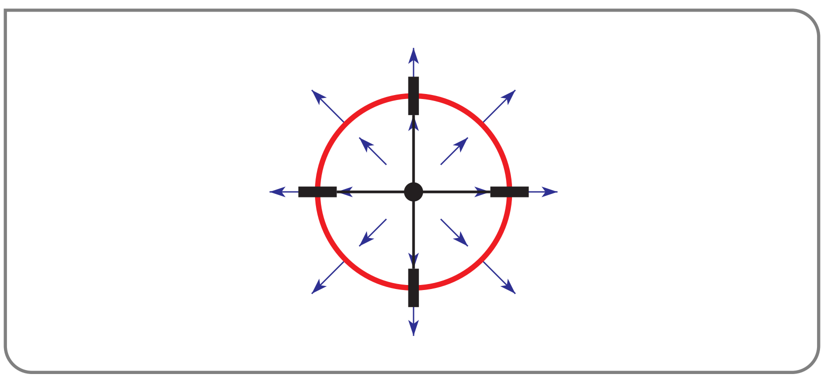
□

Here are some examples. We will use the same vector fields as in Examples 4.1.21, 4.1.22 and 4.1.23. In all examples, we shall orient the paddlewheel so that $\hat{\mathbf{n}} = \hat{\mathbf{k}}$ and sketch the top view, so that the paddlewheel looks like



Example 4.1.26

Here is a sketch of the vector field $\mathbf{v}(x,y,z) = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{\mathbf{k}}$ and a circle centered on the origin, like C_ε.



This velocity field has fluid moving parallel to the paddles, so the paddlewheel should not rotate at all. The computation

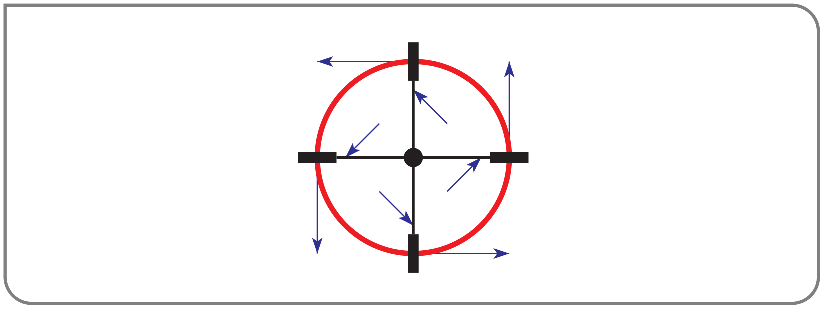
$$\nabla \times \mathbf{v}(\mathbf{0}) = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{bmatrix} = \mathbf{0} \implies \nabla \times \mathbf{v}(\mathbf{0}) \cdot \hat{\mathbf{k}} = 0$$

is consistent with our interpretation (4.1.24).

Example 4.1.26

Example 4.1.27

Here is a sketch of the vector field $\mathbf{v}(x,y,z) = -y\,\hat{\imath} + x\,\hat{\jmath}$ and a circle centered on the origin, like C_ε.



This velocity field has fluid going around in circles, counterclockwise. So the paddlewheel should rotate counterclockwise too. That is, it should have positive angular velocity. Our interpretation (4.1.24) predicts an angular velocity of half

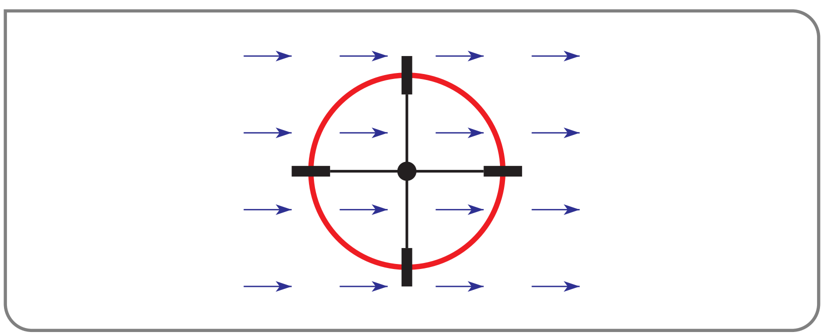
$$\nabla \times \mathbf{v}(\mathbf{0}) \cdot \hat{\mathbf{k}} = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{bmatrix} \cdot \hat{\mathbf{k}} = 2\hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 2$$

which is indeed positive[18].

Example 4.1.27

Example 4.1.28

Here is a sketch of the vector field $\mathbf{v}(x, y, z) = \hat{\imath}$ and a circle centered on the origin, like C_ε.



The fluid pushing on the top paddle tries to make the paddlewheel rotate clockwise. The fluid pushing on the bottom paddle tries to make the paddlewheel rotate counterclockwise, at the same rate. So the paddlewheel should not rotate at all. Our interpretation (4.1.24) predicts an angular velocity of

$$\frac{1}{2} \cdot \nabla \times \mathbf{v}(\mathbf{0}) \cdot \hat{\mathbf{k}} = \frac{1}{2} \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & 0 & 0 \end{bmatrix} \cdot \hat{\mathbf{k}} = \mathbf{0} \cdot \hat{\mathbf{k}} = 0$$

as expected.

Example 4.1.28

18 Even for small values of 2.

4.2▲ The Divergence Theorem

The rest of this chapter concerns three theorems: the divergence theorem, Green's theorem and Stokes' theorem. Superficially, they look quite different from each other. But, in fact, they are all *very* closely related and all three are generalizations of the fundamental theorem of calculus

$$\int_a^b \frac{\mathrm{d}f}{\mathrm{d}t}(t)\ \mathrm{d}t = f(b) - f(a)$$

The left hand side of the fundamental theorem of calculus is the integral of the derivative of a function. The right hand side involves only values of the function on the boundary of the domain of integration. The divergence theorem, Green's theorem and Stokes' theorem also have this form, but the integrals are in more than one dimension. So the derivatives are multidimensional, like the curl and divergence, and the integrands can involve vector fields.

- For the divergence theorem, the integral on the left hand side is over a (three dimensional) volume and the right hand side is an integral over the boundary of the volume, which is a surface.
- For Green's and Stokes' theorems, the integral on the left hand side is over a (two dimensional) surface and the right hand side is an integral over the boundary of the surface, which is a curve.

The divergence theorem is going to relate a volume integral over a solid V to a flux integral over the surface of V. First we need a couple of definitions concerning the allowed surfaces. In many applications solids, for example cubes, have corners and edges where the normal vector is not defined. On the other hand, to be able to compute a flux integral over a surface, we certainly need that the set of points where the normal vector is not well-defined is small enough that the existence of the flux integral is not jeopardized. This is the case for "piecewise smooth" surfaces, which we now define.

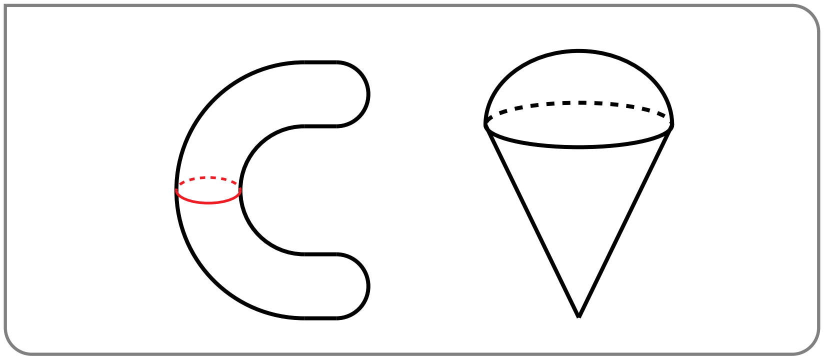
Definition 4.2.1.

(a) A surface is *smooth* if it has a parametrization $\mathbf{r}(u, v)$ with continuous partial derivatives $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$ and with $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$ nonzero.

(b) A surface is *piecewise smooth* if it consists of a finite number of smooth pieces that meet along sharp curves and at sharp corners.

Here are sketches of a smooth surface (a sausage) and a piecewise smooth surface (an ice-cream cone), followed by the divergence theorem[19].

19 It is also known as Gauss's theorem. Johann Carl Friedrich Gauss (1777–1855) was a German mathematician. Throughout the 1990's Gauss's portrait appeared on the German ten-mark banknote. In addition to Gauss's theorem, the Gaussian distribution (the bell curve), degaussing and the CGS unit for the magnetic field, and the crater Gauss on the Moon are named in his honour.



Theorem 4.2.2 (Divergence Theorem)**.**

Let

- V be a bounded solid with a piecewise smooth surface[20] ∂V
- $\mathbf{F}$ be a vector field that has continuous first partial derivatives at every point of V.

Then

$$\iint_{\partial V} \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}S = \iiint_V \boldsymbol{\nabla} \cdot \mathbf{F} \, \mathrm{d}V$$

where $\hat{\mathbf{n}}$ is the outward unit normal of ∂V.

Like the fundamental theorem of calculus, the divergence theorem expresses the integral of a derivative of a function (in this case a vector-valued function) over a region in terms of the values of the function on the boundary of the region.

Warning 4.2.3.

Note that in Theorem 4.2.2 we are assuming that the vector field $\mathbf{F}$ has continuous first partial derivatives at *every* point of V. If that is not the case, for example because $\mathbf{F}$ is not defined on all of V, then the conclusion of the divergence theorem can fail. An example is $\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^3}$, $V = \{ (x,y,z) \mid x^2 + y^2 + z^2 \leqslant 1 \}$. See Example 4.2.7.

Proof. We have to show that

$$\iint_{\partial V} \left(\mathbf{F}_1 \,\hat{\imath} + \mathbf{F}_2 \,\hat{\jmath} + \mathbf{F}_3 \,\hat{\mathbf{k}} \right) \cdot \hat{\mathbf{n}} \, \mathrm{d}S = \iiint_V \left(\frac{\partial \mathbf{F}_1}{\partial x} + \frac{\partial \mathbf{F}_2}{\partial y} + \frac{\partial \mathbf{F}_3}{\partial z} \right) \mathrm{d}V$$

Note that the left hand side is a sum of three terms — one involving $\mathbf{F}_1$, one involving $\mathbf{F}_2$ and one involving $\mathbf{F}_3$ — and the right hand side is a sum of three terms — one involving

20 We are going to consistently use the notation ∂(thing) to denote the boundary of (thing).

$\mathbf{F}_1$, one involving $\mathbf{F}_2$ and one involving $\mathbf{F}_3$. We'll just show that the $\mathbf{F}_3$ terms on the left hand side and right hand side are equal, i.e. that

$$\iint_{\partial V} \mathbf{F}_3\,\hat{\mathbf{k}}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iiint_V \frac{\partial \mathbf{F}_3}{\partial z}\,\mathrm{d}V$$

Showing that the $\mathbf{F}_1$ terms match and the $\mathbf{F}_2$ terms match is done in the same way[21].

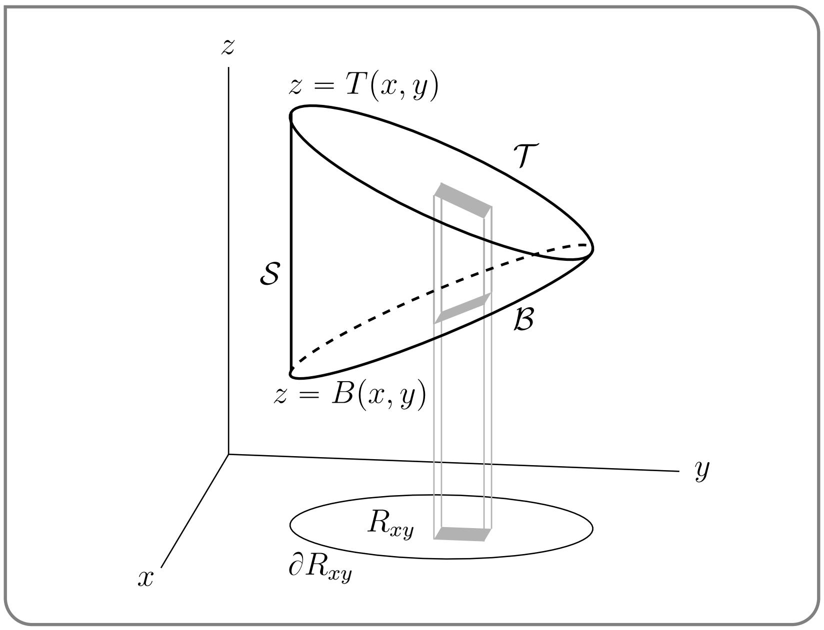
Special Geometry

We'll first assume that the solid has the special form

$$V = \big\{\ (x,y,z)\ \big|\ B(x,y) \leqslant z \leqslant T(x,y),\ \ (x,y)\in R_{xy}\ \big\}$$

where $R_{x,y}$ is some subset of the xy-plane. We can further assume that, for each $(x,y)\in R_{xy}$, we have $B(x,y)\leqslant T(x,y)$. After we're finished with this special case, we'll handle the general case.

Let's work on $\iint_{\partial V} \mathbf{F}_3\,\hat{\mathbf{k}}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$ first. As in the figure below, the surface ∂V consists of



three pieces — the top, the bottom and the side. We'll consider each in turn.

- The top is $\mathcal{T} = \big\{\ (x,y,z)\ \big|\ z = T(x,y),\ \ (x,y)\in R_{xy}\ \big\}$. By (3.3.2), on $\mathcal{T}$

  $$\hat{\mathbf{n}}\,\mathrm{d}S = +\big[-T_x(x,y)\,\hat{\boldsymbol{\imath}} - T_y(x,y)\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}\big]\,\mathrm{d}x\mathrm{d}y$$

  As $\hat{\mathbf{n}}$ is to be the outward normal, it must point upwards on $\mathcal{T}$. That's why we have chosen, and emphasised, the "+" sign. So $\hat{\mathbf{k}}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \mathrm{d}x\mathrm{d}y$ and

  $$\iint_{\mathcal{T}} \mathbf{F}_3\,\hat{\mathbf{k}}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iint_{R_{xy}} \mathbf{F}_3(x,y,T(x,y))\,\mathrm{d}x\mathrm{d}y$$

21 Mutatis mutandis.

◦ The bottom is $\mathcal{B} = \{\ (x,y,z)\ \big|\ z = B(x,y),\ \ (x,y) \in R_{xy}\ \}$. By (3.3.2), on $\mathcal{B}$

$$\hat{\mathbf{n}}\,\mathrm{d}S = -\big[-B_x(x,y)\,\hat{\boldsymbol{\imath}} - B_y(x,y)\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}\big]\,\mathrm{d}x\mathrm{d}y$$

As $\hat{\mathbf{n}}$ is to be the outward normal, it must point downwards on $\mathcal{B}$. That's why we have chosen the "−" sign. So $\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}\,\mathrm{d}S = -\mathrm{d}x\mathrm{d}y$ and

$$\iint_{\mathcal{B}} \mathbf{F}_3\,\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}\,\mathrm{d}S = -\iint_{R_{xy}} \mathbf{F}_3(x,y,B(x,y))\,\mathrm{d}x\mathrm{d}y$$

◦ The side is $\mathcal{S} = \{\ (x,y,z)\ \big|\ (x,y) \in \partial R_{xy},\ B(x,y) \leqslant z \leqslant T(x,y)\ \}$. It runs vertically. Hence on $\mathcal{S}$ the normal vector to ∂V is parallel to the xy-plane so that $\hat{\mathbf{k}} \cdot \hat{\mathbf{n}} = 0$ and

$$\iint_{\mathcal{S}} \mathbf{F}_3\,\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}\,\mathrm{d}S = 0$$

So all together

$$\begin{aligned}\iint_{\partial V} \mathbf{F}_3\,\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}\,\mathrm{d}S &= \iint_{\mathcal{T}} \mathbf{F}_3\,\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}\,\mathrm{d}S + \iint_{\mathcal{B}} \mathbf{F}_3\,\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}\,\mathrm{d}S + \iint_{\mathcal{S}} \mathbf{F}_3\,\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}\,\mathrm{d}S \\ &= \iint_{R_{xy}} \big[\mathbf{F}_3(x,y,T(x,y)) - \mathbf{F}_3(x,y,B(x,y))\big]\,\mathrm{d}x\mathrm{d}y + 0 \qquad (\partial V)\end{aligned}$$

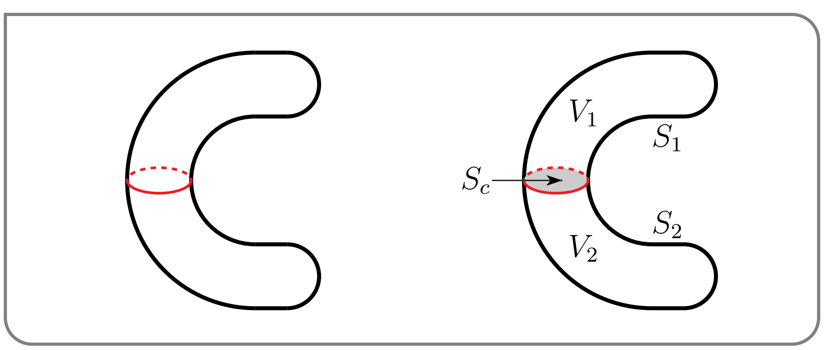
Now let us examine

$$\begin{aligned}\iiint_{V} \frac{\partial \mathbf{F}_3}{\partial z}\,\mathrm{d}V &= \iint_{R_{xy}} \mathrm{d}x\mathrm{d}y \int_{B(x,y)}^{T(x,y)} \mathrm{d}z\,\frac{\partial \mathbf{F}_3}{\partial z}(x,y,z) \\ &= \iint_{R_{xy}} \big[\mathbf{F}_3(x,y,T(x,y)) - \mathbf{F}_3(x,y,B(x,y))\big]\,\mathrm{d}x\mathrm{d}y \qquad (V)\end{aligned}$$

by the fundamental theorem of calculus. That's exactly what we had to show. The integrals (∂V) and (V) are equal.

General Geometry

Now we'll drop the assumption on V that we imposed in the "Special Geometry" section above. The key idea that makes the proof work is that we can cut up any[22] V into pieces, each of which does obey the special assumption that we just considered. Consider, for example, the sausage shaped solid in the figure on the left below.



22 We are assuming that V is "reasonable".

Call the sausage V. Cut it into two halves by running a cleaver horizontally through its centre. This splits the solid V into two halves, V_1 and V_2 as in the figure on the right above. It also splits the boundary ∂V of V into two halves S_1 and S_2, also as in the figure on the right above. Note that

- the boundary, ∂V_1, of V_1 is the union of S_1 and the shaded disk S_c (the cut introduced by the cleaver). On the cut S_c, the outward pointing normal to V_1 is $-\hat{\mathbf{k}}$.
- The boundary, ∂V_2, of V_2 is the union of S_2 and the shaded disk S_c. On the cut S_c, the outward pointing normal to V_2 is $+\hat{\mathbf{k}}$.

Now both V_1 and V_2 do satisfy the assumption of the "Special Geometry" section above. So

$$\begin{aligned}\iiint_V \frac{\partial \mathbf{F}_3}{\partial z}\,\mathrm{d}V &= \iiint_{V_1} \frac{\partial \mathbf{F}_3}{\partial z}\,\mathrm{d}V + \iiint_{V_2} \frac{\partial \mathbf{F}_3}{\partial z}\,\mathrm{d}V \\ &= \iint_{\partial V_1} \mathbf{F}_3\,\hat{\mathbf{k}}\cdot\hat{\mathbf{n}}\,\mathrm{d}S + \iint_{\partial V_2} \mathbf{F}_3\,\hat{\mathbf{k}}\cdot\hat{\mathbf{n}}\,\mathrm{d}S \\ &= \iint_{S_1} \mathbf{F}_3\,\hat{\mathbf{k}}\cdot\hat{\mathbf{n}}\,\mathrm{d}S + \iint_{S_c,\ \hat{\mathbf{n}}\downarrow} \mathbf{F}_3\,\hat{\mathbf{k}}\cdot\hat{\mathbf{n}}\,\mathrm{d}S + \iint_{S_2} \mathbf{F}_3\,\hat{\mathbf{k}}\cdot\hat{\mathbf{n}}\,\mathrm{d}S + \iint_{S_c,\ \hat{\mathbf{n}}\uparrow} \mathbf{F}_3\,\hat{\mathbf{k}}\cdot\hat{\mathbf{n}}\,\mathrm{d}S\end{aligned}$$

The second and fourth integrals are identical except that $\hat{\mathbf{n}} = -\hat{\mathbf{k}}$ in the second integral and $\hat{\mathbf{n}} = +\hat{\mathbf{k}}$ in the fourth integral. So they cancel exactly and

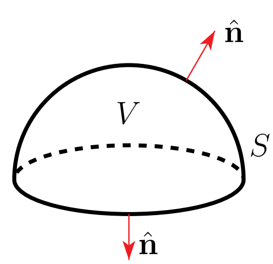
$$\iiint_V \frac{\partial \mathbf{F}_3}{\partial z}\,\mathrm{d}V = \iint_{S_1} \mathbf{F}_3\,\hat{\mathbf{k}}\cdot\hat{\mathbf{n}}\,\mathrm{d}S + \iint_{S_2} \mathbf{F}_3\,\hat{\mathbf{k}}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iint_{\partial V} \mathbf{F}_3\,\hat{\mathbf{k}}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$$

as desired. □

Example 4.2.4

Problem: Evaluate the flux integral $\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$ where $\hat{\mathbf{n}}$ is the outward normal to S, which is the surface of the hemispherical region

$$V = \{\ (x,y,z)\ \big|\ x^2+y^2+z^2 \leqslant a^2,\ z \geqslant 0\ \}$$



and

$$\mathbf{F} = xz^2\,\hat{\imath} + (x^2y - z^3)\,\hat{\jmath} + \big(2xy + y^2z + e^{\cos y}\big)\,\hat{\mathbf{k}}$$

Solution. The $e^{\cos y}$ in $\mathbf{F}$ suggests that a direct evaluation of the integral is difficult. So we'll use a little trickery to to evaluate it. Not surprisingly, considering that we have just proven the divergence theorem, the trick is to apply the divergence theorem[23]. Since

$$\begin{aligned}\nabla\cdot\mathbf{F} &= \frac{\partial \mathbf{F}_1}{\partial x}+\frac{\partial \mathbf{F}_2}{\partial y}+\frac{\partial \mathbf{F}_3}{\partial z} = \frac{\partial}{\partial x}\big(xz^2\big)+\frac{\partial}{\partial y}\big(x^2y-z^3\big)+\frac{\partial}{\partial z}\big(2xy+y^2z+e^{\cos y}\big)\\ &= z^2+x^2+y^2\end{aligned}$$

The divergence theorem tell us that

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iiint_V \big(x^2+y^2+z^2\big)\,\mathrm{d}V$$

Spherical coordinates are perfect for this integral. (See Appendix F.3, if you need to refresh your memory.)

$$\begin{aligned}\iiint_V \big(x^2+y^2+z^2\big)\,\mathrm{d}V &= \int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi/2}\mathrm{d}\varphi\int_0^a \mathrm{d}\rho\,\rho^2\sin\varphi\,\rho^2\\ &= \bigg[\int_0^{2\pi}\mathrm{d}\theta\bigg]\bigg[\int_0^{\pi/2}\sin\varphi\,\mathrm{d}\varphi\bigg]\bigg[\int_0^a\rho^4\,\mathrm{d}\rho\bigg]\\ &= \big[2\pi\big]\Big[-\cos\varphi\Big]_0^{\pi/2}\bigg[\frac{\rho^5}{5}\bigg]_0^a\\ &= \frac{2\pi a^5}{5}\end{aligned}$$

Example 4.2.4

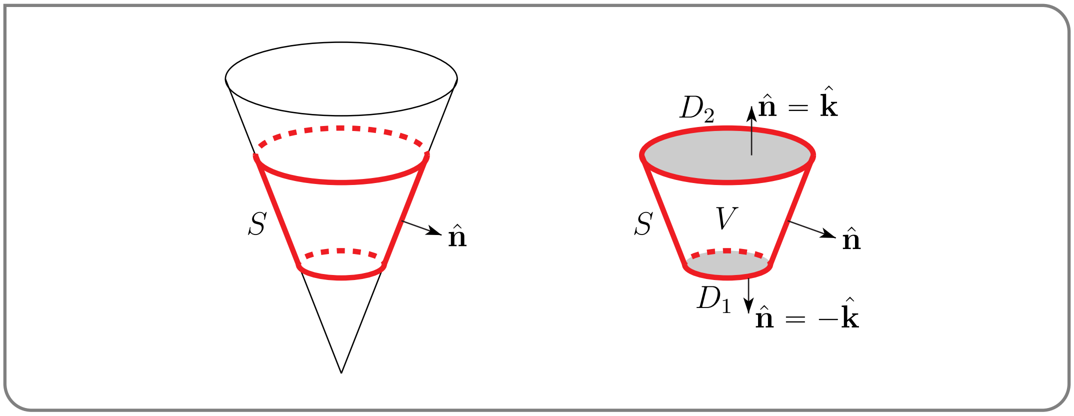
Example 4.2.5

Problem: Evaluate the flux integral $\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$ where $\hat{\mathbf{n}}$ is the outward normal to S, which is the part of the surface $z^2 = x^2+y^2$ with $1 \leqslant z \leqslant 2$, and where

$$\mathbf{F} = 3x\,\hat{\boldsymbol{\imath}} + (5y+e^{\cos x})\,\hat{\boldsymbol{\jmath}} + z\,\hat{\mathbf{k}}$$

Solution. Again the $e^{\cos x}$ in $\mathbf{F}$ suggests that a direct evaluation is difficult[24] and again we'll apply the divergence theorem. But this time S is not the boundary of a solid V. It is the portion of the cone outlined in red in the figure on the left below and does not have a top or bottom "cap". Fortunately, there is a solid V whose boundary, while not being equal

23 It's almost as though someone rigged the example with this in mind.
24 In fact, it is possible to evaluate this integral directly, if one recognizes that the ugly part of the integrand is odd under $y\to -y$ and integrates to exactly zero.



to S, at least contains S. It is (unsurprisingly)

$$V = \{ (x, y, z) \mid x^2 + y^2 \leqslant z^2,\ 1 \leqslant z \leqslant 2 \}$$

and is sketched in the figure on the right above. The boundary, ∂V, is the union of S and the two disks

$$D_1 = \{ (x, y, z) \mid x^2 + y^2 \leqslant z^2,\ z = 1 \}$$
$$D_2 = \{ (x, y, z) \mid x^2 + y^2 \leqslant z^2,\ z = 2 \}$$

So the divergence theorem gives

$$\iiint_V \nabla \cdot \mathbf{F}\, dV = \iint_{\partial V} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS + \iint_{D_1} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS + \iint_{D_2} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$$

which implies

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iiint_V \nabla \cdot \mathbf{F}\, dV - \iint_{D_1} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS - \iint_{D_2} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$$

The point of this exercise is that the left hand side, which is not easy to evaluate directly, is the integral we want, while the three integrals on the right hand side are all easy to evaluate. We do so now. The outward normal to (the horizontal disk) D_2 is $+\hat{\mathbf{k}}$. So

$$\iint_{D_2} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iint_{D_2} \mathbf{F} \cdot \hat{\mathbf{k}}\, dS = \iint_{D_2} z\, dS$$

As $z = 2$ on D_2, and D_2 is a disk of radius 2,

$$\iint_{D_1} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = 2\text{Area}(D_2) = 2\pi 2^2 = 8\pi$$

Similarly, the outward normal to (the horizontal disk) D_1 is $-\hat{\mathbf{k}}$. So

$$\iint_{D_1} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = -\iint_{D_1} \mathbf{F} \cdot \hat{\mathbf{k}}\, dS = -\iint_{D_1} z\, dS$$

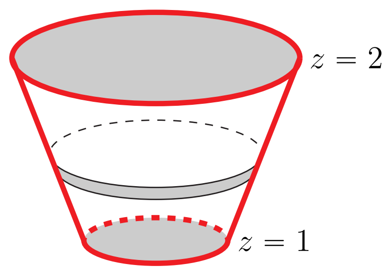
As $z = 1$ on D_1, and D_1 is a disk of radius 1,

$$\iint_{D_1} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \text{Area}(D_1) = -\pi 1^2 = -\pi$$

Finally, as $\nabla\cdot\mathbf{F} = 3+5+1 = 9$

$$\iiint_V \nabla\cdot\mathbf{F}\,\mathrm{d}V = 9\,\text{Vol}(V)$$

The volume of V can be easily computed using the first year technique[25] of slicing V into thin horizontal pancakes like that sketched in the figure below.



The pancake at height z has

- thickness $\mathrm{d}z$,
- a circular cross-section of radius z (remember that the outer boundary of V has equation $x^2+y^2 = z^2$), and hence has
- cross-sectional area πz^2 and
- volume $\pi z^2\,\mathrm{d}z$.

So

$$\iiint_V \nabla\cdot\mathbf{F}\,\mathrm{d}V = 9\,\text{Vol}(V) = 9\int_1^2 \pi z^2\,\mathrm{d}z = 9\left[\frac{\pi z^3}{3}\right]_1^2 = 9\times\pi\frac{7}{3} = 21\pi$$

and, all together

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iiint_V \nabla\cdot\mathbf{F}\,\mathrm{d}V - \iint_{D_1}\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S - \iint_{D_2}\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = 21\pi - (-\pi) - 8\pi = 14\pi$$

Example 4.2.5

Example 4.2.6

Problem: Evaluate the flux integral $\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$ where $\hat{\mathbf{n}}$ is the upward normal to S, which is the part of $z = \left(x^2+y^2\right)^2$ with $0 \leqslant z \leqslant 1$, and

$$\mathbf{F} = \left(x + e^{y^2}\right)\hat{\boldsymbol{\imath}} + (y+\cos z)\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}$$

25 You can review in §1.6 of the CLP-2 text.

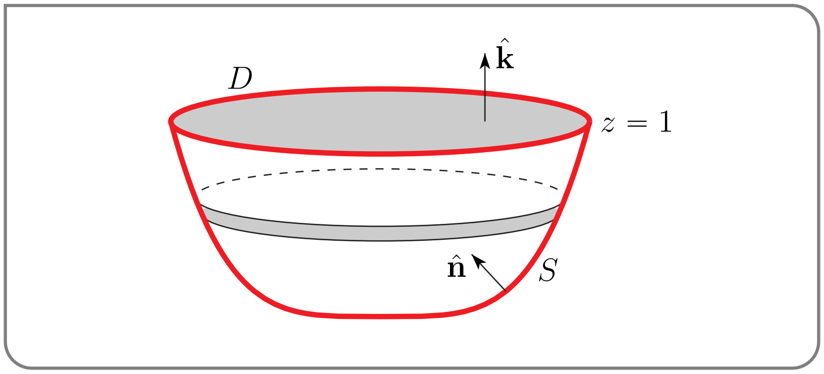
Solution. This integral can be evaluated in much the same way as we evaluated the integral of Example 4.2.5. We first define a solid V whose boundary ∂V contains S. A good, and hopefully obvious, choice is

$$V = \left\{ (x, y, z) \,\middle|\, \left(x^2 + y^2\right)^2 \leqslant z,\ \ 0 \leqslant z \leqslant 1 \right\}$$

The boundary of V is the union of S, with outward pointing normal $-\mathbf{n}$ (recall that the problem specifies that the symbol $\hat{\mathbf{n}}$ refers to the upward pointing normal) and the disk

$$D = \left\{ (x, y, z) \,\middle|\, z = 1,\ \ \left(x^2 + y^2\right)^2 \leqslant 1 \right\}$$

with outward pointing normal $\hat{\mathbf{k}}$.



So the divergence theorem gives

$$\iiint_V \boldsymbol{\nabla} \cdot \mathbf{F}\, \mathrm{d}V = -\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S + \iint_D \mathbf{F} \cdot \hat{\mathbf{k}}\, \mathrm{d}S$$

which implies

$$\begin{aligned} \iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S &= -\iiint_V \boldsymbol{\nabla} \cdot \mathbf{F}\, \mathrm{d}V + \iint_D \mathbf{F} \cdot \hat{\mathbf{k}}\, \mathrm{d}S \\ &= -\iiint_V 2\, \mathrm{d}V + \iint_D \mathrm{d}S \end{aligned}$$

D is a circular disk of radius 1, and so has area π. To evaluate the volume integral we slice V into horizontal pancakes with the pancake at height z having a circular cross-section of radius $z^{1/4}$. (Recall that the boundary of V has $\left(x^2 + y^2\right)^2 = z$.) So

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = -2\int_0^1 \pi\sqrt{z}\, \mathrm{d}z + \pi = -2\pi \times \frac{2}{3} + \pi = -\frac{\pi}{3}$$

Again, you can see that the actual integration is quite easy. All of the work (or at least all of the thinking) happens in the setup.

Example 4.2.6

Example 4.2.7

In Warning 4.2.3 we emphasised that the conclusion of the divergence Theorem 4.2.2 can fail if the vector field **F** is not defined at even a single point of V. Here is an example. Set

$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^3} \qquad \text{where } \mathbf{r} = x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + z\,\hat{\mathbf{k}}$$

and $V = \{ (x,y,z) \mid x^2 + y^2 + z^2 \leqslant 1 \}$. Then, if $(x,y,z) \neq \mathbf{0}$,

$$\begin{aligned}\nabla \cdot \mathbf{F}(x,y,z) &= \frac{\partial}{\partial x}\frac{x}{\left[x^2+y^2+z^2\right]^{3/2}} + \frac{\partial}{\partial y}\frac{y}{\left[x^2+y^2+z^2\right]^{3/2}} + \frac{\partial}{\partial z}\frac{z}{\left[x^2+y^2+z^2\right]^{3/2}} \\ &= \frac{\left[x^2+y^2+z^2\right] - x\frac{3}{2}(2x)}{\left[x^2+y^2+z^2\right]^{5/2}} + \frac{\left[x^2+y^2+z^2\right] - y\frac{3}{2}(2y)}{\left[x^2+y^2+z^2\right]^{5/2}} + \frac{\left[x^2+y^2+z^2\right] - z\frac{3}{2}(2z)}{\left[x^2+y^2+z^2\right]^{5/2}} \\ &= 0\end{aligned}$$

On the other hand, the boundary of V is the unit sphere $\partial V = \{ (x,y,z) \mid x^2+y^2+z^2 = 1 \}$. The outward unit normal to ∂V is $\hat{\mathbf{n}} = \frac{\mathbf{r}}{|\mathbf{r}|}$ so that

$$\begin{aligned}\int_{\partial V} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \int_{|\mathbf{r}|=1} \frac{\mathbf{r}}{|\mathbf{r}|^3}\cdot\frac{\mathbf{r}}{|\mathbf{r}|}\,\mathrm{d}S = \int_{|\mathbf{r}|=1}\frac{1}{|\mathbf{r}|^2}\,\mathrm{d}S = \int_{|\mathbf{r}|=1}\mathrm{d}S \\ &= 4\pi \neq 0\end{aligned}$$

Example 4.2.7

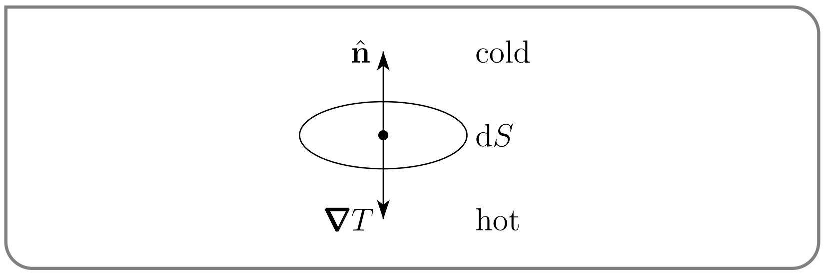
4.2.1 ▸▸ Optional — An Application of the Divergence Theorem — the Heat Equation

▸▸▸ Derivation of the Heat Equation

Let $T(x,y,z,t)$ be the temperature at time t at the point (x,y,z) in some object $\mathcal{B}$. The heat equation[26] is the partial differential equation that describes the flow of heat energy and consequently the behaviour of T. We now use the divergence theorem to derive the heat equation from two physical "laws", that we assume are valid:

26 The heat equation was formulated by the French mathematician and physicist Jean-Baptiste Joseph Fourier in 1807. He lived from 1768 to 1830, a period which included both the French revolution and the reign of Napoleon. Indeed Fourier served on his local Revolutionary Committee, was imprisoned briefly during the Terror, and was Napoleon Bonaparte's scientific advisor on his Egyptian expedition of 1798. Fourier series and the Fourier transform are named after him. Fourier is also credited with discovering the greenhouse effect.

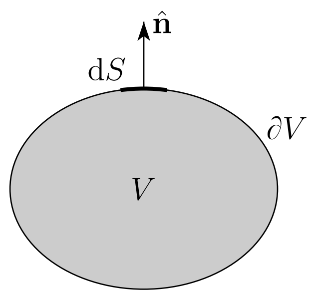
- The amount of heat energy required to raise the temperature of an object by ΔT degrees is $CM\,\Delta T$ where, M is the mass of the object and C is a positive physical constant determined by the material contained in the object. It is called the specific heat, or specific heat capacity[27], of the object.
- Think of heat energy as a moving fluid. We will rig its velocity field so that heat flows in the direction opposite to the temperature gradient. Precisely, we choose its velocity field to be $-\kappa\boldsymbol{\nabla}T(x,y,z,t)$. Here κ is another positive physical constant called the thermal conductivity of the object. So the rate at which heat is conducted across an element of surface area $\mathrm{d}S$ at (x,y,z) in the direction of its unit normal $\hat{\mathbf{n}}$ is given by $-\kappa\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}T(x,y,z,t)\,\mathrm{d}S$ at time t. (See Lemma 3.4.1.) For example, in the figure



the temperature gradient, which points in the direction of increasing temperature, is opposite $\hat{\mathbf{n}}$. Consequently the flow rate $-\kappa\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}T(x,y,z,t)\,\mathrm{d}S$ is positive, indicating flow in the direction of $\hat{\mathbf{n}}$. This is just what you would expect — heat flows from hot regions to cold regions. Also the rate of flow increases as the magnitude of the temperature gradient increases. This also makes sense (and is reminiscent of Newton's law of cooling).

Let $V \subset \mathcal{B}$ be *any* three dimensional region in the object and denote by ∂V the surface of V and by $\hat{\mathbf{n}}$ the outward normal to ∂V. The amount of heat that *enters* V across an infinitesimal piece $\mathrm{d}S$ of ∂V in an infinitesimal time interval $\mathrm{d}t$ is $-\big(-\kappa\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}T(x,y,z,t)\,\mathrm{d}S\big)\,\mathrm{d}t$. The amount of heat that enters V across all of ∂V in the time interval $\mathrm{d}t$ is given by the integral

$$\iint_{\partial V} \kappa\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}T(x,y,z,t)\,\mathrm{d}S\,\mathrm{d}t$$



In this same time interval, the temperature at a point (x,y,z) in V changes by $\frac{\partial T}{\partial t}(x,y,z,t)\,\mathrm{d}t$. If the density of the object at (x,y,z) is $\rho(x,y,z)$, the amount of heat energy required to increase the temperature of an infinitesimal volume $\mathrm{d}V$ of the object centred at (x,y,z) by $\frac{\partial T}{\partial t}(x,y,z,t)\,\mathrm{d}t$ is $C(\rho\mathrm{d}V)\,\frac{\partial T}{\partial t}(x,y,z,t)\,\mathrm{d}t$. The amount of heat energy required to increase

27 Heat is now understood to arise from the internal energy of the object. In an earlier theory, heat was viewed as measuring an invisible fluid, called the caloric. The amount of caloric that an object could hold was called its "heat capacity" by the Scottish physician and chemist Joseph Black (1728–1799).

the temperature by $\frac{\partial T}{\partial t}(x, y, z, t)\,\mathrm{d}t$ at all points (x, y, z) in V is then

$$\iiint_V C\rho \frac{\partial T}{\partial t}(x, y, z, t)\,\mathrm{d}V\,\mathrm{d}t$$

Assuming that the object is not generating or destroying[28] heat itself, this must be same as the amount of heat that entered V in the time interval $\mathrm{d}t$. That is

$$\iint_{\partial V} \kappa \hat{\mathbf{n}} \cdot \nabla T\,\mathrm{d}S\,\mathrm{d}t = \iiint_V C\rho \frac{\partial T}{\partial t}\,\mathrm{d}V\,\mathrm{d}t$$

Now we cancel the common factor of $\mathrm{d}t$. We can then rewrite the left hand side as an integral over V by applying the divergence theorem giving

$$\iiint_V \kappa \nabla \cdot \nabla T\,\mathrm{d}V = \iiint_V C\rho \frac{\partial T}{\partial t}\,\mathrm{d}V$$

As both integrals are over the same volume V, we have

$$\iiint_V \kappa \nabla \cdot \nabla T\,\mathrm{d}V - \iiint_V C\rho \frac{\partial T}{\partial t}\,\mathrm{d}V = 0 \implies \iiint_V \left[\kappa \nabla^2 T - C\rho \frac{\partial T}{\partial t}\right]\mathrm{d}V = 0 \qquad \text{(H)}$$

where $\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian. This must be true for all volumes V in the object and for all times t. We claim that this forces

$$\kappa \nabla^2 T(x, y, z, t) - C\rho \frac{\partial T}{\partial t}(x, y, z, t) = 0$$

for all (x, y, z) in the object and all t.

Suppose that to the contrary there was a point (x_0, y_0, z_0) in the object and a time t_0 with, for example, $\kappa \nabla^2 T(x_0, y_0, z_0, t_0) - C\rho \frac{\partial T}{\partial t}(x_0, y_0, z_0, t_0) > 0$. By continuity, which we are assuming, $\kappa \nabla^2 T(x, y, z, t_0) - C\rho \frac{\partial T}{\partial t}(x, y, z, t_0)$ must remain close to $\kappa \nabla^2 T(x_0, y_0, z_0, t_0) - C\rho \frac{\partial T}{\partial t}(x_0, y_0, z_0, t_0)$ when (x, y, z) is close to (x_0, y_0, z_0). So we would have

$$\kappa \nabla^2 T(x, y, z, t_0) - C\rho \frac{\partial T}{\partial t}(x, y, z, t_0) > 0$$

for all (x, y, z) in some small ball B centered on (x_0, y_0, z_0). Then, necessarily,

$$\iiint_B \left[\kappa \nabla \cdot \nabla T(x, y, z, t_0) - C\rho \frac{\partial T}{\partial t}(x, y, z, t_0)\right]\mathrm{d}V > 0$$

which violates (H) for $V = B$. This completes our derivation of the heat equation, which is

Equation 4.2.8.

$$\frac{\partial T}{\partial t}(x, y, z, t) = \alpha \nabla^2 T(x, y, z, t)$$

where $\alpha = \frac{\kappa}{C\rho}$ is called the thermal diffusivity.

28 The caloric theory of heat was itself destroyed by the cannon boring experiment of 1798. In this experiment the American/British physicist Benjamin Thompson (1753–1814) boiled water just using the heat generated by friction during the boring of a cannon.

▸▸▸ An Application of the Heat Equation

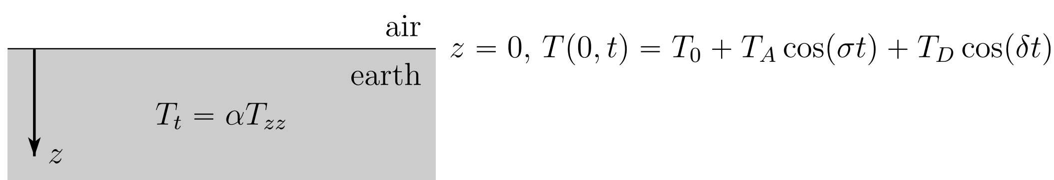
As an application, we look at the temperature a short distance below the surface of the Earth. For simplicity, we make the Earth flat[29] and we assume that the temperature, T, depends only on time, t, and the vertical coordinate, z. Then the heat equation simplifies to

$$\frac{\partial T}{\partial t}(z,t) = \alpha \frac{\partial^2 T}{\partial z^2}(z,t) \tag{HE}$$

We choose a coordinate system having the surface of the Earth at $z = 0$ and having z increase downward. We also assume that the temperature $T(0,t)$ at the surface of the Earth is primarily determined by solar heating and is given by

$$T(0,t) = T_0 + T_A \cos(\sigma t) + T_D \cos(\delta t) \tag{BC}$$

Here T_0 is the long term average of the temperature at the surface of the Earth, $T_A \cos(\sigma t)$ gives seasonal temperature variations and $T_D \cos(\delta t)$ gives daily temperature variations.



We measure time in days so that $\delta = 2\pi$ and $\sigma = \frac{2\pi}{1\text{ year}} = \frac{2\pi}{365\text{days}}$. Then $T_A \cos(\sigma t)$ has period one year and $T_D \cos(\delta t)$ has period one day. The solution to the initial value problem (HE)+(BC) can be found by separation of variables, a standard topic in courses on partial differential equations. The solution is

$$T(z,t) = T_0 + T_A e^{-\sqrt{\frac{\sigma}{2\alpha}}\, z} \cos\Big(\sigma t - \sqrt{\frac{\sigma}{2\alpha}}\, z\Big) + T_D e^{-\sqrt{\frac{\delta}{2\alpha}}\, z} \cos\Big(\delta t - \sqrt{\frac{\delta}{2\alpha}}\, z\Big) \tag{SLN}$$

Whether or not you can find this solution, you can, and should, check that (SLN) satisfies both (HE) and (BC).

Now let's see what we can learn from the solution (SLN). For any fixed z, the time average of $T(z,t)$ is T_0 (just because the average value if cosine is zero), the same as the average temperature at the surface $z = 0$. That is, under the hypotheses that we have made, the long term average temperature at any depth z is is the same as the long term average temperature at the surface.

The term $T_A e^{-\sqrt{\frac{\sigma}{2\alpha}}\, z} \cos\Big(\sigma t - \sqrt{\frac{\sigma}{2\alpha}}\, z\Big)$

- ◦ oscillates in time with a period of one year, just like $T_A \cos(\sigma t)$
- ◦ has an amplitude $T_A e^{-\sqrt{\frac{\sigma}{2\alpha}}\, z}$ which is T_A at the surface and decreases exponentially as z increases. Increasing the depth z by a distance $\sqrt{2\alpha/\sigma}$ causes the amplitude of the oscillation to decrease by a factor of $1/e$. Both of these first two bullet points are probably very consistent with your intuition. But this term also has a third property that you may find less obvious. It has

29 Insert sarcastic footnote here.

- has a time lag of $\frac{z}{\sqrt{2\alpha\sigma}}$ with respect to $T_A\cos(\sigma t)$. The surface term $T_A\cos(\sigma t)$ takes its maximum value when $t = 0,\ {}^{2\pi}/_{\sigma},\ {}^{4\pi}/_{\sigma},\ \cdots$. At depth z, the corresponding term $T_A e^{-\sqrt{\frac{\sigma}{2\alpha}}\,z}\cos\left(\sigma t - \sqrt{\sigma/2\alpha}\,z\right)$ takes its maximum value when $\sigma t - \sqrt{\sigma/2\alpha}\,z = 0,\ 2\pi, 4\pi,\ \cdots$ so that $t = {}^{z}/\sqrt{2\alpha\sigma},\ {}^{2\pi}/_{\sigma} + {}^{z}/\sqrt{2\alpha\sigma},\ {}^{4\pi}/_{\sigma} + {}^{z}/\sqrt{2\alpha\sigma},\ \cdots$.

Similarly, the term $T_D e^{-\sqrt{\frac{\delta}{2\alpha}}\,z}\cos\left(\delta t - \sqrt{\frac{\delta}{2\alpha}}\,z\right)$

- oscillates in time with a period of one day, just like $T_D\cos(\delta t)$
- has an amplitude which is T_D at the surface and decreases by a factor of ${}^{1}/_{e}$ for each increase of $\sqrt{2\alpha/\delta}$ in depth.
- has a time lag of $\frac{z}{\sqrt{2\alpha\delta}}$ with respect to $T_D\cos(\delta t)$.

For water α is approximately $0.012\ \mathrm{m}^2/\mathrm{day}$. This α gives

$$\sqrt{\frac{2\alpha}{\sigma}} \approx 1.2\,\mathrm{m} \qquad \sqrt{\frac{2\alpha}{\delta}} \approx 0.062\,\mathrm{m} \qquad \frac{z}{\sqrt{2\alpha\sigma}} \approx 49\,z\ \text{days} \qquad \frac{z}{\sqrt{2\alpha\delta}} \approx 2.6\,z\ \text{days}$$

for z measured in centimeters. So at a depth of a couple of meters, the temperature is pretty constant in time. What variation there is lags the surface variations by several months.

4.2.2 ▸▸ Variations of the Divergence Theorem

Here are a couple useful variations of the divergence theorem.

Theorem 4.2.9 (Variations on the divergence theorem)**.**

If V is a solid with surface ∂V, then

$$\iint_{\partial V} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iiint_V \boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}V$$
$$\iint_{\partial V} f\hat{\mathbf{n}}\,\mathrm{d}S = \iiint_V \boldsymbol{\nabla} f\,\mathrm{d}V$$
$$\iint_{\partial V} \hat{\mathbf{n}}\times\mathbf{F}\,\mathrm{d}S = \iiint_V \boldsymbol{\nabla}\times\mathbf{F}\,\mathrm{d}V$$

where $\hat{\mathbf{n}}$ is the outward unit normal of ∂V.

Memory Aid. All three formulae can be combined into

$$\iint_{\partial V} \hat{\mathbf{n}} * \tilde{F}\,\mathrm{d}S = \iiint_V \boldsymbol{\nabla} * \tilde{F}\,\mathrm{d}V$$

where $*$ can be either $\cdot$, $\times$ or nothing. When $* = \cdot$ or $* = \times$, then $\tilde{F} = \mathbf{F}$. When $*$ is nothing, $\tilde{F} = f$.

Proof. The first formula is exactly the divergence theorem and was proven in Theorem 4.2.2. To prove the second formula, set $\mathbf{F} = f\mathbf{a}$, where $\mathbf{a}$ is any constant vector, and apply the divergence theorem.

$$\begin{aligned}\iint_{\partial V} f\mathbf{a}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \iiint_V \boldsymbol{\nabla}\cdot(f\mathbf{a})\,\mathrm{d}V \\ &= \iiint_V \big[(\boldsymbol{\nabla} f)\cdot\mathbf{a} + f\underbrace{\boldsymbol{\nabla}\cdot\mathbf{a}}_{=0}\big]\,\mathrm{d}V \qquad \text{(by the vector identity Theorem 4.1.4.c)} \\ &= \iiint_V (\boldsymbol{\nabla} f)\cdot\mathbf{a}\,\mathrm{d}V\end{aligned}$$

To get the third line, we just used that $\mathbf{a}$ is a constant, so that all of its derivatives are zero. Rewrite

$$\iiint_V (\boldsymbol{\nabla} f)\cdot\mathbf{a}\,\mathrm{d}V = \iiint_V \mathbf{a}\cdot(\boldsymbol{\nabla} f)\,\mathrm{d}V$$

Since $\mathbf{a}$ is a constant, we can factor it out of both integrals, so

$$\begin{aligned}\mathbf{a}\cdot\iint_{\partial V} f\hat{\mathbf{n}}\,\mathrm{d}S &= \mathbf{a}\cdot\iiint_V \boldsymbol{\nabla} f\,\mathrm{d}V \\ \Longrightarrow \mathbf{a}\cdot\Big\{\iint_{\partial V} f\hat{\mathbf{n}}\,\mathrm{d}S - \iiint_V \boldsymbol{\nabla} f\,\mathrm{d}V\Big\} &= 0\end{aligned}$$

In particular, choosing $\mathbf{a} = \hat{\imath}$, $\hat{\jmath}$ and $\hat{\mathbf{k}}$, we see that all three components of the vector $\iint_{\partial V} f\hat{\mathbf{n}}\,\mathrm{d}S - \iiint_V \boldsymbol{\nabla} f\,\mathrm{d}V$ are zero. So

$$\iint_{\partial V} f\hat{\mathbf{n}}\,\mathrm{d}S - \iiint_V \boldsymbol{\nabla} f\,\mathrm{d}V = 0$$

which is what we wanted show.

To prove the third formula, apply the divergence theorem, but with $\mathbf{F}$ replaced by $\mathbf{a}\times\mathbf{F}$, where $\mathbf{a}$ is any constant vector.

$$\begin{aligned}\iint_{\partial V} (\mathbf{a}\times\mathbf{F})\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \iiint_V \boldsymbol{\nabla}\cdot(\mathbf{a}\times\mathbf{F})\,\mathrm{d}V \\ &= \iiint_V \big[\mathbf{F}\cdot\underbrace{(\boldsymbol{\nabla}\times\mathbf{a})}_{=\mathbf{0}} - \mathbf{a}\cdot(\boldsymbol{\nabla}\times\mathbf{F})\big]\,\mathrm{d}V \quad \text{(by the vector identity Theorem 4.1.4.d)} \\ &= -\iiint_V \mathbf{a}\cdot(\boldsymbol{\nabla}\times\mathbf{F})\,\mathrm{d}V = -\mathbf{a}\cdot\iiint_V \boldsymbol{\nabla}\times\mathbf{F}\,\mathrm{d}V\end{aligned}$$

To get the third line, we again used that $\mathbf{a}$ is a constant, so that all of its derivatives are zero. For all vectors $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ (in case you don't remember this, it was Lemma 4.1.8.a) so that

$$(\mathbf{a}\times\mathbf{F})\cdot\hat{\mathbf{n}} = \mathbf{a}\cdot(\mathbf{F}\times\hat{\mathbf{n}})$$

and

$$\mathbf{a}\cdot\iint_{\partial V}\mathbf{F}\times\mathbf{n}\,\mathrm{d}S = -\mathbf{a}\cdot\iiint_{V}\nabla\times\mathbf{F}\,\mathrm{d}V$$
$$\implies \mathbf{a}\cdot\left\{\iint_{\partial V}\mathbf{F}\times\mathbf{n}\,\mathrm{d}S + \iiint_{V}\nabla\times\mathbf{F}\,\mathrm{d}V\right\} = 0$$

In particular, choosing $\mathbf{a} = \hat{\imath}$, $\hat{\jmath}$ and $\hat{\mathbf{k}}$, we see that all three components of the vector $\iint_{\partial V}\mathbf{F}\times\mathbf{n}\,\mathrm{d}S + \iiint_{V}\nabla\times\mathbf{F}\,\mathrm{d}V$ are zero. So

$$\iiint_{V}\nabla\times\mathbf{F}\,\mathrm{d}V = -\iint_{\partial V}\mathbf{F}\times\mathbf{n}\,\mathrm{d}S = \iint_{\partial V}\hat{\mathbf{n}}\times\mathbf{F}\,\mathrm{d}S$$

which is what we wanted show. □

4.2.3 ▸▸ An Application of the Divergence Theorem — Buoyancy

In this section, we use the divergence theorem to show that when you immerse an object in a fluid the net effect of fluid pressure acting on the surface of the object is a vertical force (called the buoyant force) whose magnitude equals the weight of fluid displaced by the object. This is known as Archimedes' principle[30].

We shall also show that the buoyant force acts through the "centre of buoyancy" which is the centre of mass of the fluid displaced by the object. The design of self-righting[31] boats exploits the fact that the centre of buoyancy and the centre of gravity, where gravity acts, need not be the same.

We start by computing the total force due to the pressure of the fluid pushing on the object. Recall that pressure

- is the force per unit surface area that the fluid exerts on the object
- acts perpendicularly to the surface
- pushes on the object

Thus the force due to pressure that acts on an infinitesimal piece of the object's surface at $\mathbf{r} = (x, y, z)$ with surface area $\mathrm{d}S$ and outward normal $\hat{\mathbf{n}}$ is $-p(\mathbf{r})\,\hat{\mathbf{n}}\mathrm{d}S$. The minus sign is there because pressure is directed into the object. If the object fills the volume V and has surface ∂V, then the total force on the object due to fluid pressure, called the buoyant force, is

$$\mathbf{B} = -\iint_{\partial V} p(\mathbf{r})\,\hat{\mathbf{n}}\,\mathrm{d}S$$

We now wish to apply a variant of the divergence theorem to rewrite $\mathbf{B} = -\iiint_V \nabla p\,\mathrm{d}V$. But there is a problem with this: $p(\mathbf{r})$ is the fluid pressure at $\mathbf{r}$ and is only defined where there is fluid. In particular, there is no fluid[32] inside the object, so $p(\mathbf{r})$ is not defined for any $\mathbf{r}$ in the interior of V.

30 The interested reader should do a net search for the story of Archimedes and the golden crown.
31 The first design of a self-righting boat was entered by William Wouldhave in a lifeboat design competition organised by South Shield's Law House committee in 1789.
32 A cup of tea in the galley doesn't count.

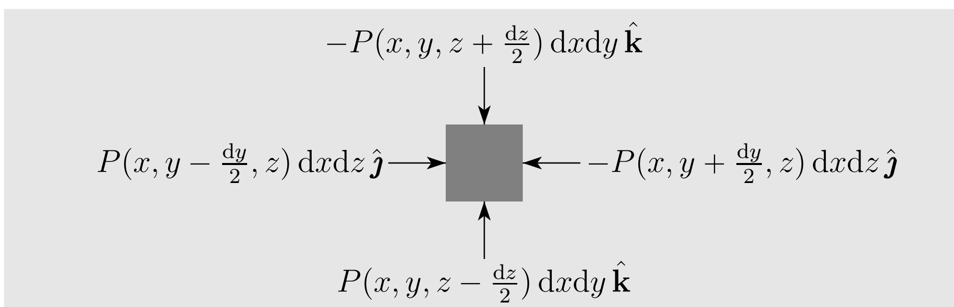
So we pretend that we remove the object from the fluid and we call $P(\mathbf{r})$ the fluid pressure at $\mathbf{r}$ when there is no object in the fluid. We also make the assumption that at any point $\mathbf{r}$ outside of the object, the pressure at $\mathbf{r}$ does not depend on whether the object is in the fluid or not. In other words, we assume that

$$p(\mathbf{r}) = \begin{cases} P(\mathbf{r}) & \text{if } \mathbf{r} \text{ is not in } V \\ \text{not defined} & \text{if } \mathbf{r} \text{ is in the } V \end{cases}$$

This assumption is only an approximation to reality, but, in practice, it is a very good approximation. So, by Theorem 4.2.9,

$$\mathbf{B} = -\iint_{\partial V} p(\mathbf{r})\,\hat{\mathbf{n}}\,\mathrm{d}S = -\iint_{\partial V} P(\mathbf{r})\,\hat{\mathbf{n}}\,\mathrm{d}S = -\iiint_{V} \nabla P(\mathbf{r})\,\mathrm{d}V \tag{4.2.1}$$

Our next job is to compute ∇P. Concentrate on an infinitesimal cube of fluid whose edges are parallel to the coordinate axes. Call the lengths of the edges $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$ and the position of the centre of the cube (x, y, z). The forces applied to the various faces of the cube by the pressure of fluid outside the cube are illustrated in the figure



The total force due to the pressure acting on the cube is the sum

$$\begin{aligned}
&-P\left(x + \frac{\mathrm{d}x}{2}, y, z\right) dy\mathrm{d}z\,\hat{\boldsymbol{\imath}} + P\left(x - \frac{\mathrm{d}x}{2}, y, z\right) dy\mathrm{d}z\,\hat{\boldsymbol{\imath}} \\
&-P\left(x, y + \frac{dy}{2}, z\right) \mathrm{d}x\mathrm{d}z\,\hat{\boldsymbol{\jmath}} + P\left(x, y - \frac{dy}{2}, z\right) \mathrm{d}x\mathrm{d}z\,\hat{\boldsymbol{\jmath}} \\
&-P\left(x, y, z + \frac{\mathrm{d}z}{2}\right) \mathrm{d}x dy\,\hat{\mathbf{k}} + P\left(x, y, z - \frac{\mathrm{d}z}{2}\right) \mathrm{d}x dy\,\hat{\mathbf{k}}
\end{aligned}$$

of the forces acting on the six faces. Consider the $\hat{\boldsymbol{\imath}}$ component and rewrite it as

$$\begin{aligned}
-P\left(x + \frac{\mathrm{d}x}{2}, y, z\right) dy\mathrm{d}z\,\hat{\boldsymbol{\imath}} &+ P\left(x - \frac{\mathrm{d}x}{2}, y, z\right) dy\mathrm{d}z\,\hat{\boldsymbol{\imath}} \\
&= -\frac{P(x + \frac{\mathrm{d}x}{2}, y, z) - P(x - \frac{\mathrm{d}x}{2}, y, z)}{\mathrm{d}x}\,\hat{\boldsymbol{\imath}}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= -\frac{\partial P}{\partial x}(x, y, z)\,\hat{\boldsymbol{\imath}}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z
\end{aligned}$$

Doing this for the other components as well, we see that the total force due to the pressure acting on the cube is

$$-\Big\{\frac{\partial P}{\partial x}(x,y,z)\,\hat{\imath} + \frac{\partial P}{\partial y}(x,y,z)\,\hat{\jmath} + \frac{\partial P}{\partial z}(x,y,z)\,\hat{\mathbf{k}}\Big\}\mathrm{d}x\mathrm{d}y\mathrm{d}z = -\nabla P(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

We shall assume that the only other force acting on the cube is gravity and that the fluid is stationary (or at least not accelerating). Hence the total force acting on the cube is zero. If the fluid has density ρ_f, then the cube has mass $\rho_f\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$ so that the force of gravity is $-g\rho_f\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\,\hat{\mathbf{k}}$. The vanishing of the total force now tells us that

$$-\nabla P(\mathbf{r})\,\mathrm{d}x\mathrm{d}y\mathrm{d}z - g\rho_f\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\,\hat{\mathbf{k}} = 0 \implies \nabla P(\mathbf{r}) = -g\rho_f\,\hat{\mathbf{k}}$$

Subbing this into (4.2.1) gives

$$\mathbf{B} = g\,\hat{\mathbf{k}}\iiint_V \rho_f\,\mathrm{d}V = gM_f\,\hat{\mathbf{k}}$$

where $M_f = \iiint_V \rho_f\,\mathrm{d}V$ is the mass of the fluid displaced by the object — *not* the mass of the object itself. Thus the buoyant force acts straight up and has magnitude equal to gM_f, which is also the magnitude of the force of gravity acting on the fluid displaced by the object. In other words, it is the weight of the displaced fluid. This is exactly Archimedes' principle.

We next consider the rotational motion of our submerged object. The physical law that determines the rotational motion of a rigid body about a point $\mathbf{r}_0$ is analogous to the familiar Newton's law, $m\frac{d\mathbf{v}}{\mathrm{d}t} = \mathbf{F}$, that determines the translational motion of the object. For the rotational law of motion,

- the mass m is replaced by a physical quantity, characteristic of the object, called the moment of inertia, and
- the ordinary velocity $\mathbf{v}$ is replaced by the angular velocity, which is a vector whose length is the rate of rotation (i.e. angle rotated per unit time) and whose direction is parallel to the axis of rotation (with the sign determined by a right hand rule), and
- the force $\mathbf{F}$ is replaced by a vector called the torque about $\mathbf{r}_0$. A force $\mathbf{F}$ applied at $\mathbf{r} = (x,y,z)$ produces the torque[33] $(\mathbf{r}-\mathbf{r}_0)\times\mathbf{F}$ about $\mathbf{r}_0$.

This is derived in the optional §4.2.4, entitled "Torque", and is all that we need to know about rotational motion of rigid bodies in this discussion.

Fix any point $\mathbf{r}_0$. The total torque about $\mathbf{r}_0$ produced by force of pressure acting on the surface of the submerged object is

$$\mathbf{T} = \iint_{\partial V}(\mathbf{r}-\mathbf{r}_0)\times\big(-p(\mathbf{r})\hat{\mathbf{n}}\big)\,\mathrm{d}S = \iint_{\partial V}\hat{\mathbf{n}}\times\big(P(\mathbf{r})\,(\mathbf{r}-\mathbf{r}_0)\big)\,\mathrm{d}S$$

Recall that in these integrals $\mathbf{r} = (x,y,z)$ is the position of the infinitesimal piece $\mathrm{d}S$ of the surface S. Applying the cross product variant of the divergence theorem in Theorem 4.2.9,

33 This is what Archimedes was referring to when he said "Give me a lever and a place to stand and I will move the earth."

followed by the vector identity Theorem 4.1.5.c, gives

$$\begin{aligned}\mathbf{T} &= \iiint_V \boldsymbol{\nabla} \times \big(P(\mathbf{r})\,(\mathbf{r}-\mathbf{r}_0)\big)\,\mathrm{d}V = \iiint_V \Big\{\boldsymbol{\nabla} P(\mathbf{r}) \times (\mathbf{r}-\mathbf{r}_0) + P(\mathbf{r})\,\underbrace{\boldsymbol{\nabla}\times(\mathbf{r}-\mathbf{r}_0)}_{=\mathbf{0}}\Big\}\,\mathrm{d}V \\ &= \iiint_V \boldsymbol{\nabla} P(\mathbf{r}) \times (\mathbf{r}-\mathbf{r}_0)\,\mathrm{d}V\end{aligned}$$

since $\boldsymbol{\nabla} \times \mathbf{r}_0 = 0$, because $\mathbf{r}_0$ is a constant, and

$$\boldsymbol{\nabla} \times \mathbf{r} = \det\begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{bmatrix} = 0$$

We have already found that $\boldsymbol{\nabla} P(\mathbf{r}) = -g\rho_f\hat{\mathbf{k}}$. Substituting it in gives

$$\begin{aligned}\mathbf{T} &= -\iiint_V g\rho_f\hat{\mathbf{k}} \times (\mathbf{r}-\mathbf{r}_0)\,\mathrm{d}V & &= -g\hat{\mathbf{k}} \times \iiint_V \rho_f(\mathbf{r}-\mathbf{r}_0)\,\mathrm{d}V \\ &= -g\hat{\mathbf{k}} \times \Big\{\iiint_V \mathbf{r}\rho_f\,\mathrm{d}V - \mathbf{r}_0\iiint_V \rho_f\,\mathrm{d}V\Big\} &&= -g\Big\{\iiint_V \rho_f\,\mathrm{d}V\Big\}\hat{\mathbf{k}} \times \Big\{\frac{\iiint_V \mathbf{r}\rho_f\,\mathrm{d}V}{\iiint_V \rho_f\,\mathrm{d}V} - \mathbf{r}_0\Big\} \\ &= -\mathbf{B} \times \Big\{\frac{\iiint_V \mathbf{r}\rho_f\,\mathrm{d}V}{\iiint_V \rho_f\,\mathrm{d}V} - \mathbf{r}_0\Big\} &&= \Big\{\frac{\iiint_V \mathbf{r}\rho_f\,\mathrm{d}V}{\iiint_V \rho_f\,\mathrm{d}V} - \mathbf{r}_0\Big\} \times \mathbf{B}\end{aligned}$$

So the torque generated at $\mathbf{r}_0$ by pressure over the entire surface is the same as the torque generated at $\mathbf{r}_0$ by a force $\mathbf{B}$ applied at the single point

$$\mathbf{C}_{\mathbf{B}} = \frac{\iiint_V \mathbf{r}\rho_f\,\mathrm{d}V}{\iiint_V \rho_f\,\mathrm{d}V}$$

This point is called the centre of buoyancy. It is the centre of mass of the displaced fluid.

The moral of the above discussion is that the buoyant force, $\mathbf{B}$, on a rigid body

- ○ acts straight upward,
- ○ has magnitude equal to the weight of the displaced fluid and
- ○ acts at the centre of buoyancy, which is the centre of mass of the displaced fluid.

As above, denoting by ρ_b the density of the object, the torque about $\mathbf{r}_0$ due to gravity acting on the object is

$$\iiint_V (\mathbf{r}-\mathbf{r}_0) \times (-g\rho_b\hat{\mathbf{k}})\,\mathrm{d}V = \Big\{\frac{\iiint_V \mathbf{r}\rho_b\,\mathrm{d}V}{\iiint_V \rho_b\,\mathrm{d}V} - \mathbf{r}_0\Big\} \times \left(-g\Big\{\iiint_V \rho_b\,\mathrm{d}V\Big\}\,\hat{\mathbf{k}}\right)$$

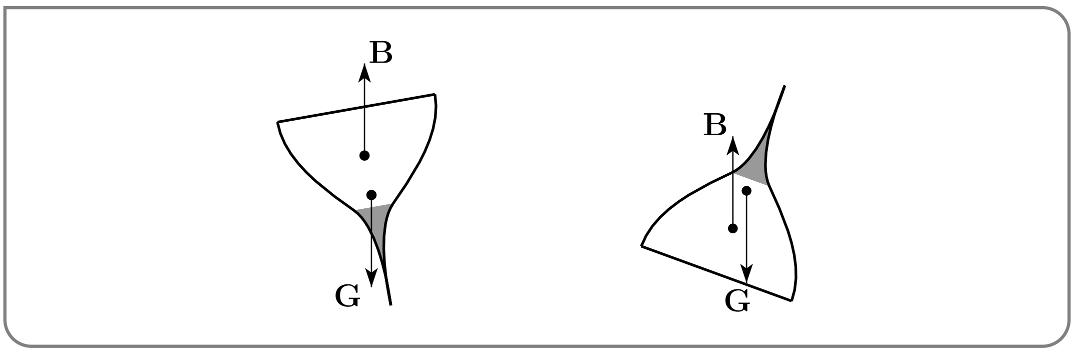
So the gravitational force, $\mathbf{G}$,

- ○ acts straight down,
- ○ has magnitude equal to the weight $gM_b = g\iiint_V \mathbf{r}\rho_b\,\mathrm{d}V$ (where ρ_b is the density of the object) of the object and

- acts at the centre of mass, $\mathbf{C_G} = \frac{\iiint_V \mathbf{r}\rho_b \, \mathrm{d}V}{\iiint_V \rho_b \, \mathrm{d}V}$, of the object.

Because the mass distribution of the object need not be the same as the mass distribution of the displaced fluid, buoyancy and gravity may act at two different points. This is exploited in the design of self-righting boats.

These boats are constructed with a heavy, often lead (which is cheap and dense), keel. As a result, the centre of gravity is lower in the boat than the center of buoyancy, which, because the displaced fluid has constant density, is at the geometric centre of the boat. As the figure below illustrates, a right side up configuration of such a boat is stable, while an upside down configuration is unstable. The boat rotates so as to keep the centre of gravity straight below the centre of buoyancy. To see this pretend that you are holding on to the boat with one hand holding the centre of buoyancy and the other hand holding the centre of gravity. Use your hands to apply forces in the directions of the arrows and think about how the boat will respond.

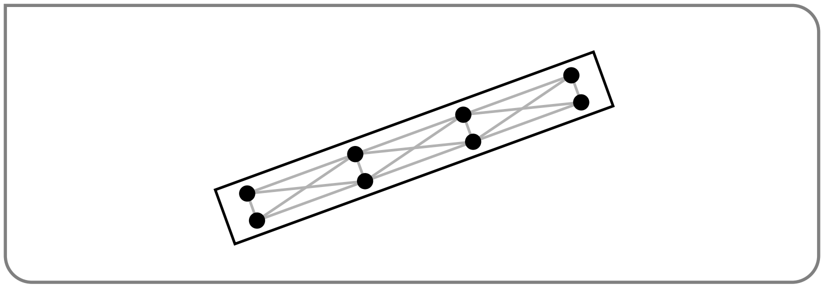


4.2.4 ▸▸ Optional — Torque

In this section, we derive the properties of torque that we used in the last section. Newton's law of motion says that the position $\mathbf{r}(t)$ of a single particle moving under the influence of a force $\mathbf{F}$ obeys $m\mathbf{r}''(t) = \mathbf{F}$. Similarly, the positions $\mathbf{r}_i(t)$, $1 \leqslant i \leqslant n$, of a set of particles moving under the influence of forces $\mathbf{F}_i$ obey $m\mathbf{r}_i''(t) = \mathbf{F}_i$, $1 \leqslant i \leqslant n$. Very often systems of interest consist of some small number of rigid bodies. Suppose that we are interested in the motion of a single rigid body, say a piece of wood. The piece of wood is made up of a huge number[34] of atoms. So the system of equations determining the motion of all of the individual atoms in the piece of wood is huge. On the other hand, we shall see that because the piece of wood is rigid, its configuration is completely determined by the position of, for example, its centre of mass and its orientation (we won't get into what precisely is meant by "orientation", but it is certainly determined by, for example, the positions of a few of the corners of the piece of wood). To be precise, we shall extract from the huge system of equations that determine the motion of all of the individual atoms, a small system of equations that determine the motion of the centre of mass and the orientation. We'll do so now.

Imagine a piece of wood moving in $\mathbb{R}^3$. Furthermore, imagine that the piece of wood

34 Just 12 grams of carbon contains about 6×10^{23} atoms.



consists of a huge number of particles joined by a huge number of weightless but very strong[35] steel rods. The steel rod joining particle number one to particle number two just represents a force acting between particles number one and two. Suppose that

- there are n particles, with particle number i having mass m_i,
- at time t, particle number i has position $\mathbf{r}_i(t)$,
- at time t, the external force (gravity and the like) acting on particle number i is $\mathbf{F}_i(t)$, and
- at time t, the force acting on particle number i, due to the steel rod joining particle number i to particle number j is $\mathbf{F}_{i,j}(t)$. If there is no steel rod joining particles number i and j, just set $\mathbf{F}_{i,j}(t) = 0$. In particular, $\mathbf{F}_{i,i}(t) = 0$.

The only assumptions that we shall make about the steel rod forces are

(A1) for each $i \neq j$, $\mathbf{F}_{i,j}(t) = -\mathbf{F}_{j,i}(t)$. In words, the steel rod joining particles i and j applies equal and opposite forces to particles i and j.

(A2) for each $i \neq j$, there is a function $M_{i,j}(t)$ such that $\mathbf{F}_{i,j}(t) = M_{i,j}(t)\big[\mathbf{r}_i(t) - \mathbf{r}_j(t)\big]$. In words, the force due to the rod joining particles i and j acts parallel to the line joining particles i and j. For (A1) to be true, that is to have $M_{i,j}(t)\big[\mathbf{r}_i(t) - \mathbf{r}_j(t)\big] = -M_{j,i}(t)\big[\mathbf{r}_j(t) - \mathbf{r}_i(t)\big]$, we need $M_{i,j}(t) = M_{j,i}(t)$.

Newton's law of motion, applied to particle number i, now tells us that

$$m_i\mathbf{r}_i''(t) = \mathbf{F}_i(t) + \sum_{j=1}^{n} \mathbf{F}_{i,j}(t) \qquad (N_i)$$

Adding up all of the equations (N_i), for $i = 1,\ 2,\ 3,\ \cdots,\ n$ gives

$$\sum_{i=1}^{n} m_i\mathbf{r}_i''(t) = \sum_{i=1}^{n} \mathbf{F}_i(t) + \sum_{1\leqslant i,j\leqslant n} \mathbf{F}_{i,j}(t) \qquad (\Sigma N_i)$$

The sum $\sum\limits_{1\leqslant i,j\leqslant n} \mathbf{F}_{i,j}(t)$ contains $\mathbf{F}_{1,2}(t)$ exactly once and it also contains $\mathbf{F}_{2,1}(t)$ exactly once and these two terms cancel exactly, by assumption (A1). In this way, all terms in $\sum\limits_{1\leqslant i,j\leqslant n} \mathbf{F}_{i,j}(t)$ with $i \neq j$ exactly cancel. All terms with $i = j$ are assumed to be zero. So $\sum\limits_{1\leqslant i,j\leqslant n} \mathbf{F}_{i,j}(t) = 0$ and the equation (ΣN_i) simplifies to

$$\sum_{i=1}^{n} m_i\mathbf{r}_i''(t) = \sum_{i=1}^{n} \mathbf{F}_i(t) \qquad (\Sigma N_i)$$

35 Mathematicians and their idealizations! Really the rods just represent the atomic/chemical forces that hold the wood together.

Phew! Denote by $M = \sum_{i=1}^{n} m_i$ the total mass of the body, by $\mathbf{R}(t) = \frac{1}{M}\sum_{i=1}^{n} m_i\mathbf{r}_i(t)$ the centre of mass[36] of the body and by $\mathbf{F}(t) = \sum_{i=1}^{n} \mathbf{F}_i(t)$ the total external force acting on the system. In this notation, equation (ΣN_i) can be written as

Equation 4.2.10.

$$M\mathbf{R}''(t) = \mathbf{F}(t)$$

The upshot is that the centre of mass of the system moves just like a single particle of mass M subject to the total external force. This is why we can often replace an extended object by a point mass at its centre of mass.

Now take the cross product of $\mathbf{r}_i(t)$ and equation (N_i) and sum over i. This gives

$$\sum_{i=1}^{n} m_i\, \mathbf{r}_i(t) \times \mathbf{r}_i''(t) = \sum_{i=1}^{n} \mathbf{r}_i(t) \times \mathbf{F}_i(t) + \sum_{1\leqslant i,j\leqslant n} \mathbf{r}_i(t) \times \mathbf{F}_{i,j}(t) \qquad (\Sigma\mathbf{r}_i \times N_i)$$

By the assumption (A2)

$$\begin{aligned}\mathbf{r}_1(t) \times \mathbf{F}_{1,2}(t) &= M_{1,2}(t)\, \mathbf{r}_1(t) \times \big[\mathbf{r}_1(t) - \mathbf{r}_2(t)\big]\\ \mathbf{r}_2(t) \times \mathbf{F}_{2,1}(t) &= M_{2,1}(t)\, \mathbf{r}_2(t) \times \big[\mathbf{r}_2(t) - \mathbf{r}_1(t)\big]\\ &= -M_{1,2}(t)\, \mathbf{r}_2(t) \times \big[\mathbf{r}_1(t) - \mathbf{r}_2(t)\big]\end{aligned}$$

so that

$$\mathbf{r}_1(t) \times \mathbf{F}_{1,2}(t) + \mathbf{r}_2(t) \times \mathbf{F}_{2,1}(t) = M_{1,2}(t)\, \big[\mathbf{r}_1(t) - \mathbf{r}_2(t)\big] \times \big[\mathbf{r}_1(t) - \mathbf{r}_2(t)\big] = 0$$

because the cross product of any two parallel vectors is zero.

The last equation says that the $i = 1$, $j = 2$ term in $\sum_{1\leqslant i,j\leqslant n} \mathbf{r}_i(t) \times \mathbf{F}_{i,j}(t)$ exactly cancels the $i = 2$, $j = 1$ term. In this way all of the terms in $\sum_{1\leqslant i,j\leqslant n} \mathbf{r}_i(t) \times \mathbf{F}_{i,j}(t)$ with $i \neq j$ cancel. Each term with $i = j$ is exactly zero because $\mathbf{F}_{ii} = 0$. So $\sum_{1\leqslant i,j\leqslant n} \mathbf{r}_i(t) \times \mathbf{F}_{i,j}(t) = 0$ and $(\Sigma\mathbf{r}_i \times N_i)$ simplifies to

$$\sum_{i=1}^{n} m_i\, \mathbf{r}_i(t) \times \mathbf{r}_i''(t) = \sum_{i=1}^{n} \mathbf{r}_i(t) \times \mathbf{F}_i(t) \qquad (\Sigma\mathbf{r}_i \times N_i)$$

At this point it makes sense to define vectors

$$\begin{aligned}\mathbf{L}(t) &= \sum_{i=1}^{n} m_i\, \mathbf{r}_i(t) \times \mathbf{r}_i'(t)\\ \mathbf{T}(t) &= \sum_{i=1}^{n} \mathbf{r}_i(t) \times \mathbf{F}_i(t)\end{aligned}$$

because, in this notation, $(\Sigma\mathbf{r}_i \times N_i)$ becomes

36 Note that this is just the weighted average (no pun intended) of the positions of the particles.

Equation 4.2.11.

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{L}(t) = \mathbf{T}(t)$$

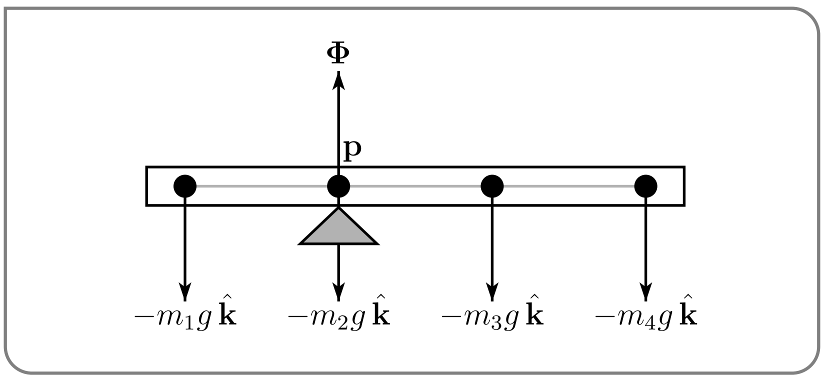
Equation (4.2.11) plays the rôle of Newton's law of motion for rotational motion. $\mathbf{T}(t)$ is called the torque and plays the rôle of "rotational force". $\mathbf{L}(t)$ is called the angular momentum (about the origin) and is a measure of the rate at which the piece of wood is rotating. For example, if a particle of mass m is traveling in a circle of radius ρ in the xy-plane at ω radians per unit time, then $\mathbf{r}(t) = \rho\cos(\omega t)\hat{\imath} + \rho\sin(\omega t)\hat{\jmath}$ and

$$\begin{aligned} m\mathbf{r}(t)\times\mathbf{r}'(t) &= m\big[\rho\cos(\omega t)\hat{\imath} + \rho\sin(\omega t)\hat{\jmath}\big] \times \big[-\omega\rho\sin(\omega t)\hat{\imath} + \omega\rho\cos(\omega t)\hat{\jmath}\big] \\ &= m\rho^2\,\omega\,\hat{\mathbf{k}} \end{aligned}$$

is proportional to ω, which is the rate of rotation about the origin and is in the direction $\hat{\mathbf{k}}$, which is normal to the plane containing the circle.

In any event, in order for the piece of wood to remain stationary, equations (4.2.10) and (4.2.11) force $\mathbf{F}(t) = \mathbf{T}(t) = 0$.

Now suppose that the piece of wood is a seesaw[37] that is supported on a fulcrum at $\mathbf{p}$. The forces consist of gravity, $-m_i g\hat{\mathbf{k}}$, acting on particle number i, for each $1 \leqslant i \leqslant n$, and the force $\mathbf{\Phi}$ imposed by the fulcrum that is pushing up on the particle at $\mathbf{p}$. The total



external force is $\mathbf{F} = \mathbf{\Phi} - \sum_{i=1}^{n} m_i g\hat{\mathbf{k}} = \mathbf{\Phi} - Mg\hat{\mathbf{k}}$. If the seesaw is to remain stationary, this must be zero so that $\mathbf{\Phi} = Mg\hat{\mathbf{k}}$.

The total torque (about the origin) is

$$\mathbf{T} = \mathbf{p}\times\mathbf{\Phi} - \sum_{i=1}^{n} m_i g\mathbf{r}_i \times \hat{\mathbf{k}} = g\Big(M\mathbf{p} - \sum_{i=1}^{n} m_i\mathbf{r}_i\Big)\times\hat{\mathbf{k}}$$

If the seesaw is to remain stationary, this must also be zero. This will be the case if the fulcrum is placed at

$$\mathbf{p} = \frac{1}{M}\sum_{i=1}^{n} m_i\mathbf{r}_i$$

which is just the centre of mass of the piece of wood.

37 Or teeter-totter for those who speak a different English dialect.

More generally, suppose that the external forces acting on the piece of wood consist of $\mathbf{F}_i$, acting on particle number i, for each $1 \leqslant i \leqslant n$, and a "fulcrum force" $\mathbf{\Phi}$ acting on a particle at $\mathbf{p}$. The total external force is $\mathbf{F} = \mathbf{\Phi} + \sum_{i=1}^{n} \mathbf{F}_i$. If the piece of wood is to remain stationary, this must be zero so that $\mathbf{\Phi} = -\sum_{i=1}^{n} \mathbf{F}_i$. The total torque (about the origin) is

$$\mathbf{T} = \mathbf{p} \times \mathbf{\Phi} + \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{F}_i = \sum_{i=1}^{n} (\mathbf{r}_i - \mathbf{p}) \times \mathbf{F}_i$$

If the piece of wood is to remain stationary, this must also be zero. That is, the torque about point $\mathbf{p}$ due to all of the forces $\mathbf{F}_i$, $1 \leqslant i \leqslant n$, must be zero.

4.2.5 ▸▸ Optional — Solving Poisson's Equation

In this section we shall use the divergence theorem to find a formula for the solution of Poisson's equation

$$\nabla^2 \varphi = 4\pi\rho$$

Here $\rho = \rho(\mathbf{r})$ is a given (continuous) function and φ is the unknown function that we wish to find. This equation arises, for example, in electrostatics, where ρ is the charge density and φ is the electric potential.

The main step in finding this solution formula will be to consider an

arbitrary (smooth) function φ and an
arbitrary (smooth) region V in $\mathbb{R}^3$ and an
arbitrary point $\mathbf{r}_0$ in the interior of V

and to find an auxiliary formula which expresses $\varphi(\mathbf{r}_0)$ in terms of

$\nabla^2\varphi(\mathbf{r})$, with $\mathbf{r}$ running over V and
$\nabla\varphi(\mathbf{r})$ and $\varphi(\mathbf{r})$, with $\mathbf{r}$ running only over ∂V.

This auxiliary formula, which we shall derive below, is

$$\varphi(\mathbf{r}_0) = -\frac{1}{4\pi}\left\{ \iiint_V \frac{\nabla^2\varphi(\mathbf{r})}{|\mathbf{r}-\mathbf{r}_0|}\, d^3\mathbf{r} - \iint_{\partial V} \varphi(\mathbf{r}) \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} \cdot \hat{\mathbf{n}}\, \mathrm{d}S - \iint_{\partial V} \frac{\nabla\varphi(\mathbf{r})}{|\mathbf{r}-\mathbf{r}_0|} \cdot \hat{\mathbf{n}}\, \mathrm{d}S \right\} \qquad (V)$$

When we take the limit as V expands to fill all of $\mathbb{R}^3$ then, assuming that φ and $\nabla\varphi$ go to zero sufficiently quickly[38] at ∞, the two integrals over ∂V will converge to zero and we will end up with the formula

$$\varphi(\mathbf{r}_0) = -\frac{1}{4\pi} \iiint_{\mathbb{R}^3} \frac{\nabla^2\varphi(\mathbf{r})}{|\mathbf{r}-\mathbf{r}_0|}\, d^3\mathbf{r}$$

38 Suppose, for example, that, for large $|\mathbf{r}-\mathbf{r}_0|$, $|\varphi(\mathbf{r})|$ is bounded by a constant times $1/|\mathbf{r}-\mathbf{r}_0|$ and $|\nabla\varphi(\mathbf{r})|$ is bounded by a constant times $1/|\mathbf{r}-\mathbf{r}_0|^2$. Then, if ∂V is the sphere of radius R centred on $\mathbf{r}_0$, ∂V has surface area $4\pi R^2$ and the two integrals over ∂V are bounded by a constant times $1/R$.

This expresses φ evaluated at an arbitrary point, $\mathbf{r}_0$, of $\mathbb{R}^3$ in terms of $\nabla^2\varphi(\mathbf{r})$, with $\mathbf{r}$ running over $\mathbb{R}^3$, which is exactly what we want, since $\nabla^2\varphi = 4\pi\rho$ for any solution of Poisson's equation. So once we have proven (V) we will have proven[39]

Theorem 4.2.12.

Assume that $\rho(\mathbf{r})$ is continuous and decays sufficiently quickly as $\mathbf{r}\to\infty$. If φ obeys $\nabla^2\varphi = 4\pi\rho$ on $\mathbb{R}^3$, and φ and $\nabla\varphi$ decay sufficiently quickly as $\mathbf{r}\to\infty$, then

$$\varphi(\mathbf{r}_0) = -\iiint_{\mathbb{R}^3} \frac{\rho(\mathbf{r})}{|\mathbf{r}-\mathbf{r}_0|}\, d^3\mathbf{r}$$

for all $\mathbf{r}_0$ in $\mathbb{R}^3$.

Let

$$\begin{aligned}\mathbf{r}(x,y,z) &= x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{\mathbf{k}}\\ \mathbf{r}_0 &= x_0\,\hat{\imath} + y_0\,\hat{\jmath} + z_0\,\hat{\mathbf{k}}\end{aligned}$$

We shall exploit three properties of the function $\frac{1}{|\mathbf{r}-\mathbf{r}_0|}$. The first two properties are

$$\nabla\frac{1}{|\mathbf{r}-\mathbf{r}_0|} = -\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} \tag{P1}$$

$$\nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}_0|} = -\nabla\cdot\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} = 0 \tag{P2}$$

and are valid for all $\mathbf{r}\neq\mathbf{r}_0$. Verification of the first property is a simple one line computation. Verification of the second property is a simple three line computation. (See the solution to Question 6 in Section 4.1 of the CLP-4 problem book.)

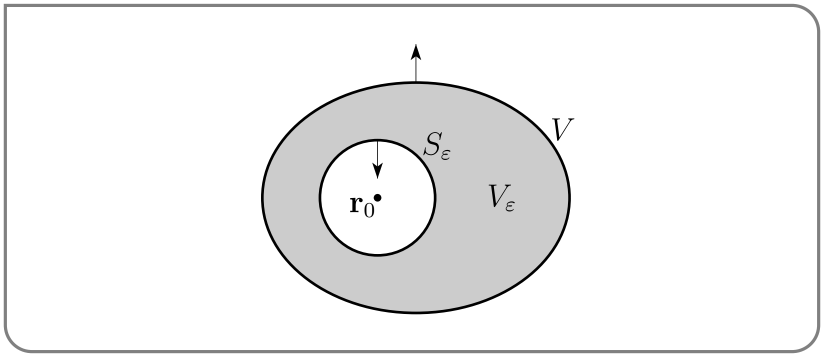
The other property of $\frac{1}{|\mathbf{r}-\mathbf{r}_0|}$ that we shall use is the following. Let S_ε be the sphere of radius ε centered on $\mathbf{r}_0$. Then, for any continuous function $\psi(\mathbf{r})$,

$$\begin{aligned}\lim_{\varepsilon\to0+}\iint_{S_\varepsilon}\frac{\psi(\mathbf{r})}{|\mathbf{r}-\mathbf{r}_0|^p}\,\mathrm{d}S &= \lim_{\varepsilon\to0+}\frac{1}{\varepsilon^p}\iint_{S_\varepsilon}\psi(\mathbf{r})\,\mathrm{d}S = \lim_{\varepsilon\to0+}\frac{\psi(\mathbf{r}_0)}{\varepsilon^p}\iint_{S_\varepsilon}\mathrm{d}S = \lim_{\varepsilon\to0+}\frac{\psi(\mathbf{r}_0)}{\varepsilon^p}4\pi\varepsilon^2\\ &= \begin{cases}4\pi\psi(\mathbf{r}_0) & \text{if } p=2\\ 0 & \text{if } p<2\\ \text{undefined} & \text{if } p>2\end{cases}\end{aligned} \tag{P3}$$

Derivation of (V):

Here is the derivation of (V). Let V_ε be the part of V outside of S_ε. Note that the

39 Note that the theorem does not claim that the φ defined in the theorem obeys $\nabla^2\varphi = 4\pi\rho$. It does, but the proof is beyond our scope.



boundary ∂V_ε of V_ε consists of two parts — the boundary ∂V of V and the sphere S_ε — and that the unit outward normal to ∂V_ε on S_ε is $-\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|}$, because it points towards $\mathbf{r}_0$ and hence outside of V_ε.

Recall the vector identity Theorem 4.1.7.d, which says

$$\nabla \cdot (f\nabla g - g\nabla f) = f\,\nabla^2 g - g\,\nabla^2 f$$

Applying this identity with $f = \frac{1}{|\mathbf{r}-\mathbf{r}_0|}$ and $g = \varphi$ gives

$$\begin{aligned}\nabla \cdot \Big(\frac{1}{|\mathbf{r}-\mathbf{r}_0|}\nabla\varphi - \varphi\nabla\frac{1}{|\mathbf{r}-\mathbf{r}_0|}\Big) &= \frac{\nabla^2\varphi}{|\mathbf{r}-\mathbf{r}_0|} - \varphi\,\overbrace{\nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}_0|}}^{=0 \text{ by (P2)}}\\ &= \frac{\nabla^2\varphi}{|\mathbf{r}-\mathbf{r}_0|}\end{aligned}$$

which is the integrand of the first integral on the right hand side of (V). So, by the divergence theorem

$$\begin{aligned}\iiint\limits_{V_\varepsilon} \frac{\nabla^2\varphi}{|\mathbf{r}-\mathbf{r}_0|}dV &= \iiint\limits_{V_\varepsilon} \nabla\cdot\Big(\frac{1}{|\mathbf{r}-\mathbf{r}_0|}\nabla\varphi - \varphi\nabla\frac{1}{|\mathbf{r}-\mathbf{r}_0|}\Big)dV\\ &= \iint\limits_{\partial V}\Big(\frac{1}{|\mathbf{r}-\mathbf{r}_0|}\nabla\varphi - \varphi\nabla\frac{1}{|\mathbf{r}-\mathbf{r}_0|}\Big)\cdot\hat{\mathbf{n}}\,\mathrm{d}S\\ &\qquad + \iint\limits_{S_\varepsilon}\Big(\frac{1}{|\mathbf{r}-\mathbf{r}_0|}\nabla\varphi - \varphi\nabla\frac{1}{|\mathbf{r}-\mathbf{r}_0|}\Big)\cdot\Big(-\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|}\Big)\mathrm{d}S \qquad \text{(M)}\end{aligned}$$

To see the connection between (M) and the rest of (V), note that,

- by (P1), the first term on the right hand side of (M) is

$$\iint\limits_{\partial V}\Big(\frac{1}{|\mathbf{r}-\mathbf{r}_0|}\nabla\varphi - \varphi\nabla\frac{1}{|\mathbf{r}-\mathbf{r}_0|}\Big)\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iint\limits_{\partial V}\frac{\nabla\varphi(\mathbf{r})}{|\mathbf{r}-\mathbf{r}_0|}\cdot\hat{\mathbf{n}}\,\mathrm{d}S + \iint\limits_{\partial V}\varphi(\mathbf{r})\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}\cdot\hat{\mathbf{n}}\,\mathrm{d}S \qquad \text{(R1)}$$

  which is 4π times the second and third terms on the right hand side of (V),

- and substituting in $\nabla\frac{1}{|\mathbf{r}-\mathbf{r}_0|} = -\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}$, from (P1), and applying (P3) with $p = 2$, the limit of the second term on the right hand side of (M) is

$$\lim_{\varepsilon\to 0+}\iint_{S_\varepsilon}\Big(\frac{1}{|\mathbf{r}-\mathbf{r}_0|}\nabla\varphi - \varphi\nabla\frac{1}{|\mathbf{r}-\mathbf{r}_0|}\Big)\cdot\Big(-\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|}\Big)\mathrm{d}S$$

$$= -\lim_{\varepsilon\to 0+}\iint_{B_\varepsilon}\big[\nabla\varphi\cdot(\mathbf{r}-\mathbf{r}_0)+\varphi\big]\frac{1}{|\mathbf{r}-\mathbf{r}_0|^2}\mathrm{d}S$$

$$= -4\pi\Big[\nabla\varphi\cdot(\mathbf{r}-\mathbf{r}_0)+\varphi\Big]_{\mathbf{r}=\mathbf{r}_0}$$

$$= -4\pi\varphi(\mathbf{r}_0) \tag{R2}$$

So applying[40] $\lim_{\varepsilon\to 0+}$ to (M) and substituting in (R1) and (R2) gives

$$\iiint_V \frac{\nabla^2\varphi}{|\mathbf{r}-\mathbf{r}_0|}\,dV = \iint_{\partial V}\frac{\nabla\varphi(\mathbf{r})}{|\mathbf{r}-\mathbf{r}_0|}\cdot\hat{\mathbf{n}}\,\mathrm{d}S + \iint_{\partial V}\varphi(\mathbf{r})\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}\cdot\hat{\mathbf{n}}\,\mathrm{d}S - 4\pi\varphi(\mathbf{r}_0)$$

which is exactly equation (V).

4.3▲ Green's Theorem

Our next variant of the fundamental theorem of calculus is Green's[41] theorem, which relates an integral, of a derivative of a (vector-valued) function, over a region in the xy-plane, with an integral of the function over the curve bounding the region. First we need to define some properties of curves.

40 You might worry about the singularity in $\frac{\nabla^2\varphi}{|\mathbf{r}-\mathbf{r}_0|}$ when applying $\lim_{\varepsilon\to 0+}$ to $\iiint_{V_\varepsilon}\frac{\nabla^2\varphi}{|\mathbf{r}-\mathbf{r}_0|}dV$. That this singularity is harmless may be seen using spherical coordinates centred on $\mathbf{r}_0$. Then $\mathrm{d}V$ contains a factor of $|\mathbf{r}-\mathbf{r}_0|^2$ (see §F.3), which completely eliminates the singularity.

41 George Green (1793–1841) was a British mathematical physicist. He spent much of the early part of his life working in his father's bakery and grain mill. He was finally admitted as an undergraduate to Cambridge in 1832, aged nearly forty.

Definition 4.3.1.

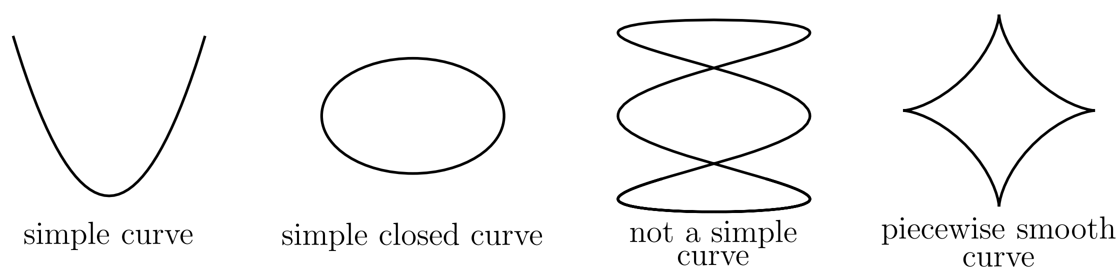
(a) A curve C with parametrization $\mathbf{r}(t)$, $a \leqslant t \leqslant b$, is said to be *closed* if $\mathbf{r}(a) = \mathbf{r}(b)$.

(b) A curve C is said to be *simple* if it does not cross itself. More precisely, if $\mathbf{r}(t)$, $a \leqslant t \leqslant b$, is a parametrization of the curve and if $a \leqslant t_1, t_2 \leqslant b$ obey $t_1 \neq t_2$ and $\{t_1, t_2\} \neq \{a, b\}$, then $\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$. That is, if $\mathbf{r}(t_1) = \mathbf{r}(t_2)$, then either $t_1 = t_2$ or $t_1 = a$, $t_2 = b$, or $t_1 = b$, $t_2 = a$.

(c) A curve C is *piecewise smooth* if it has a parametrization $\mathbf{r}(t)$ which

- is continuous and which
- is differentiable except possibly at finitely many points with
- the derivative being continuous and nonzero except possibly at finitely many points.

Here are sketches of some examples.

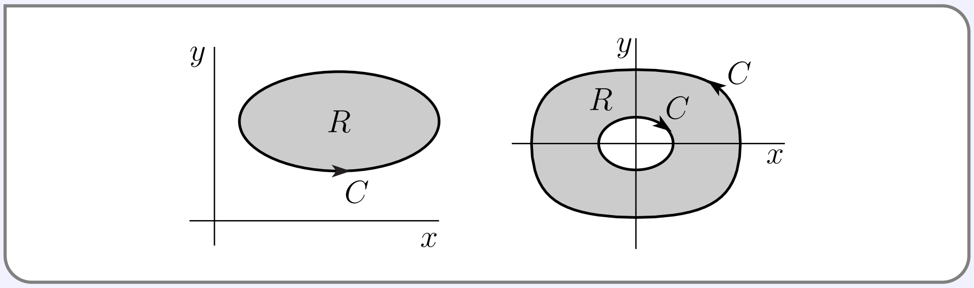


And here is Green's theorem.

Theorem 4.3.2 (Green's Theorem).

Let

- R be a finite region in the xy-plane,
- the boundary, C, of R consist of a finite number of piecewise smooth, simple closed curves
  - that are oriented (i.e. arrows are put on C) consistently with R in the sense that if you walk along C in the direction of the arrows, then R is on your left



- $F_1(x,y)$ and $F_2(x,y)$ have continuous first partial derivatives at *every* point of R.

Then

$$\oint_C \left[F_1(x,y)\,\mathrm{d}x + F_2(x,y)\,\mathrm{d}y\right] = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathrm{d}x\mathrm{d}y$$

Warning 4.3.3.

Note that in Theorem 4.3.2 we are assuming that F_1 and F_2 have continuous first partial derivatives at *every* point of R. If that is not the case, for example because F_1 or F_2 is not defined on all of R, then the conclusion of Green's theorem can fail. An example is $F_1 = -\frac{y}{x^2+y^2}$, $F_2 = \frac{x}{x^2+y^2}$, $R = \{\ (x,y)\ |\ x^2+y^2 \leqslant 1\ \}$. See Examples 4.3.7 and 4.3.8.

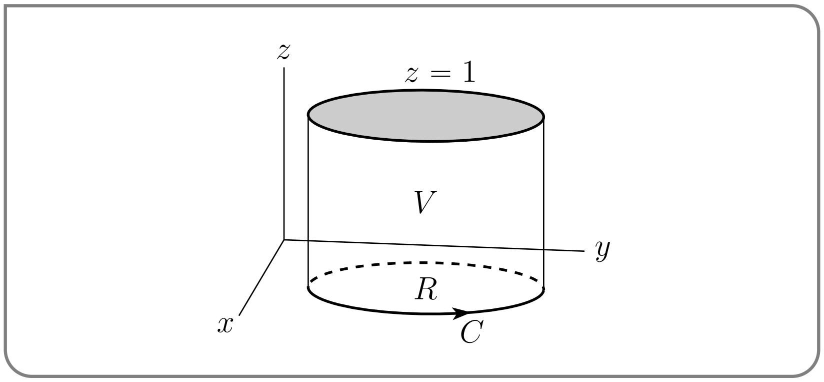
Here are three notational remarks before we start the proof.

- One way to remember the integrand on the right hand side is to write it as $(\boldsymbol{\nabla}\times\mathbf{F})\cdot\hat{\mathbf{k}}$.
- Many people use M instead of F_1 and N instead of F_2. Then Green's theorem becomes $\oint_C \left[M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y\right] = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathrm{d}x\mathrm{d}y$
- The symbol $\oint_C$ is just an alternate notation for $\int_C$ that is sometimes used when C is a closed curve. See Notation 2.4.1.

Proof. We prove the result by reformulating it as a divergence theorem statement. To that end, we define

$$V = \{\, (x,y,z) \mid (x,y) \in R,\ \ 0 \leqslant z \leqslant 1 \,\}$$
$$\mathbf{G}(x,y,z) = F_2(x,y)\,\hat{\boldsymbol{\imath}} - F_1(x,y)\,\hat{\boldsymbol{\jmath}}$$

Notice that V is exactly the volume obtained by expanding R vertically upward by one unit.



The definition of $\mathbf{G}$ does *not* contain a typo — the x-component of $\mathbf{G}$ really is F_2 and the y-component of $\mathbf{G}$ really is $-F_1$. (More or less the reverse of what you would normally write down.)

These definitions have been rigged so that the divergence theorem applied to $\mathbf{G}$ and V, namely

$$\iint_{\partial V} \mathbf{G} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \iiint_V \boldsymbol{\nabla} \cdot \mathbf{G}\, \mathrm{d}V$$

gives us exactly Green's theorem, as we shall now see.

Since $\boldsymbol{\nabla} \cdot \mathbf{G} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$, the right hand side is just

$$\begin{aligned}
\iiint_V \boldsymbol{\nabla} \cdot \mathbf{G}\, \mathrm{d}V &= \iint_R \mathrm{d}x\mathrm{d}y \int_0^1 \mathrm{d}z\ \boldsymbol{\nabla} \cdot \mathbf{G} \\
&= \iint_R \mathrm{d}x\mathrm{d}y \int_0^1 \mathrm{d}z\ \left(\frac{\partial F_2}{\partial x}(x,y) - \frac{\partial F_1}{\partial y}(x,y) \right) \\
&= \iint_R \mathrm{d}x\mathrm{d}y\ \left(\frac{\partial F_2}{\partial x}(x,y) - \frac{\partial F_1}{\partial y}(x,y) \right)
\end{aligned}$$

because the integrand is independent of z. This is exactly the right hand side of Green's theorem.

Now for the left hand side. The boundary, ∂V, of V is the union of the (flat) bottom, the (flat) top and the (curved) side. The outward unit normal on the (horizontal, flat) top is $+\hat{\mathbf{k}}$ and the outward unit normal on the (horizontal, flat) bottom is $-\hat{\mathbf{k}}$ so that

$$\begin{aligned}\iint_{\partial V} \mathbf{G}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \iint_{\text{top}} \mathbf{G}\cdot\hat{\mathbf{k}}\,\mathrm{d}S + \iint_{\text{bottom}} \mathbf{G}\cdot(-\hat{\mathbf{k}})\,\mathrm{d}S + \iint_{\text{side}} \mathbf{G}\cdot\hat{\mathbf{n}}\,\mathrm{d}S\\ &= \iint_{\text{side}} \mathbf{G}\cdot\hat{\mathbf{n}}\,\mathrm{d}S\end{aligned}$$

We have used the fact that the $\hat{\mathbf{k}}$ component of $\mathbf{G}$ is exactly zero to discard the integrals over the top and bottom of ∂V. To evaluate the integral over the side, we'll parametrize the side. Suppose that $\mathbf{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath}$, $a \leqslant t \leqslant b$, is a parametrization of C, with the arrow in the figure above giving the direction of increasing t. Then we can use

$$\mathbf{R}(t,z) = \mathbf{r}(t) + z\,\hat{\mathbf{k}} = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z\,\hat{\mathbf{k}} \qquad a \leqslant t \leqslant b,\ 0 \leqslant z \leqslant 1$$

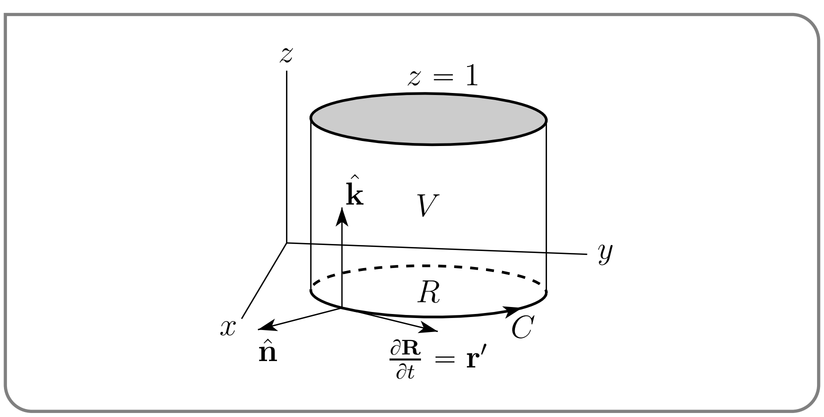
as a parametrization of the side. We'll use (3.3.1) to determine $\hat{\mathbf{n}}\,\mathrm{d}S$ for the side. Since

$$\begin{aligned}\frac{\partial \mathbf{R}}{\partial t}(t,z) &= x'(t)\,\hat{\imath} + y'(t)\,\hat{\jmath}\\ \frac{\partial \mathbf{R}}{\partial z}(t,z) &= \hat{\mathbf{k}}\end{aligned}$$

(3.3.1) gives

$$\begin{aligned}\hat{\mathbf{n}}\,\mathrm{d}S &= \frac{\partial \mathbf{R}}{\partial t}(t,z) \times \frac{\partial \mathbf{R}}{\partial z}(t,z)\,\mathrm{d}t\mathrm{d}z\\ &= \big(x'(t)\,\hat{\imath} + y'(t)\,\hat{\jmath}\big) \times \hat{\mathbf{k}}\,\mathrm{d}t\mathrm{d}z\\ &= \big(-x'(t)\,\hat{\jmath} + y'(t)\,\hat{\imath}\big)\,\mathrm{d}t\mathrm{d}z\end{aligned}$$

Note that with this choice of $\pm$ sign (that is, $\frac{\partial \mathbf{R}}{\partial t} \times \frac{\partial \mathbf{R}}{\partial z}\,\mathrm{d}t\mathrm{d}z$ rather than $-\frac{\partial \mathbf{R}}{\partial t} \times \frac{\partial \mathbf{R}}{\partial z}\,\mathrm{d}t\mathrm{d}z$), the vector $\hat{\mathbf{n}}$ really is the *outward* pointing normal, as we see from the sketch



We can now compute the surface integral directly.

$$\begin{aligned}
\iint_{\partial V} \mathbf{G}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \iint_{\text{side}} \mathbf{G}\cdot\hat{\mathbf{n}}\,\mathrm{d}S \\
&= \int_a^b \mathrm{d}t \int_0^1 \mathrm{d}z\ \mathbf{G}\big(\mathbf{R}(t,z)\big)\cdot\big(-x'(t)\,\hat{\boldsymbol{\jmath}} + y'(t)\,\hat{\boldsymbol{\imath}}\big) \\
&= \int_a^b \mathrm{d}t \int_0^1 \mathrm{d}z\ \big(F_2(x(t),y(t))\,\hat{\boldsymbol{\imath}} - F_1(x(t),y(t))\,\hat{\boldsymbol{\jmath}}\big)\cdot\big(-x'(t)\,\hat{\boldsymbol{\jmath}} + y'(t)\,\hat{\boldsymbol{\imath}}\big) \\
&= \int_a^b \mathrm{d}t\ \big[F_2(x(t),y(t))\,y'(t) + F_1(x(t),y(t))\,x'(t)\big] \\
&\qquad\qquad \text{since the integrand is independent of } z \\
&= \oint_C \big[F_1(x,y)\,\mathrm{d}x + F_2(x,y)\,\mathrm{d}y\big]
\end{aligned}$$

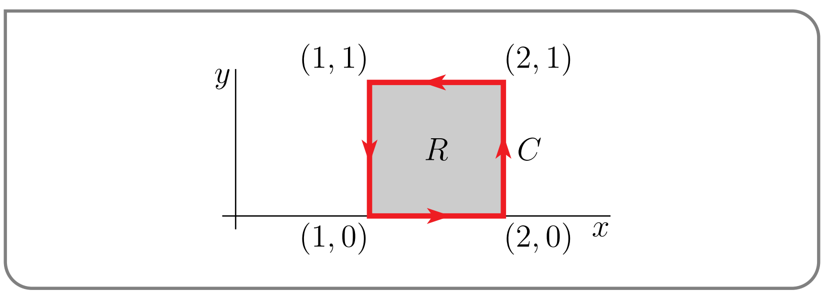
This is exactly the left hand side of Green's theorem. □

Example 4.3.4

Problem: Evaluate

$$\oint_C \big[(x-xy)\,\mathrm{d}x + (y^3+1)\,\mathrm{d}y\big]$$

where C is the curve given in the figure



Solution. Let $R = \{\,(x,y)\,\big|\,1\leqslant x\leqslant 2,\ 0\leqslant y\leqslant 1\,\}$. By Green's theorem

$$\begin{aligned}
\oint_C \big[(x-xy)\,\mathrm{d}x + (y^3+1)\,\mathrm{d}y &= \iint_R \Big[\frac{\partial}{\partial x}(y^3+1) - \frac{\partial}{\partial y}(x-xy)\Big]\mathrm{d}x\mathrm{d}y \\
&= \int_1^2 \mathrm{d}x \int_0^1 \mathrm{d}y\ x = \frac{x^2}{2}\bigg|_1^2 = \frac{3}{2}
\end{aligned}$$

Example 4.3.4

Here is a simple corollary of Green's theorem that tells how to compute the area enclosed by a curve in the xy-plane.

Corollary 4.3.5.

Let

- R be a finite region in the xy-plane whose boundary
- C consists of a finite number of piecewise smooth, simple closed curves.
- Orient C (i.e. put arrows on C) so that if you walk along C in the direction of the arrows, then R is on your left.

Then

$$\text{Area}(R) = \oint_C x\,\mathrm{d}y = -\oint_C y\,\mathrm{d}x = \frac{1}{2}\oint_C [x\,\mathrm{d}y - y\,\mathrm{d}x]$$

Proof. This is just Green's theorem applied first with $\mathbf{F} = x\,\hat{\boldsymbol{\jmath}}$, then with $\mathbf{F} = -y\,\hat{\boldsymbol{\imath}}$ and finally with $\mathbf{F} = \frac{1}{2}\big[-y\,\hat{\boldsymbol{\imath}} + x\,\hat{\boldsymbol{\jmath}}\big]$. For all three of these $\mathbf{F}$'s,

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$$

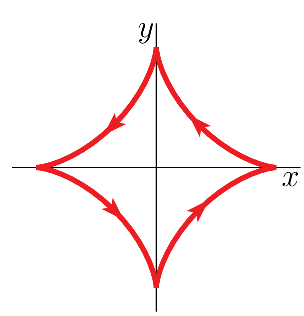
so that Green's theorem gives

$$\oint_C \big[F_1(x,y)\,\mathrm{d}x + F_2(x,y)\,\mathrm{d}y\big] = \iint_R \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\,\mathrm{d}x\mathrm{d}y = \iint_R \mathrm{d}x\mathrm{d}y = \text{Area}(R)$$

□

Example 4.3.6

In this example we will use Green's theorem to compute the area enclosed by the astroid $x^{2/3} + y^{2/3} = a^{2/3}$.



In Example 1.1.9 we found the parametrization

$$\mathbf{r}(t) = x(t)\,\hat{\boldsymbol{\imath}} + y(t)\,\hat{\boldsymbol{\jmath}} = a\cos^3 t\,\hat{\boldsymbol{\imath}} + a\sin^3 t\,\hat{\boldsymbol{\jmath}} \qquad 0 \leqslant t \leqslant 2\pi$$

for the astroid. So, by Corollary 4.3.5,

$$\begin{aligned}
\text{Area} &= \frac{1}{2}\oint_C \big[x\mathrm{d}y - y\mathrm{d}x\big] = \frac{1}{2}\int_0^{2\pi} \big[x(t)y'(t) - y(t)x'(t)\big]\mathrm{d}t \\
&= \frac{3a^2}{2}\int_0^{2\pi} \big[\cos^3 t\sin^2 t\cos t + \sin^3 t\cos^2 t\sin t\big]\mathrm{d}t \\
&= \frac{3a^2}{2}\int_0^{2\pi} \cos^2 t\sin^2 t\big[\cos^2 t + \sin^2 t\big]\mathrm{d}t \\
&= \frac{3a^2}{2}\int_0^{2\pi} \cos^2 t\sin^2 t\ \mathrm{d}t \\
&= \frac{3a^2}{8}\int_0^{2\pi} \sin^2(2t)\ \mathrm{d}t = \frac{3a^2}{16}\int_0^{2\pi}[1-\cos(4t)]\ \mathrm{d}t \\
&= \frac{3}{8}a^2\pi
\end{aligned}$$

Example 4.3.6

Example 4.3.7 (Trick Question)

Problem: Evaluate

$$\oint_C \mathbf{B}\cdot \mathrm{d}\mathbf{r}$$

where

$$\mathbf{B} = \frac{-y\,\hat{\imath} + x\,\hat{\jmath}}{x^2+y^2}$$

and C is the curve

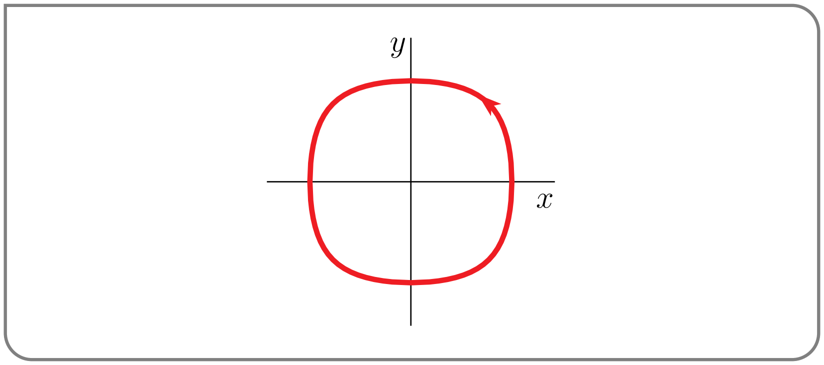
$$\begin{aligned}
x(t) &= \sin(\cos t) \\
y(t) &= \sin(\sin t) \\
z(t) &= 0
\end{aligned}$$

with $0 \leqslant t \leqslant 2\pi$.

Solution. First let's think about the curve C. If the curve were just $X(t) = \cos t$, $Y(t) = \sin t$, $Z(t) = 0$, it would be the unit circle centred on the origin in the xy-plane, traversed counterclockwise. For $-\frac{\pi}{2} \leqslant u \leqslant \frac{\pi}{2}$, the function $\sin u$ increases monotonically with u and is of the same sign as u so that, since $|\sin t|, |\cos t| \leqslant 1 < \frac{\pi}{2}$,

- $x(t) = \sin\big(\cos t\big)$ has the same sign as $X(t) = \cos t$ and is increasing at precisely the same t's as is $X(t)$
- $y(t) = \sin\big(\sin t\big)$ has the same sign as $Y(t) = \sin t$ and is increasing at precisely the same t's as is $Y(t)$

So the extra sine in our parametrization of C just distorts the circle, straightening the sides a little as depicted here.



It looks like our problem is a straightforward Green's theorem problem like Example 4.3.4. Let's just try using the strategy of Example 4.3.4. Because

$$\begin{aligned}\frac{\partial \mathbf{B}_2}{\partial x} - \frac{\partial \mathbf{B}_1}{\partial y} &= \frac{\partial}{\partial x}\frac{x}{x^2+y^2} - \frac{\partial}{\partial y}\frac{-y}{x^2+y^2}\\ &= \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}\\ &= \frac{(x^2+y^2)-2x^2+(x^2+y^2)-2y^2}{(x^2+y^2)^2}\\ &= 0\end{aligned}$$

it looks like Green's theorem gives us, trivially,

$$\oint_C \mathbf{B}\cdot \mathrm{d}\mathbf{r} = \oint_C \left[\mathbf{B}_1\,\mathrm{d}x + \mathbf{B}_2\,\mathrm{d}y\right] = \iint_R \left(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}\right)\,\mathrm{d}x\mathrm{d}y = 0$$

where R is the region inside our curve C.

That was easy — but it's also very **wrong**! Our next steps are to

- verify that $\oint_C \mathbf{B}\cdot\mathrm{d}\mathbf{r} \neq 0$, and
- explain why we got the wrong answer, and
- modify our computation so as to give the correct answer. We'll do this in Example 4.3.8.

Verification that $\oint_C \mathbf{B}\cdot\mathrm{d}\mathbf{r} \neq 0$*:*
Since

$$\begin{aligned}x'(t) &= -\cos(\cos t)\ \sin t\\ y'(t) &= \cos(\sin t)\ \cos t\\ z'(t) &= 0\end{aligned}$$

our integral is

$$\begin{aligned}\oint_C \mathbf{B}\cdot\mathrm{d}\mathbf{r} = \oint_C \left[\mathbf{B}_1\,\mathrm{d}x + \mathbf{B}_2\,\mathrm{d}y\right] &= \int_0^{2\pi}\left[\mathbf{B}_1\big(x(t),y(t)\big)\,x'(t) + \mathbf{B}_2\,\big(x(t),y(t)\big)\,y'(t)\right]\mathrm{d}t\\ &= \int_0^{2\pi}\frac{\sin(\sin t)\ \cos(\cos t)\ \sin t + \sin(\cos t)\ \cos(\sin t)\ \cos t}{\sin^2(\cos t)+\sin^2(\sin t)}\mathrm{d}t\end{aligned}$$

This is a very ugly looking integral[42]. But even if we can't evaluate the integral, we can

42 Indeed!

see that the integrand is strictly positive, and that forces $\oint_C \mathbf{B}\cdot\mathbf{r} > 0$. Because

$$0 \leqslant |\sin t|, |\cos t| \leqslant 1 < \frac{\pi}{2}$$

- $\cos(\cos t) > 0$, and $\sin(\sin t)$ has the same sign as $\sin t$, and $\sin(\sin t)$ is zero if and only if $\sin t = 0$. So the first term in the numerator,

  $$\cos(\cos t)\ \sin(\sin t)\ \sin t \geqslant 0$$

  and is zero if and only if $\sin t = 0$
- $\cos(\sin t) > 0$, and $\sin(\cos t)$ has the same sign as $\cos t$, and $\sin(\cos t)$ is zero if and only if $\cos t = 0$. So the second term in the numerator,

  $$\cos(\sin t)\ \sin(\cos t)\ \cos t \geqslant 0$$

  and is zero if and only if $\cos t = 0$.
- There is no t for which both $\sin t$ and $\cos t$ are simultaneously zero. So the whole numerator

  $$\sin(\sin t)\ \cos(\cos t)\ \sin t + \sin(\cos t)\ \cos(\sin t)\ \cos t > 0$$

  is strictly positive.

Since the integrand is strictly positive, the integral is strictly positive.

Why we got the wrong answer:

In our initial and wrong calculation above, we assumed that $\frac{\partial B_2}{\partial x}(x, y) - \frac{\partial B_1}{\partial y}(x, y) = 0$ at all points (x, y) of the region R inside C. That's not true. While it is true for most points, it is not true for *all* points. The vector field $\mathbf{B}(x, y)$ is not defined at $(x, y) = (0, 0)$. So $\frac{\partial B_2}{\partial x}(x, y) - \frac{\partial B_1}{\partial y}(x, y)$ is also not defined at $(x, y) = (0, 0)$. That's enough to invalidate Green's theorem. Read the statement of Theorem 4.3.2 again carefully.

Example 4.3.7

Example 4.3.8 (Example 4.3.7, again.)

Problem: Evaluate

$$\oint_C \mathbf{B}\cdot d\mathbf{r}$$

where

$$\mathbf{B} = \frac{-y\,\hat{\boldsymbol{\imath}} + x\,\hat{\boldsymbol{\jmath}}}{x^2 + y^2}$$

and C is the curve

$$x(t) = \sin(\cos t)$$
$$y(t) = \sin(\sin t)$$
$$z(t) = 0$$

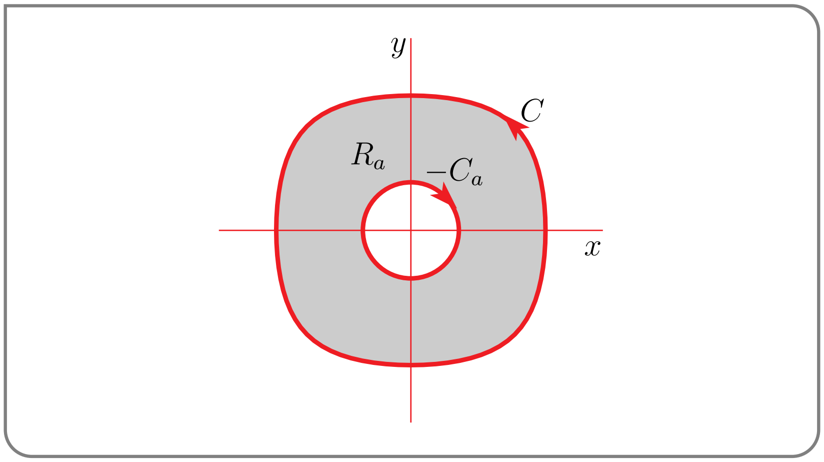
with $0 \leqslant t \leqslant 2\pi$.

Solution. This is the same integral that we computed incorrectly in Example 4.3.7. We'll use two ingredients to compute $\oint_C \mathbf{B}\cdot d\mathbf{r}$ correctly.

- Let $a > 0$ and denote by C_a the counterclockwise oriented circle in the xy-plane that is of radius a and is centered on the origin. We can explicitly compute $\oint_{C_a} \mathbf{B} \cdot d\mathbf{r}$. To do so just parametrize C_a by $x(t) = a\cos t$, $y(t) = a\sin t$, $z(t) = 0$. Then $x'(t) = -a\sin t$, $y'(t) = a\cos t$ and

$$\oint_{C_a} \mathbf{B} \cdot d\mathbf{r} = \int_0^{2\pi} \left[\frac{-a\sin t\,\hat{\imath} + a\cos t\,\hat{\jmath}}{a^2\cos^2 t + a^2\sin^2 t}\right] \cdot \left[-a\sin t\,\hat{\imath} + a\cos t\,\hat{\jmath}\right] dt = \int_0^{2\pi} dt = 2\pi$$

- Pick an a that is small enough that C_a lies entirely inside C and apply Green's theorem with the region, R_a, that is between C and C_a. The curve bounding R_a has two



components — C and C_a, but now C_a is oriented clockwise. (Recall that, in Green's theorem, when you walk along a boundary curve in the direction of the arrow, R_a has to be on your left.). Use $-C_a$ to denote C_a oriented clockwise. $\frac{\partial B_2}{\partial x}(x,y) - \frac{\partial B_1}{\partial y}(x,y)$ really is zero at *all* points (x,y) of the region R_a. So Green's theorem gives

$$\begin{aligned} 0 &= \iint_{R_a} \left(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}\right) dxdy = \oint_C \mathbf{B} \cdot d\mathbf{r} + \oint_{-C_a} \mathbf{B} \cdot d\mathbf{r} \\ &= \oint_C \mathbf{B} \cdot d\mathbf{r} - \oint_{C_a} \mathbf{B} \cdot d\mathbf{r} \end{aligned}$$

and so

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = \oint_{C_a} \mathbf{B} \cdot d\mathbf{r} = 2\pi$$

Example 4.3.8

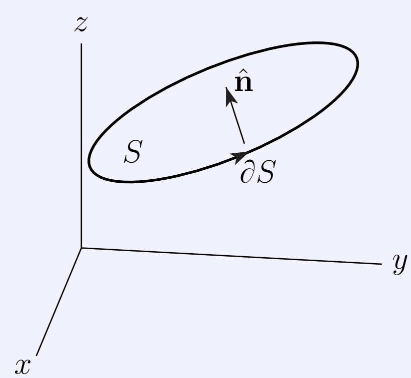
4.4▲ Stokes' Theorem

Our last variant of the fundamental theorem of calculus is Stokes'[43] theorem, which is like Green's theorem, but in three dimensions. It relates an integral over a finite surface in $\mathbb{R}^3$ with an integral over the curve bounding the surface.

Theorem 4.4.1 (Stokes' Theorem).

Let

- S be a piecewise smooth oriented surface (i.e. a unit normal $\hat{\mathbf{n}}$ has been chosen at each point of S and this choice depends continuously on the point)
- the boundary, ∂S, of the surface S consist of a finite number of piecewise smooth, simple curves that are oriented consistently with $\hat{\mathbf{n}}$ in the sense that
  - if you walk along ∂S in the direction of the arrow on ∂S,
  - with the vector from your feet to your head having direction $\hat{\mathbf{n}}$
  - then S is on your left hand side.



- $\mathbf{F}$ be a vector field that has continuous first partial derivatives at every point of S.

Then

$$\oint_{\partial S} \mathbf{F}\cdot d\mathbf{r} = \iint_S \boldsymbol{\nabla}\times\mathbf{F}\cdot\hat{\mathbf{n}}\,dS$$

Note that

- in Stokes' theorem, S must be an oriented surface. In particular, S may not be a Möbius strip. (See Example 3.5.3.)

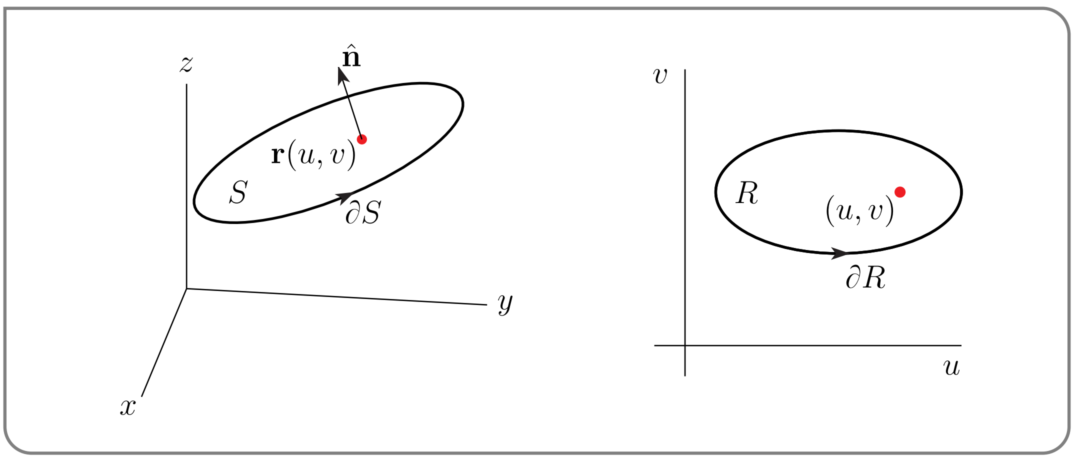
43 Sir George Gabriel Stokes (1819–1903) was an Irish physicist and mathematician. In addition to Stokes' theorem, he is known for the Navier-Stokes equations of fluid dynamics and for his work on the wave theory of light. He gave evidence to the Royal Commission on the Use of Iron in Railway Structures after the Dee bridge disaster of 1847.

- If S is part of the xy-plane, then Stokes' theorem reduces to Green's theorem. Our proof of Stokes' theorem will consist of rewriting the integrals so as to allow an application of Green's theorem.
- If ∂S is simple closed curve and
  - when you look at ∂S from high on the z-axis, it is oriented counterclockwise (look at the figure in Theorem 4.4.1), then
  - $\hat{\mathbf{n}}$ is upward pointing, i.e. has positive z-component, at least near ∂S.

Proof. Write $\mathbf{F} = F_1\,\hat{\boldsymbol{\imath}} + F_2\,\hat{\boldsymbol{\jmath}} + F_3\,\hat{\mathbf{k}}$. Both integrals involve F_1 terms and F_2 terms and F_3 terms. We shall show that the F_1 terms in the two integrals agree. In other words, we shall assume that $\mathbf{F} = F_1\hat{\boldsymbol{\imath}}$. The proofs that the F_2 and F_3 terms also agree are similar. For simplicity, we'll assume[44] that the boundary of S consists of just a single curve, and that we can

- pick a parametrization of S with
$$S = \left\{ \mathbf{r}(u,v) = \big(x(u,v), y(u,v), z(u,v)\big) \;\middle|\; (u,v) \text{ in } R \subset \mathbb{R}^2 \right\}$$
and with $\mathbf{r}(u,v)$ orientation preserving in the sense that $\hat{\mathbf{n}}\,\mathrm{d}S = +\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\,\mathrm{d}u\,\mathrm{d}v$. Also
- pick a parametrization of the curve, ∂R, bounding R as $\big(u(t), v(t)\big)$, $a \leqslant t \leqslant b$, in such a way that when you walk along ∂R in the direction of increasing t, then R is on your left.

Then the curve ∂S bounding S can be parametrized as $\mathbf{R}(t) = \mathbf{r}\big(u(t), v(t)\big)$, $a \leqslant t \leqslant b$.



The orientation of $\mathbf{R}(t)$:

We'll now verify that the direction of increasing t for the parametrization $\mathbf{R}(t)$ of ∂S is the direction of the arrow on ∂S in the figure on the left above. By continuity, it suffices to check the orientation at a single point.

Find a point (u_0, v_0) on ∂R where the forward pointing tangent vector is a positive multiple of $\hat{\boldsymbol{\imath}}$. The horizontal arrow on ∂R in the figure on the left below is at such a point. Suppose that $t = t_0$ at this point — in other words, suppose that $(u_0, v_0) = \big(u(t_0), v(t_0)\big)$.

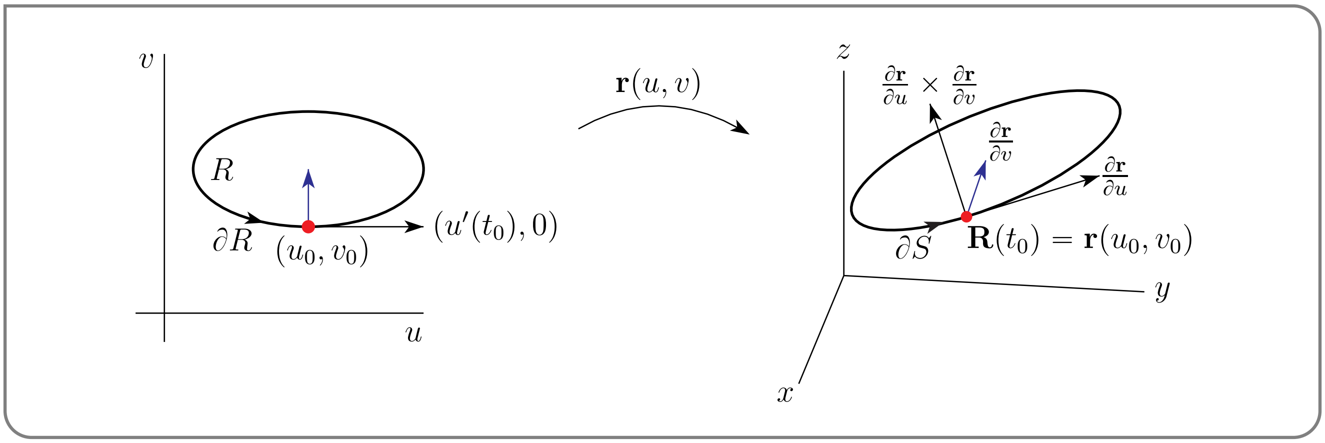
44 Otherwise, decompose S into simpler pieces, analogously to what we did in the proof of the divergence theorem.

Because the forward pointing tangent vector to ∂R at (u_0, v_0), namely $\big(u'(t_0), v'(t_0)\big)$, is a positive multiple of $\hat{\imath}$, we have $u'(t_0) > 0$ and $v'(t_0) = 0$. The tangent vector to ∂S at $\mathbf{R}(t_0) = \mathbf{r}\big(u_0, v_0\big)$, pointing in the direction of increasing t, is

$$\begin{aligned}\mathbf{R}'(t_0) &= \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}\big(u(t), v(t)\big)\big|_{t=t_0} = u'(t_0)\frac{\partial \mathbf{r}}{\partial u}(u_0, v_0) + v'(t_0)\frac{\partial \mathbf{r}}{\partial v}(u_0, v_0) \\ &= u'(t_0)\frac{\partial \mathbf{r}}{\partial u}(u_0, v_0)\end{aligned}$$

and so is a positive multiple of $\frac{\partial \mathbf{r}}{\partial u}(u_0, v_0)$. See the figure on the right below.

If we now walk along a path in the uv-plane which starts at (u_0, v_0), holds u fixed at u_0 and increases v, we move into the interior of R starting at (u_0, v_0). Correspondingly, if we walk along the path, $\mathbf{r}(u_0, v)$, in $\mathbb{R}^3$ with v starting at v_0 and increasing, we move into the interior of S. The forward tangent to this new path, $\frac{\partial \mathbf{r}}{\partial v}(u_0, v_0)$, points from $\mathbf{r}(u_0, v_0)$ into the interior of S. It's the blue arrow in the figure on the right below.



Now imagine that you are walking along ∂S in the direction of increasing t. At time t_0 you are at $\mathbf{R}(t_0)$. You point your right arm straight ahead of you. So it is pointing in the direction $\frac{\partial \mathbf{r}}{\partial u}(u_0, v_0)$. You point your left arm out sideways into the interior of S. It is pointing in the direction $\frac{\partial \mathbf{r}}{\partial v}(u_0, v_0)$. If the direction of increasing t is the same as the forward direction of the orientation of ∂S, then the vector from our feet to our head, which is $\frac{\partial \mathbf{r}}{\partial u}(u_0, v_0) \times \frac{\partial \mathbf{r}}{\partial v}(u_0, v_0)$, should be pointing in the same direction as $\hat{\mathbf{n}}$. And since $\hat{\mathbf{n}}\,\mathrm{d}S = +\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\,\mathrm{d}u\,\mathrm{d}v$, it is.

Now, with our parametrization and orientation sorted out, we can examine the integrals.

The surface integral:

Since $F = F_1\,\hat{\imath}$, so that

$$\boldsymbol{\nabla} \times \mathbf{F} = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & 0 & 0 \end{bmatrix} = \left(0, \frac{\partial F_1}{\partial z}, -\frac{\partial F_1}{\partial y}\right)$$

and

$$\hat{\mathbf{n}}\,\mathrm{d}S = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\,\mathrm{d}u\,\mathrm{d}v = \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{bmatrix}$$
$$= \left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}\right)\hat{\boldsymbol{\imath}} + \left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right)\hat{\boldsymbol{\jmath}} + \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)\hat{\mathbf{k}}$$

and

$$\iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S = \iint_R \left(0, \frac{\partial F_1}{\partial z}, -\frac{\partial F_1}{\partial y}\right) \cdot \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\,\mathrm{d}u\,\mathrm{d}v$$
$$= \iint_R \left\{\frac{\partial F_1}{\partial z}\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right) - \frac{\partial F_1}{\partial y}\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)\right\}\mathrm{d}u\,\mathrm{d}v$$

Now we examine the line integral and show that it equals this one.

The line integral:

$$\oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}\Big(\mathbf{r}\big(u(t), v(t)\big)\Big) \cdot \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}\big(u(t), v(t)\big)\,\mathrm{d}t$$
$$= \int_a^b \mathbf{F}\Big(\mathbf{r}\big(u(t), v(t)\big)\Big) \cdot \left[\frac{\partial \mathbf{r}}{\partial u}\big(u(t), v(t)\big)\frac{\mathrm{d}u}{\mathrm{d}t}(t) + \frac{\partial \mathbf{r}}{\partial v}\big(u(t), v(t)\big)\frac{\mathrm{d}v}{\mathrm{d}t}(t)\right]\mathrm{d}t$$

We can write this as the line integral

$$\oint_{\partial R} M(u,v)\,\mathrm{d}u + N(u,v)\,\mathrm{d}v = \int_a^b \left[M\big(u(t), v(t)\big)\,\frac{\mathrm{d}u}{\mathrm{d}t}(t) + N\big(u(t), v(t)\big)\,\frac{\mathrm{d}v}{\mathrm{d}t}(t)\right]\mathrm{d}t$$

around ∂R, if we choose

$$M(u,v) = \mathbf{F}\big(\mathbf{r}(u,v)\big) \cdot \frac{\partial \mathbf{r}}{\partial u}(u,v) = F_1\big(x(u,v), y(u,v), z(u,v)\big)\frac{\partial x}{\partial u}(u,v)$$
$$N(u,v) = \mathbf{F}\big(\mathbf{r}(u,v)\big) \cdot \frac{\partial \mathbf{r}}{\partial v}(u,v) = F_1\big(x(u,v), y(u,v), z(u,v)\big)\frac{\partial x}{\partial v}(u,v)$$

Finally, we show that the surface integral equals the line integral:

By Green's Theorem, we have

$$\oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \oint_{\partial R} M(u,v)\,\mathrm{d}u + N(u,v)\,\mathrm{d}v$$
$$= \iint_R \left\{\frac{\partial N}{\partial u} - \frac{\partial M}{\partial v}\right\}\mathrm{d}u\mathrm{d}v$$
$$= \iint_R \left\{\frac{\partial}{\partial u}\big[F_1\big(x(u,v), y(u,v), z(u,v)\big)\big]\frac{\partial x}{\partial v} + F_1\frac{\partial^2 x}{\partial u \partial v}\right.$$
$$\left. -\frac{\partial}{\partial v}\big[F_1\big(x(u,v), y(u,v), z(u,v)\big)\big]\frac{\partial x}{\partial u} - F_1\frac{\partial^2 x}{\partial v \partial u}\right\}\mathrm{d}u\mathrm{d}v$$

$$
\begin{aligned}
&= \iint\limits_R \Big\{ \Big(\frac{\partial F_1}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial F_1}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial u} \Big) \frac{\partial x}{\partial v} + F_1 \frac{\partial^2 x}{\partial u \partial v} \\
&\qquad - \Big(\frac{\partial F_1}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial F_1}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial v} \Big) \frac{\partial x}{\partial u} - F_1 \frac{\partial^2 x}{\partial v \partial u} \Big\} \, \mathrm{d}u\mathrm{d}v \\
&= \iint\limits_R \Big\{ \Big(\frac{\partial F_1}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial u} \Big) \frac{\partial x}{\partial v} - \Big(\frac{\partial F_1}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial v} \Big) \frac{\partial x}{\partial u} \Big\} \, \mathrm{d}u \, \mathrm{d}v \\
&= \iint\limits_S \boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}S
\end{aligned}
$$

which is the conclusion that we wanted. □

Before we move on to some examples, here are a couple of remarks.

- Stokes' theorem says that $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S \boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}S$ for any (suitably oriented) surface whose boundary is C. So if S_1 and S_2 are two different (suitably oriented) surfaces having the same boundary curve C, then

$$
\iint\limits_{S_1} \boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}S = \iint\limits_{S_2} \boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}S
$$

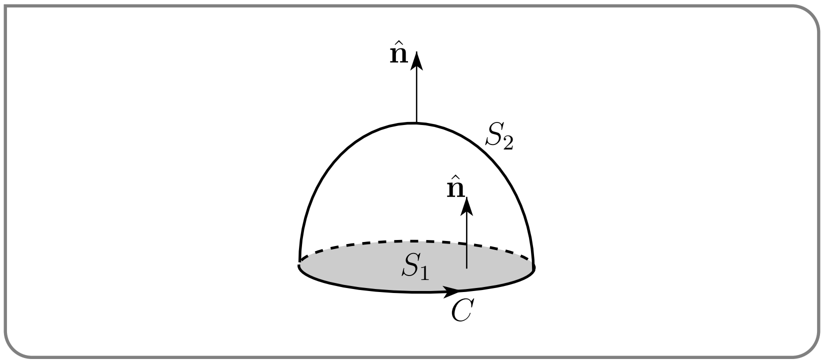
  For example, if C is the unit circle

$$
C = \{ (x, y, z) \mid x^2 + y^2 = 1, \ z = 0 \}
$$

  oriented counterclockwise when viewed from above, then both

$$
\begin{aligned}
S_1 &= \{ (x, y, z) \mid x^2 + y^2 \leqslant 1, \ z = 0 \} \\
S_2 &= \{ (x, y, z) \mid z \geqslant 0, \ x^2 + y^2 + z^2 = 1 \}
\end{aligned}
$$

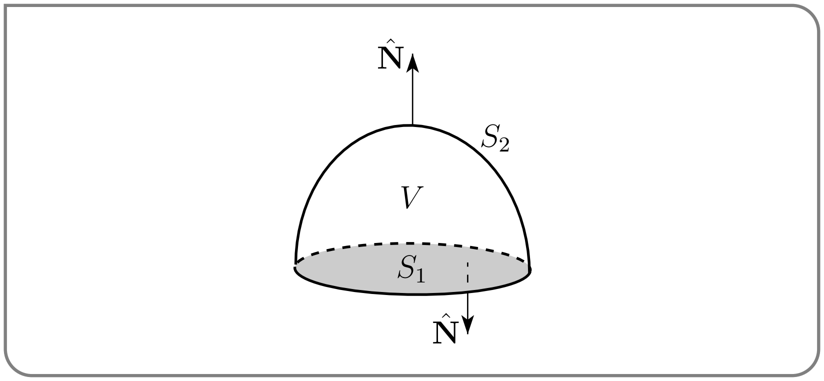
  with upward pointing unit normal vectors, have boundary C. So Stokes' tells us that $\iint_{S_1} \boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}S = \iint_{S_2} \boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}S$.



  It should not be a surprise that $\iint_{S_1} \boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}S = \iint_{S_2} \boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}S$, for the following reason. Let

$$
V = \{ (x, y, z) \mid x^2 + y^2 + z^2 \leqslant 1, \ z \geqslant 0 \}
$$

  be the solid between S_1 and S_2. The boundary ∂V of V is the union of S_1 and S_2.



But beware that the outward pointing normal to ∂V (call it $\hat{\mathbf{N}}$) is $+\hat{\mathbf{n}}$ on S_2 and $-\hat{\mathbf{n}}$ on S_1. So the divergence theorem gives

$$\begin{aligned}\iint_{S_2} \boldsymbol{\nabla}\times\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S - \iint_{S_1} \boldsymbol{\nabla}\times\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \iint_{S_2} \boldsymbol{\nabla}\times\mathbf{F}\cdot\hat{\mathbf{N}}\,\mathrm{d}S + \iint_{S_1} \boldsymbol{\nabla}\times\mathbf{F}\cdot\hat{\mathbf{N}}\,\mathrm{d}S \\ &= \iint_{\partial V} \boldsymbol{\nabla}\times\mathbf{F}\cdot\hat{\mathbf{N}}\,\mathrm{d}S \\ &= \iiint_V \boldsymbol{\nabla}\cdot\big(\boldsymbol{\nabla}\times\mathbf{F}\big)\,\mathrm{d}V \qquad \text{by the divergence theorem} \\ &= 0\end{aligned}$$

by the vector identity Theorem 4.1.7.a.

- As a second remark, suppose that the vector field $\mathbf{F}$ obeys $\boldsymbol{\nabla}\times\mathbf{F} = \mathbf{0}$ everywhere. Then Stokes' theorem forces $\oint_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = 0$ are around all closed curves C, which implies that $\mathbf{F}$ is conservative, by Theorem 2.4.7. So Stokes' theorem provides another proof of Theorem 2.4.8.

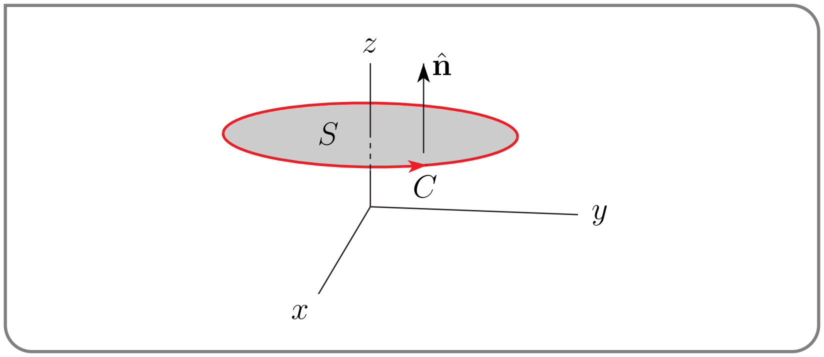
Here is an easy example which shows that Stokes' can be very useful when $\boldsymbol{\nabla}\times\mathbf{F}$ simplifies.

Example 4.4.2

Problem: Evaluate $\oint_C \mathbf{F}\cdot\mathrm{d}\mathbf{r}$ where $\mathbf{F} = \big[2z + \sin\big(x^{146}\big)\big]\,\hat{\boldsymbol{\imath}} - 5z\,\hat{\boldsymbol{\jmath}} - 5y\,\hat{\mathbf{k}}$ and the curve C is the circle $x^2 + y^2 = 4$, $z = 1$, oriented counterclockwise when viewed from above.

Solution. The x^{146} in $\mathbf{F}$ will probably make a direct evaluation of the integral difficult. So we'll use Stokes' theorem. To do so we need a surface S with $\partial S = C$. The simplest is just the flat disk

$$S = \big\{\ (x,y,z)\ \big|\ x^2 + y^2 \leqslant 4,\ \ z = 1\ \big\}$$



Since

$$\begin{aligned}\nabla\times\mathbf{F} &= \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2z+\sin\left(x^{146}\right) & -5z & -5y\end{bmatrix}\\ &= \hat{\imath}\det\begin{bmatrix}\frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ -5z & -5y\end{bmatrix} - \hat{\jmath}\det\begin{bmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial z}\\ 2z+\sin\left(x^{146}\right) & -5y\end{bmatrix} + \hat{\mathbf{k}}\det\begin{bmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\\ 2z+\sin\left(x^{146}\right) & -5z\end{bmatrix}\\ &= 2\hat{\jmath}\end{aligned}$$

and the normal to S is $\hat{\mathbf{k}}$, Stokes' theorem gives

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \nabla\times\mathbf{F}\cdot\hat{\mathbf{n}}\, dS = \iint_S (2\hat{\jmath})\cdot\hat{\mathbf{k}}\, dS = 0$$

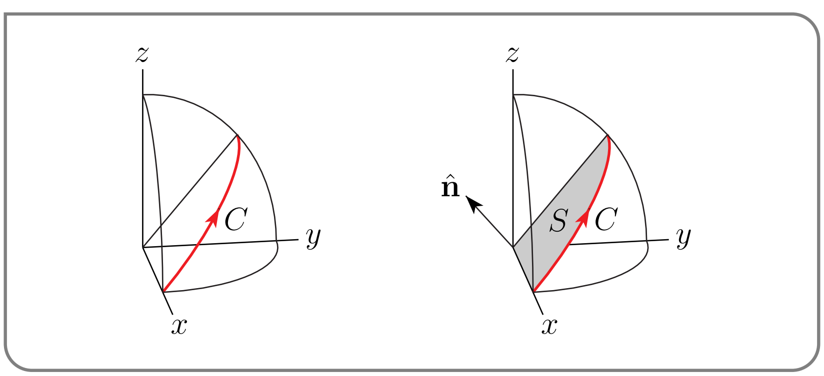
Example 4.4.2

Now we'll repeat the last example with a harder curve.

Example 4.4.3

Problem: Evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F} = \left[2z+\sin\left(x^{146}\right)\right]\hat{\imath} - 5z\,\hat{\jmath} - 5y\,\hat{\mathbf{k}}$ and the curve C is the intersection of $x^2+y^2+z^2 = 4$ and $z = y$, oriented counterclockwise when viewed from above.

Solution. The surface $x^2+y^2+z^2 = 4$ is the sphere of radius 2 centred on the origin and $z = y$ is a plane which contains the origin. So C, being the intersection of a sphere with a plane through the centre of the sphere, is a circle, with centre $(0,0,0)$ and radius 2. The part of the circle in the first octant is sketched on the left below. The x^{146} in $\mathbf{F}$ will probably



make a direct evaluation of the integral difficult. So we'll use Stokes' theorem. To do so we need a surface S with $\partial S = C$. The simplest is the flat disk

$$S = \{ (x,y,z) \mid x^2+y^2+z^2 \leqslant 4,\ \ z = y \}$$

The first octant of S is shown in the figure on the right above. We saw in the last Example 4.4.2 that

$$\nabla \times \mathbf{F} = 2\hat{\boldsymbol{\jmath}}$$

So Stokes' theorem gives

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F}\cdot \hat{\mathbf{n}}\, dS = 2\iint_S \hat{\boldsymbol{\jmath}}\cdot\hat{\mathbf{n}}\, dS$$

We'll evaluate the integral $2\iint_S \hat{\boldsymbol{\jmath}}\cdot\hat{\mathbf{n}}\, dS$ in two ways. The first way is more efficient, but also requires more insight. Since $\nabla(z-y) = \hat{\mathbf{k}} - \hat{\boldsymbol{\jmath}}$, the upward unit normal to the plane $z - y = 0$, and hence to S, is $\hat{\mathbf{n}} = \frac{1}{\sqrt{2}}(\hat{\mathbf{k}} - \hat{\boldsymbol{\jmath}})$. Consequently the integrand

$$\hat{\boldsymbol{\jmath}}\cdot\hat{\mathbf{n}} = \hat{\boldsymbol{\jmath}}\cdot\Big(\frac{-\hat{\boldsymbol{\jmath}}+\hat{\mathbf{k}}}{\sqrt{2}}\Big) = -\frac{1}{\sqrt{2}}$$

is a constant and we do not need a formula for $\hat{\mathbf{n}}\,dS$:

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= 2\iint_S \hat{\boldsymbol{\jmath}}\cdot\hat{\mathbf{n}}\, dS = -\sqrt{2}\iint_S dS = -\sqrt{2}\,\mathrm{Area}(S) = -\sqrt{2}\pi\, 2^2\\ &= -4\sqrt{2}\pi\end{aligned}$$

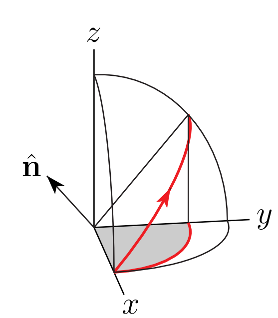
Alternatively, we can evaluate the integral $\iint_S \hat{\boldsymbol{\jmath}}\cdot\hat{\mathbf{n}}\, dS$ using our normal protocol. As S is part of the plane $z = f(x,y) = y$,

$$\hat{\mathbf{n}}\,dS = \pm\big(-f_x\,\hat{\boldsymbol{\imath}} - f_y\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}\big)\,dxdy = \pm(-\hat{\boldsymbol{\jmath}}+\hat{\mathbf{k}})\,dxdy$$

To get the upward pointing normal pointing normal, we take the $+$ sign so that $\hat{\mathbf{n}}\,dS = (-\hat{\boldsymbol{\jmath}}+\hat{\mathbf{k}})\,dxdy$. As (x,y,z) runs over

$$\begin{aligned}S &= \{ (x,y,z) \mid x^2+y^2+z^2 \leqslant 4,\ \ z = y \} = \{ (x,y,z) \mid x^2+2y^2 \leqslant 4,\ \ z = y \}\\ &= \{ (x,y,z) \mid \tfrac{x^2}{4}+\tfrac{y^2}{2} \leqslant 1,\ \ z = y \}\end{aligned}$$

(x,y) runs over the elliptical disk $R = \{ (x,y) \mid \frac{x^2}{4}+\frac{y^2}{2} \leqslant 1 \}$. The part of this ellipse in the first octant is the shaded region in the figure below. This ellipse has semiaxes $a = 2$



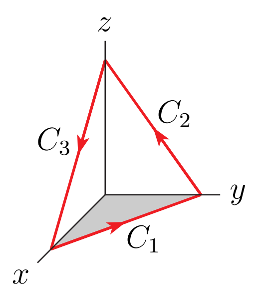
and $b = \sqrt{2}$ and hence area $\pi ab = 2\sqrt{2}\pi$. So

$$\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = 2\iint_S \hat{\boldsymbol{\jmath}}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = 2\iint_R \hat{\boldsymbol{\jmath}}\cdot(-\hat{\boldsymbol{\jmath}}+\hat{\mathbf{k}})\,\mathrm{d}x\mathrm{d}y = -2\iint_R \mathrm{d}x\mathrm{d}y = -2\mathrm{Area}(R)$$
$$= -4\sqrt{2}\pi$$

Example 4.4.3

Example 4.4.4

Problem: Evaluate $\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r}$ where $\mathbf{F} = (x+y)\,\hat{\boldsymbol{\imath}} + 2(x-z)\,\hat{\boldsymbol{\jmath}} + (y^2+z)\,\hat{\mathbf{k}}$ and C is the oriented curve obtained by going from $(2,0,0)$ to $(0,3,0)$ to $(0,0,6)$ and back to $(2,0,0)$ along straight line segments.



Solution 1. In this first solution, we'll evaluate the integral directly. The first line segment (C_1 in the figure above) may be parametrized as

$$\mathbf{r}(t) = (2,0,0) + t\big\{(0,3,0)-(2,0,0)\big\} = \big(2-2t\,,\,3t\,,\,0\big) \qquad 0 \leqslant t \leqslant 1$$

So the integral along this segment is

$$\int_0^1 \mathbf{F}(\mathbf{r}(t))\cdot\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\,\mathrm{d}t = \int_0^1 \big(2+t\,,\,2(2-2t)\,,\,(3t)^2\big)\cdot(-2\,,\,3\,,\,0)\,\mathrm{d}t = \int_0^1 (8-14t)\,\mathrm{d}t$$
$$= \Big[8t-7t^2\Big]_0^1 = 1$$

The second line segment (C_2 in the figure above) may be parametrized as

$$\mathbf{r}(t) = (0,3,0) + t\big\{(0,0,6)-(0,3,0)\big\} = \big(0\,,\,3-3t\,,\,6t\big) \qquad 0 \leqslant t \leqslant 1$$

. So the integral along this segment is

$$\int_0^1 \mathbf{F}(\mathbf{r}(t))\cdot\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\,\mathrm{d}t = \int_0^1 \big(3(1-t)\,,\,-12t\,,\,9(1-t)^2+6t\big)\cdot(0,-3,6)\,\mathrm{d}t$$
$$= \int_0^1 [36t+54(1-t)^2+36t]\,\mathrm{d}t = \Big[18t^2-18(1-t)^3+18t^2\Big]_0^1$$
$$= 54$$

The final line segment (C_3 in the figure above) may be parametrized as

$$\mathbf{r}(t) = (0,0,6) + t\big\{(2,0,0) - (0,0,6)\big\} = (2t\,,\,0\,,\,6-6t) \qquad 0 \leqslant t \leqslant 1$$

So the line integral along this segment is

$$\begin{aligned}\int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\,\mathrm{d}t &= \int_0^1 \big(2t\,,\,4t - 12(1-t)\,,\,6(1-t)\big) \cdot (2,0,-6)\,\mathrm{d}t \\ &= \int_0^1 [4t - 36(1-t)]\,\mathrm{d}t = \Big[2t^2 + 18(1-t)^2\Big]_0^1 = -16\end{aligned}$$

The full line integral is

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 1 + 54 - 16 = 39$$

Solution 2. This time we shall apply Stokes' Theorem. The curl of $\mathbf{F}$ is

$$\nabla \times \mathbf{F} = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2(x-z) & y^2+z \end{bmatrix} = (2y+2)\hat{\imath} - (0-0)\hat{\jmath} + (2-1)\hat{\mathbf{k}} = 2(y+1)\hat{\imath} + \hat{\mathbf{k}}$$

The curve C is a triangle and so is contained in a plane. Any plane has an equation of the form $Ax + By + Cz = D$. Our plane does not pass through the origin (look at the figure above) so the D must be nonzero. Consequently we may divide $Ax + By + Cz = D$ through by D giving an equation of the form $ax + by + cz = 1$.

- Because $(2,0,0)$ lies on the plane, $a = \frac{1}{2}$.
- Because $(0,3,0)$ lies on the plane, $b = \frac{1}{3}$.
- Because $(0,0,6)$ lies on the plane, $c = \frac{1}{6}$.

So the triangle is contained in the plane $\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$. It is the boundary of the surface S that consists of the portion of the plane $\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$ that obeys $x \geqslant 0$, $y \geqslant 0$ and $z \geqslant 0$. Rewrite the equation of the plane as $z = 6 - 3x - 2y$. For this surface

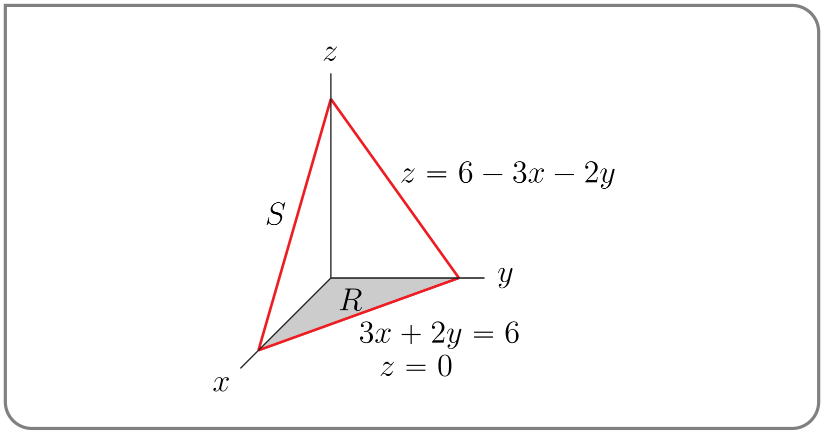
$$\hat{\mathbf{n}}\,\mathrm{d}S = (3\hat{\imath} + 2\hat{\jmath} + \hat{\mathbf{k}})\,\mathrm{d}x\,\mathrm{d}y$$

by (3.3.2), and we can write

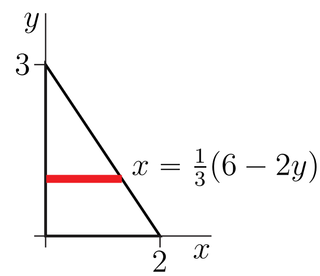
$$\begin{aligned} S &= \big\{\ (x,y,z)\ \big|\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0,\ z = 6 - 3x - 2y\ \big\} \\ &= \big\{\ (x,y,z)\ \big|\ x \geqslant 0, y \geqslant 0,\ 6 - 3x - 2y \geqslant 0,\ z = 6 - 3x - 2y\ \big\}\end{aligned}$$

As (x,y,z) runs over S, (x,y) runs over the triangle

$$\begin{aligned} R &= \big\{\ (x,y,z)\ \big|\ x \geqslant 0,\ y \geqslant 0,\ 3x + 2y \leqslant 6\ \big\} \\ &= \big\{\ (x,y,z)\ \big|\ x \geqslant 0,\ 0 \leqslant y \leqslant \tfrac{3}{2}(2-x)\ \big\}\end{aligned}$$

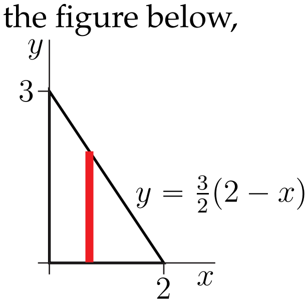


Using horizontal strips as in the figure below,



$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \boldsymbol{\nabla}\times\mathbf{F}\cdot\hat{\mathbf{n}}\, dS$$
$$= \iint_R [2(y+1)\hat{\imath}+\hat{\mathbf{k}}]\cdot[3\hat{\imath}+2\hat{\jmath}+\hat{\mathbf{k}}]\, dx\, dy$$
$$= \iint_R [6y+7]\, dx\, dy = \int_0^3 dy \int_0^{\frac{1}{3}(6-2y)} dx\, [6y+7]$$
$$= \int_0^3 dy\, \frac{1}{3}[6y+7][6-2y] = \frac{1}{3}\int_0^3 dy\, [-12y^2+22y+42]$$
$$= \frac{1}{3}\Big[-4y^3+11y^2+42y\Big]_0^3 = \big[-4\times 9+11\times 3+42\big] = 39$$

Alternatively, using vertical strips as in the figure below,



$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R [6y+7]\, dx\, dy$$
$$= \int_0^2 dx \int_0^{\frac{3}{2}(2-x)} dy\, [6y+7]$$
$$= \int_0^2 dx\, \Big[3\frac{3^2}{2^2}(2-x)^2 + 7\frac{3}{2}(2-x)\Big] = \Big[-\frac{27}{4}\frac{1}{3}(2-x)^3 - \frac{21}{2}\frac{1}{2}(2-x)^2\Big]_0^2$$
$$= \frac{9}{4}8 + \frac{21}{4}4 = 39$$

Example 4.4.4

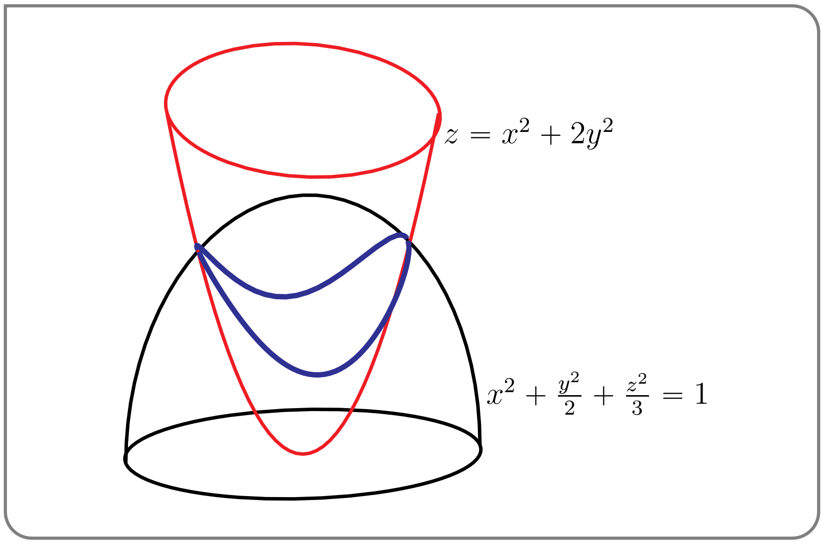
Example 4.4.5

Problem: Evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F} = (\cos x + y + z)\,\hat{\imath} + (x+z)\,\hat{\jmath} + (x+y)\,\hat{\mathbf{k}}$ and C is the intersection of the surfaces

$$x^2 + \frac{y^2}{2} + \frac{z^2}{3} = 1 \qquad \text{and} \qquad z = x^2 + 2y^2$$

oriented counterclockwise when viewed from above.

Solution. First, let's sketch the curve. $x^2 + \frac{y^2}{2} + \frac{z^2}{3} = 1$ is an ellipsoid centred on the origin and $z = x^2 + 2y^2$ is an upward opening paraboloid that passes through the origin. They are sketched in the figure below. The paraboloid is red. Their intersection, the curve C, is



the blue curve in the figure. It looks like a deformed[45] circle.

One could imagine parametrizing C. For example, substituting $x^2 = z - 2y^2$ into the equation of the ellipsoid gives $-\frac{3}{2}y^2 + \frac{1}{3}\left(z + \frac{3}{2}\right)^2 = \frac{7}{4}$. This can be solved to give y as a function of z and then $x^2 = z - 2y^2$ also gives x as a function of z. However this would clearly yield, at best, a really messy integral. So let's try Stokes' theorem.

In fact, since

$$\begin{aligned} \nabla \times \mathbf{F} &= \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos x + y + z & x + z & x + y \end{bmatrix} \\ &= \hat{\imath}(1-1) - \hat{\jmath}(1-1) + \hat{\mathbf{k}}(1-1) \\ &= \mathbf{0} \end{aligned}$$

This $\mathbf{F}$ is conservative! (In fact $\mathbf{F} = \nabla\big(\sin x + xy + xz + yz\big)$.) As C is a closed curve, $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$.

Example 4.4.5

Example 4.4.6

Problem: Evaluate $\iint_S \mathbf{G} \cdot \hat{\mathbf{n}}\, dS$ where $\mathbf{G} = (2x)\,\hat{\imath} + (2z - 2x)\,\hat{\jmath} + (2x - 2z)\,\hat{\mathbf{k}}$ and

$$S = \left\{ (x, y, z) \mid z = \big(1 - x^2 - y^2\big)\big(1 - y^3\big) \cos x \, e^y,\ x^2 + y^2 \leqslant 1 \right\}$$

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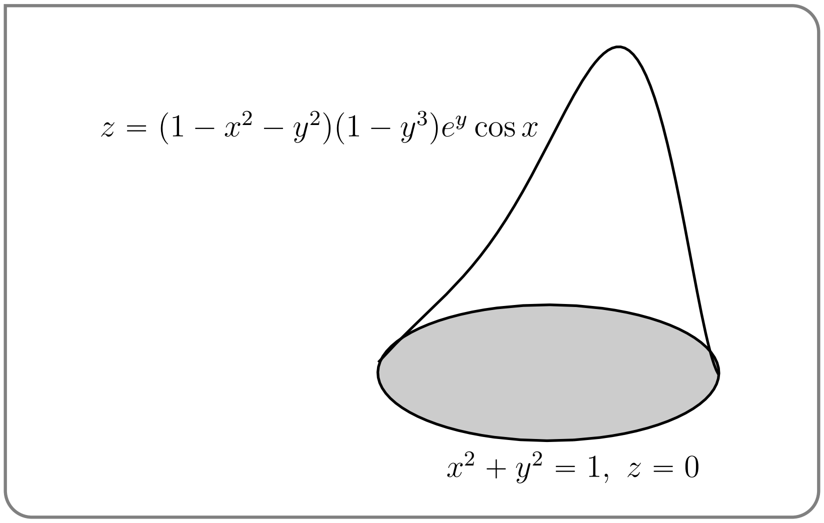
with upward pointing normal

Solution 1. The surface S is sketched below. It is a pretty weird surface. About the only



simple thing about it is that its boundary, ∂S, is the circle $x^2 + y^2 = 1, z = 0$. It is clear that we should not try to evaluate the integral directly[46]. In this solution we will combine the divergence theorem with the observation that

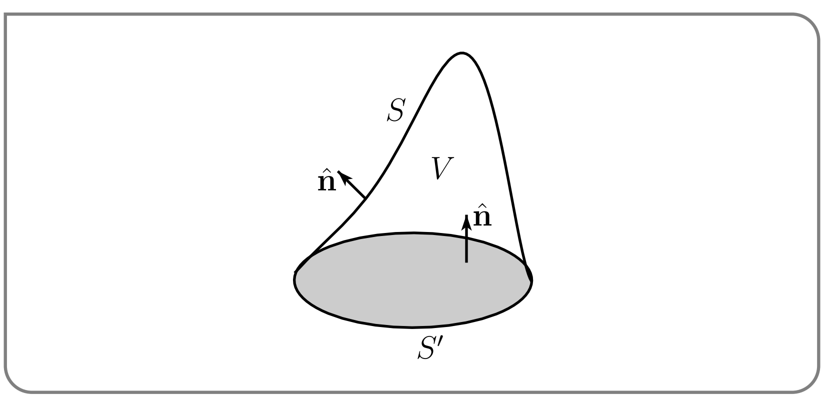
$$\nabla \cdot \mathbf{G} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(2z - 2x) + \frac{\partial}{\partial z}(2x - 2z) = 0$$

to avoid ever having work with the surface S. Here is an outline of what we will do.

- We first select a simple surface S' whose boundary $\partial S'$ is also the circle $x^2 + y^2 = 1$, $z = 0$. A nice simple choice of S', and the surface that we will use, is the disk

$$S' = \{ \ (x, y, z) \mid x^2 + y^2 = 1, \ z = 0 \ \}$$

- Then we define V to be the solid whose top surface is S and whose bottom surface is S'. So the boundary of V is the union of S and S'.



46 That way lies pain.

- For S', we will use the upward pointing normal $\hat{\mathbf{n}} = \hat{\mathbf{k}}$, which is *minus* the outward pointing normal to ∂V on S'. So the divergence theorem says that

$$\iiint_V \nabla \cdot \mathbf{G}\, \mathrm{d}V = \iint_S \mathbf{G} \cdot \hat{\mathbf{n}}\, \mathrm{d}S - \iint_{S'} \mathbf{G} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$$

  The left hand side is zero because, as we have already seen, $\nabla \cdot \mathbf{G} = 0$. So

$$\iint_S \mathbf{G} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \iint_{S'} \mathbf{G} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$$

- Finally, we compute $\iint_{S'} \mathbf{G} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$.

We saw an argument like this (with $\mathbf{G} = \nabla \times \mathbf{F}$) in the first remark following the proof of Theorem 4.4.1.

So all that we have to do now is compute

$$\iint_S \mathbf{G} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \iint_{S'} \mathbf{G} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \iint_{S'} \mathbf{G} \cdot \hat{\mathbf{k}}\, \mathrm{d}S = \iint_{\substack{x^2+y^2\leqslant 1\\ z=0}} (2x - 2z)\, \mathrm{d}x\mathrm{d}y = \iint_{\substack{x^2+y^2\leqslant 1\\ z=0}} (2x)\, \mathrm{d}x\mathrm{d}y$$
$$= 0$$

simply because the integrand is odd under $x \to -x$.

Solution 2. In this second solution we'll use Stokes' theorem instead of the divergence theorem. To do so, we have to express $\mathbf{G}$ in the form $\nabla \times \mathbf{F}$. So the first thing to do is to check if $\mathbf{G}$ passes the screening test, Theorem 4.1.12, for the existence of vector potentials. That is, to check if $\nabla \cdot \mathbf{G} = 0$. It is. We saw this in Solution 1 above.

Next, we have to find a vector potential. In fact, we have already found, in Example 4.1.15, that

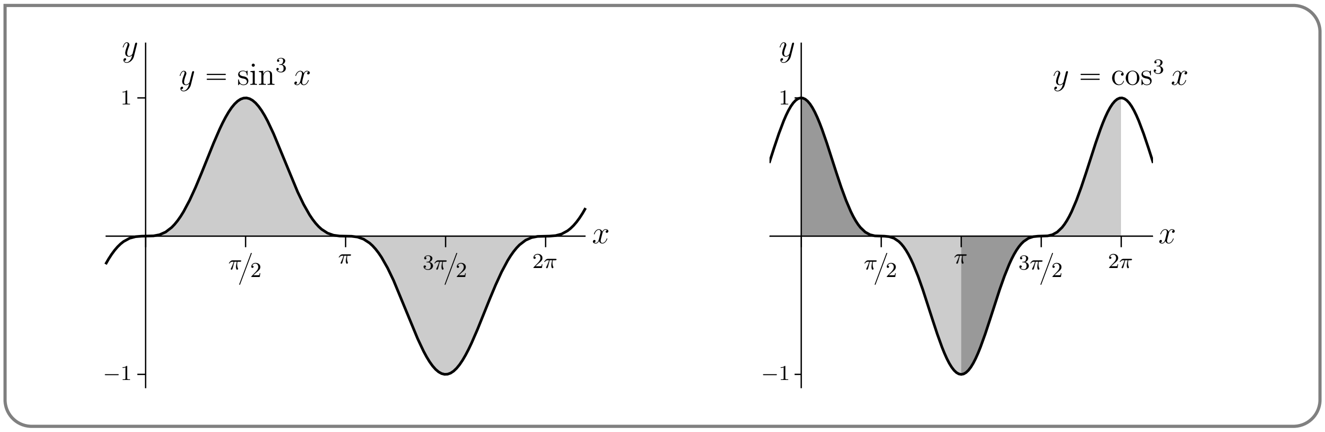
$$\mathbf{F} = (z^2 - 2xz)\hat{\imath} + (x^2 - 2xz)\hat{\jmath}$$

is a vector potential for $\mathbf{G}$, which we can quickly check.

Parametrizing C by $\mathbf{r}(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath}$, $0 \leqslant t \leqslant 2\pi$, Stokes' theorem gives (recalling that $z = 0$ on C so that $\mathbf{F}\big(\mathbf{r}(t)\big) = x^2\,\hat{\jmath}\Big|_{x=\cos t} = \cos^2 t$)

$$\iint_S \mathbf{G} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^{2\pi} \mathbf{F}\big(\mathbf{r}(t)\big) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\, \mathrm{d}t$$
$$= \int_0^{2\pi} \big(\cos^2 t\big)(\cos t)\, \mathrm{d}t$$

Of course this integral can be evaluated by using that one antiderivative of the integrand $\cos^3 t = \big(1 - \sin^2 t\big)\cos t$ is $\sin t - \frac{1}{3}\sin^3 t$ and that this antiderivative is zero at $t = 0$ and at $t = 2\pi$. But it is easier to observe that the integral of any odd power of $\sin t$ or $\cos t$ over any full period is zero. Look, for example, at the graphs of $\sin^3 x$ and $\cos^3 x$, below.



Either way

$$\iint_S \mathbf{G} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = 0$$

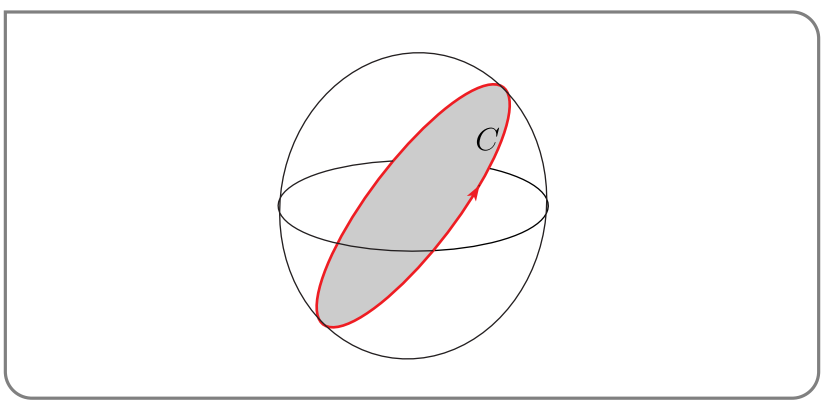
Example 4.4.6

Example 4.4.7

In this example we compute, in three different ways, $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ where

$$\mathbf{F} = (z - y)\,\hat{\boldsymbol{\imath}} - (x + z)\,\hat{\boldsymbol{\jmath}} - (x + y)\,\hat{\mathbf{k}}$$

and C is the curve $x^2 + y^2 + z^2 = 4$, $z = y$ oriented counterclockwise when viewed from above.

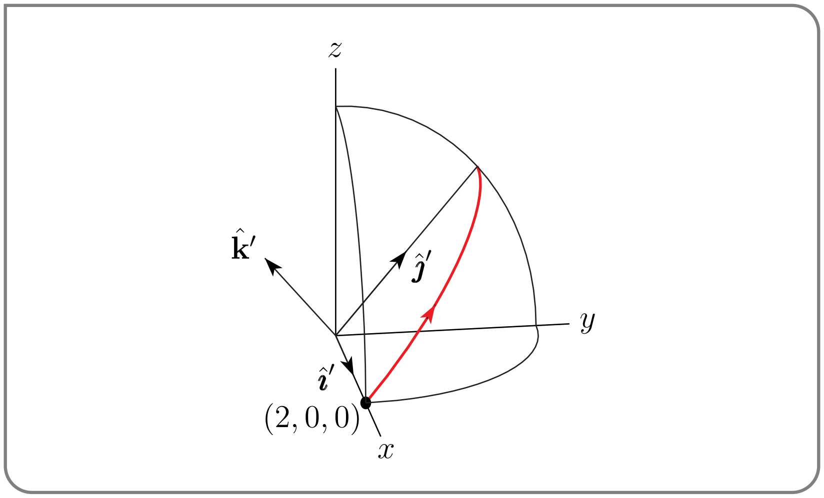


Direct Computation:
In this first computation, we parametrize the curve C and compute $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ directly. The plane $z = y$ passes through the origin, which is the centre of the sphere $x^2 + y^2 + z^2 = 4$. So C is a circle which, like the sphere, has radius 2 and centre $(0, 0, 0)$. We use a parametrization of the form

$$\mathbf{r}(t) = \mathbf{c} + \rho \cos t\, \hat{\boldsymbol{\imath}}' + \rho \sin t\, \hat{\boldsymbol{\jmath}}' \qquad 0 \leqslant t \leqslant 2\pi$$

where

- $\mathbf{c} = (0,0,0)$ is the centre of C,
- $\rho = 2$ is the radius of C and
- $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$ are two vectors that
  (a) are unit vectors,
  (b) are parallel to the plane $z = y$ and
  (c) are mutually perpendicular.



The trickiest part is finding suitable vectors $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$:

- The point $(2,0,0)$ satisfies both $x^2+y^2+z^2=4$ and $z=y$ and so is on C. We may choose $\hat{\boldsymbol{\imath}}'$ to be the unit vector in the direction from the centre $(0,0,0)$ of the circle towards $(2,0,0)$. Namely $\hat{\boldsymbol{\imath}}' = (1,0,0)$.
- Since the plane of the circle is $z-y=0$, the vector $\boldsymbol{\nabla}(z-y) = (0,-1,1)$ is perpendicular to the plane of C. So $\hat{\mathbf{k}}' = \frac{1}{\sqrt{2}}(0,-1,1)$ is a unit vector normal to $z=y$. Then $\hat{\boldsymbol{\jmath}}' = \hat{\mathbf{k}}' \times \hat{\boldsymbol{\imath}}' = \frac{1}{\sqrt{2}}(0,-1,1)\times(1,0,0) = \frac{1}{\sqrt{2}}(0,1,1)$ is a unit vector that is perpendicular to $\hat{\boldsymbol{\imath}}'$ and $\hat{\mathbf{k}}'$. Since $\hat{\boldsymbol{\jmath}}'$ is perpendicular to $\hat{\mathbf{k}}'$, it is parallel to $z=y$.

Substituting in $\mathbf{c} = (0,0,0)$, $\rho = 2$, $\hat{\boldsymbol{\imath}}' = (1,0,0)$ and $\hat{\boldsymbol{\jmath}}' = \frac{1}{\sqrt{2}}(0,1,1)$ gives

$$\mathbf{r}(t) = 2\cos t\,(1,0,0) + 2\sin t\,\frac{1}{\sqrt{2}}(0,1,1) = 2\Big(\cos t, \frac{\sin t}{\sqrt{2}}, \frac{\sin t}{\sqrt{2}}\Big) \qquad 0 \leqslant t \leqslant 2\pi$$

To check that this parametrization is correct, note that $x = 2\cos t$, $y = \sqrt{2}\sin t$, $z = \sqrt{2}\sin t$ satisfies both $x^2+y^2+z^2=4$ and $z=y$.

At $t=0$, $\mathbf{r}(0) = (2,0,0)$. As t increases, $z(t) = \sqrt{2}\sin t$ increases and $\mathbf{r}(t)$ moves upwards towards $\mathbf{r}\big(\frac{\pi}{2}\big) = (0,\sqrt{2},\sqrt{2})$. This is the desired counterclockwise direction (when viewed from above). Now that we have a parametrization, we can set up the integral.

$$\begin{aligned}
\mathbf{r}(t) &= \big(2\cos t, \sqrt{2}\sin t, \sqrt{2}\sin t\big)\\
\mathbf{r}'(t) &= \big(-2\sin t, \sqrt{2}\cos t, \sqrt{2}\cos t\big)\\
\mathbf{F}\big(\mathbf{r}(t)\big) &= \big(z(t)-y(t), -x(t)-z(t), -x(t)-y(t)\big)\\
&= \big(\sqrt{2}\sin t - \sqrt{2}\sin t, -2\cos t - \sqrt{2}\sin t, -2\cos t - \sqrt{2}\sin t\big)\\
&= -\big(0, 2\cos t + \sqrt{2}\sin t, 2\cos t + \sqrt{2}\sin t\big)\\
\mathbf{F}\big(\mathbf{r}(t)\big)\cdot\mathbf{r}'(t) &= -\big[4\sqrt{2}\cos^2 t + 4\cos t\sin t\big] = -\big[2\sqrt{2}\cos(2t) + 2\sqrt{2} + 2\sin(2t)\big]
\end{aligned}$$

by the double angle formulae $\sin(2t) = 2\sin t\,\cos t$ and $\cos(2t) = 2\cos^2 t - 1$. So

$$\begin{aligned}\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \int_0^{2\pi} \mathbf{F}\big(\mathbf{r}(t)\big)\cdot \mathbf{r}'(t)\,\mathrm{d}t \\ &= \int_0^{2\pi} -\big[2\sqrt{2}\cos(2t) + 2\sqrt{2} + 2\sin(2t)\big]\,\mathrm{d}t \\ &= -\Big[\sqrt{2}\sin(2t) + 2\sqrt{2}t - \cos(2t)\Big]_0^{2\pi} \\ &= -4\sqrt{2}\pi\end{aligned}$$

Oof! Let's do it an easier way.

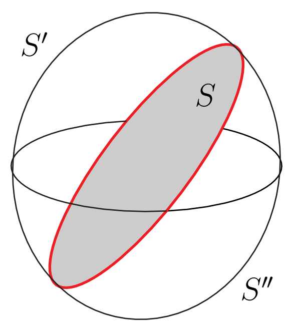
Stokes' Theorem

To apply Stokes' theorem we need to express C as the boundary ∂S of a surface S. As

$$C = \{\ (x,y,z)\ |\ x^2+y^2+z^2 = 4,\ z = y\ \}$$

is a closed curve, this is possible. In fact there are many possible choices of S with $\partial S = C$. Three possible S's (sketched below) are

$$\begin{aligned}S &= \{\ (x,y,z)\ |\ x^2+y^2+z^2 \leqslant 4,\ z = y\ \} \\ S' &= \{\ (x,y,z)\ |\ x^2+y^2+z^2 = 4,\ z \geqslant y\ \} \\ S'' &= \{\ (x,y,z)\ |\ x^2+y^2+z^2 = 4,\ z \leqslant y\ \}\end{aligned}$$



The first of these, which is part of a plane, is likely to lead to simpler computations than the last two, which are parts of a sphere. So we choose what looks like the simpler way.

In preparation for application of Stokes' theorem, we compute $\nabla\times\mathbf{F}$ and $\hat{\mathbf{n}}\,\mathrm{d}S$. For the latter, we apply the formula $\hat{\mathbf{n}}\,\mathrm{d}S = \pm(-f_x, -f_y, 1)\,\mathrm{d}x\mathrm{d}y$ (of Equation (3.3.2)) to the surface $z = f(x,y) = y$. We use the $+$ sign to give the normal a positive $\hat{\mathbf{k}}$ component.

$$\begin{aligned}\nabla\times\mathbf{F} &= \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z-y & -x-z & -x-y\end{bmatrix} \\ &= \hat{\imath}\big(-1-(-1)\big) - \hat{\jmath}\big(-1-1\big) + \hat{\mathbf{k}}\big(-1-(-1)\big) \\ &= 2\hat{\jmath} \\ \hat{\mathbf{n}}\,\mathrm{d}S &= (0,-1,1)\,\mathrm{d}x\mathrm{d}y \\ \nabla\times\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= (0,2,0)\cdot(0,-1,1)\,\mathrm{d}x\mathrm{d}y = -2\,\mathrm{d}x\mathrm{d}y\end{aligned}$$

The integration variables are x and y and, by definition, the domain of integration is

$$R = \{\ (x,y)\ \big|\ (x,y,z) \text{ is in } S \text{ for some } z\ \}$$

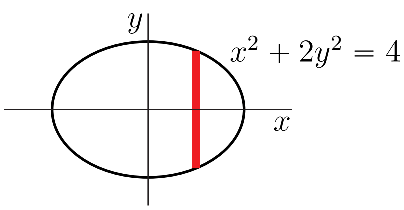
To determine precisely what this domain of integration is, we observe that since $z = y$ on S, $x^2 + y^2 + z^2 \leqslant 4$ is the same as $x^2 + 2y^2 \leqslant 4$ on S, so that

$$S = \{\ (x,y,z)\ \big|\ x^2 + 2y^2 \leqslant 4,\ z = y\ \} \implies R = \{\ (x,y)\ \big|\ x^2 + 2y^2 \leqslant 4\ \}$$

So the domain of integration is an ellipse with semimajor axis $a = 2$, semiminor axis $b = \sqrt{2}$ and area $\pi ab = 2\sqrt{2}\pi$. The integral is then

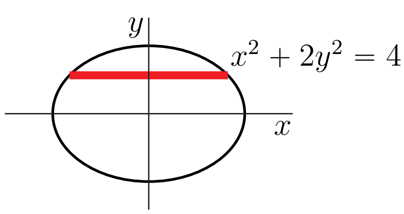
$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S \boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \iint_R (-2)\, \mathrm{d}x\mathrm{d}y = -2\ \text{Area}\,(R) = -4\sqrt{2}\pi$$

Remark (Limits of integration):
If the integrand were more complicated, we would have to evaluate the integral over R by expressing it as an iterated integrals with the correct limits of integration. First suppose that we slice up R using thin vertical slices. On each such slice, x is essentially constant and y runs from $-\sqrt{(4-x^2)/2}$ to $\sqrt{(4-x^2)/2}$. The leftmost such slice would have $x = -2$ and the rightmost such slice would have $x = 2$. So the correct limits with this slicing are



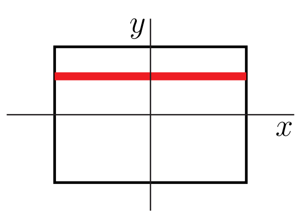
$$\iint_R f(x,y)\, \mathrm{d}x\mathrm{d}y = \int_{-2}^{2} \mathrm{d}x \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \mathrm{d}y\ f(x,y)$$

If, instead, we slice up R using thin horizontal slices, then, on each such slice, y is essentially constant and x runs from $-\sqrt{4-2y^2}$ to $\sqrt{4-2y^2}$. The bottom such slice would have $y = -\sqrt{2}$ and the top such slice would have $y = \sqrt{2}$. So the correct limits with this slicing are



$$\iint_R f(x,y)\, \mathrm{d}x\mathrm{d}y = \int_{-\sqrt{2}}^{\sqrt{2}} \mathrm{d}y \int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} \mathrm{d}x\ f(x,y)$$

Note that the integral with limits



$$\int_{-\sqrt{2}}^{\sqrt{2}} \mathrm{d}y \int_{-2}^{2} \mathrm{d}x\ f(x,y)$$

corresponds to a slicing with x running from -2 to 2 on **every** slice. This corresponds to a rectangular domain of integration, not what we have here.

Stokes' Theorem, Again:

Since the integrand is just a constant (after Stoking — not the original integrand) and S is so simple (because we chose it wisely), we can evaluate the integral $\iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$ without ever determining $\mathrm{d}S$ explicitly and without ever setting up any limits of integration. We already know that $\nabla \times \mathbf{F} = 2\hat{\boldsymbol{\jmath}}$. Since S is the level surface $z - y = 0$, the gradient $\nabla(z-y) = -\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}$ is normal to S. So $\hat{\mathbf{n}} = \frac{1}{\sqrt{2}}(-\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}})$ and

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \iint_S (2\hat{\boldsymbol{\jmath}}) \cdot \frac{1}{\sqrt{2}}(-\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}})\, \mathrm{d}S = \iint_S -\sqrt{2}\, \mathrm{d}S = -\sqrt{2}\ \text{Area}\,(S)$$

As S is a circle of radius 2, $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -4\sqrt{2}\pi$, yet again.

Example 4.4.7

Example 4.4.8

In Warning 4.1.17, we stated that if a vector field fails to pass the screening test $\nabla \cdot \mathbf{B} = 0$ at even a single point, for example because the vector field is not defined at that point, then $\mathbf{B}$ can fail to have a vector potential. An example is the point source

$$\mathbf{B}(x,y,z) = \frac{\hat{\mathbf{r}}(x,y,z)}{r(x,y,z)^2}$$

of Example 3.4.2. Here, as usual,

$$r(x,y,z) = \sqrt{x^2+y^2+z^2} \qquad \hat{\mathbf{r}}(x,y,z) = \frac{x\hat{\imath} + y\hat{\jmath} + z\hat{\mathbf{k}}}{\sqrt{x^2+y^2+z^2}}$$

This vector field is defined on all of $\mathbb{R}^3$, except for the origin, and its divergence

$$\begin{aligned}
\nabla \cdot \mathbf{B} &= \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right) + \frac{\partial}{\partial z}\left(\frac{z}{(x^2+y^2+z^2)^{3/2}}\right) \\
&= \left(\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3x^2}{(x^2+y^2+z^2)^{5/2}}\right) + \left(\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3y^2}{(x^2+y^2+z^2)^{5/2}}\right) \\
&\qquad + \left(\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3z^2}{(x^2+y^2+z^2)^{5/2}}\right) \\
&= \frac{3}{(x^2+y^2+z^2)^{3/2}} - \frac{3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}}
\end{aligned}$$

is zero everywhere except at the origin, where it is not defined.

This vector field cannot have a vector potential on its domain of definition, i.e. on $\mathbb{R}^3 \setminus \{(0,0,0)\} = \{\, (x,y,z) \mid (x,y,z) \neq (0,0,0) \,\}$. To see this, suppose to the contrary that it did have a vector potential $\mathbf{A}$. Then its flux through any closed surface[47] (i.e. surface without a boundary) S would be

$$\iint_S \mathbf{B} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \iint_S \nabla \times \mathbf{A} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \oint_{\partial S} \mathbf{A} \cdot \mathrm{d}\mathbf{r} = 0$$

47 If you are uncomfortable with the surface not having a boundary, poke a very small hole in the surface, giving it a very small boundary. Then take the limit as the hole tends to zero.

by Stokes' theorem, since ∂S is empty. But we found in Example 3.4.2, with $m = 1$, that the flux of $\mathbf{B}$ through any sphere centred on the origin is 4π.

Example 4.4.8

4.4.1 ▸▸ The Interpretation of Div and Curl Revisited

In sections 4.1.4 and 4.1.5 we derived interpretations of the divergence and of the curl. Now that we have the divergence theorem and Stokes' theorem, we can simplify those derivations a lot.

▸▸▸ Divergence

Let $\varepsilon > 0$ be a tiny positive number, and then let

$$B_\varepsilon(x_0, y_0, z_0) = \left\{\ (x, y, z)\ \middle|\ (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < \varepsilon^2\ \right\}$$

be a tiny ball of radius ε centred on the point (x_0, y_0, z_0). Denote by

$$S_\varepsilon(x_0, y_0, z_0) = \left\{\ (x, y, z)\ \middle|\ (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = \varepsilon^2\ \right\}$$

its surface. Because $B_\varepsilon(x_0, y_0, z_0)$ is really small, $\boldsymbol{\nabla} \cdot \mathbf{v}$ is essentially constant in $B_\varepsilon(x_0, y_0, z_0)$ and we essentially have

$$\iiint\limits_{B_\varepsilon(x_0,y_0,z_0)} \boldsymbol{\nabla} \cdot \mathbf{v}\, \mathrm{d}V = \boldsymbol{\nabla} \cdot \mathbf{v}(x_0, y_0, z_0)\ \mathrm{Vol}\big(B_\varepsilon(x_0, y_0, z_0)\big)$$

Of course we are really making an approximation here, based on the assumption that $\mathbf{v}(x, y, z)$ is continuous and so takes values very close to $\mathbf{v}(x_0, y_0, z_0)$ everywhere on the domain of integration. The approximation gets better and better as $\varepsilon \to 0$ and a more precise statement is

$$\boldsymbol{\nabla} \cdot \mathbf{v}(x_0, y_0, z_0) = \lim_{\varepsilon \to 0} \frac{\iiint_{B_\varepsilon(x_0,y_0,z_0)} \boldsymbol{\nabla} \cdot \mathbf{v}\, \mathrm{d}V}{\mathrm{Vol}\big(B_\varepsilon(x_0, y_0, z_0)\big)}$$

By the divergence theorem, we also have

$$\iiint\limits_{B_\varepsilon(x_0,y_0,z_0)} \boldsymbol{\nabla} \cdot \mathbf{v}\, \mathrm{d}V = \iint\limits_{S_\varepsilon(x_0,y_0,z_0)} \mathbf{v} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$$

Think of the vector field $\mathbf{v}$ as the velocity of a moving fluid which has density one. We have already seen, in §3.4, that the flux integral for a velocity field has the interpretation

$$\iint\limits_{S_\varepsilon(x_0,y_0,z_0)} \mathbf{v} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \begin{cases} \text{the volume of fluid leaving } B_\varepsilon(x_0, y_0, z_0) \text{ through} \\ S_\varepsilon(x_0, y_0, z_0) \text{ per unit time} \end{cases}$$

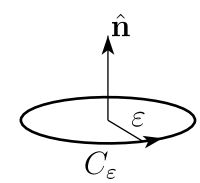
We conclude that, as we said in (4.1.19),

$$\begin{aligned}\nabla \cdot \mathbf{v}(x_0, y_0, z_0) &= \lim_{\varepsilon \to 0} \frac{\text{the rate at which fluid is exiting } B_\varepsilon(x_0, y_0, z_0)}{\text{Vol}\big(B_\varepsilon(x_0, y_0, z_0)\big)} \\ &= \begin{cases}\text{rate at which fluid is exiting an infinitesimal sphere} \\ \text{centred at } (x_0, y_0, z_0)\text{, per unit time, per unit volume}\end{cases} \\ &= \text{strength of the source at } (x_0, y_0, z_0)\end{aligned}$$

If our world is filled with an incompressible fluid, a fluid whose density is constant and so never expands or compresses, we will have $\nabla \cdot \mathbf{v} = 0$.

▸▸▸ Curl

Again let $\varepsilon > 0$ be a tiny positive number and let $D_\varepsilon(x_0, y_0, z_0)$ be a tiny flat circular disk of radius ε centred on the point (x_0, y_0, z_0) and denote by $C_\varepsilon(x_0, y_0, z_0)$ its boundary circle. Let $\hat{\mathbf{n}}$ be a unit normal vector to D_ε. It tells us the orientation of D_ε. Give the circle C_ε the corresponding orientation using the right hand rule. That is, if the fingers of your right hand are pointing in the corresponding direction of motion along C_ε and your palm is facing D_ε, then your thumb is pointing in the direction $\hat{\mathbf{n}}$.



Because $D_\varepsilon(x_0, y_0, z_0)$ is really small, $\nabla \times \mathbf{v}$ is essentially constant on $D_\varepsilon(x_0, y_0, z_0)$ and we essentially have

$$\begin{aligned}\iint\limits_{D_\varepsilon(x_0,y_0,z_0)} \nabla \times \mathbf{v} \cdot \hat{\mathbf{n}}\, \mathrm{d}S &= \nabla \times \mathbf{v}(x_0, y_0, z_0) \cdot \hat{\mathbf{n}}\ \text{Area}\big(D_\varepsilon(x_0, y_0, z_0)\big) \\ &= \pi\varepsilon^2\, \nabla \times \mathbf{v}(x_0, y_0, z_0) \cdot \hat{\mathbf{n}}\end{aligned}$$

Again, this is really an approximate statement which gets better and better as $\varepsilon \to 0$. A more precise statement is

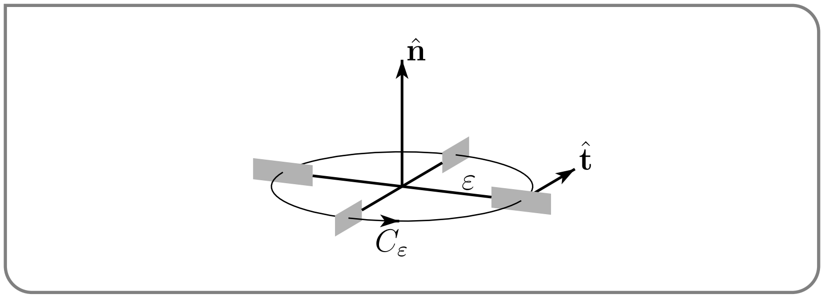
$$\nabla \times \mathbf{v}(x_0, y_0, z_0) \cdot \hat{\mathbf{n}} = \lim_{\varepsilon \to 0} \frac{\iint_{D_\varepsilon(x_0,y_0,z_0)} \nabla \times \mathbf{v} \cdot \hat{\mathbf{n}}\, \mathrm{d}S}{\pi\varepsilon^2}$$

By Stokes' theorem, we also have

$$\iint\limits_{D_\varepsilon(x_0,y_0,z_0)} \nabla \times \mathbf{v} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \oint\limits_{C_\varepsilon(x_0,y_0,z_0)} \mathbf{v} \cdot d\mathbf{r}$$

Again, think of the vector field $\mathbf{v}$ as the velocity of a moving fluid. Then $\oint_{C_\varepsilon} \mathbf{v} \cdot d\mathbf{r}$ is called the circulation of $\mathbf{v}$ around C_ε.

To measure the circulation experimentally, place a small paddle wheel in the fluid, with the axle of the paddle wheel pointing along $\hat{\mathbf{n}}$ and each of the paddles perpendicular to C_ε and centred on C_ε. Each paddle moves tangentially to C_ε. It would like to move with



the same speed as the tangential speed $\mathbf{v} \cdot \hat{\mathbf{t}}$ (where $\hat{\mathbf{t}}$ is the forward pointing unit tangent vector to C_ε at the location of the paddle) of the fluid at its location. But all paddles are rigidly fixed to the axle of the paddle wheel and so must all move with the same speed. That common speed will be the average value of $\mathbf{v} \cdot \hat{\mathbf{t}}$ around C_ε. If $\mathrm{d}s$ represents an element of arc length of C_ε, the average value of $\mathbf{v} \cdot \hat{\mathbf{t}}$ around C_ε is

$$\overline{v_T} = \frac{1}{2\pi\varepsilon} \oint_{C_\varepsilon} \mathbf{v} \cdot \hat{\mathbf{t}}\, \mathrm{d}s = \frac{1}{2\pi\varepsilon} \oint_{C_\varepsilon} \mathbf{v} \cdot \mathrm{d}\mathbf{r}$$

since $\mathrm{d}\mathbf{r}$ has direction $\hat{\mathbf{t}}$ and length $\mathrm{d}s$ so that $\mathrm{d}\mathbf{r} = \hat{\mathbf{t}}\mathrm{d}s$, and since $2\pi\varepsilon$ is the circumference of C_ε. If the paddle wheel rotates at Ω radians per unit time, each paddle travels a distance $\Omega\varepsilon$ per unit time (remember that ε is the radius of C_ε). That is, $\overline{v_T} = \Omega\varepsilon$. Combining all this information,

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{v}(x_0, y_0, z_0) \cdot \hat{\mathbf{n}} &= \lim_{\varepsilon \to 0} \frac{\iint_{D_\varepsilon(x_0,y_0,z_0)} \boldsymbol{\nabla} \times \mathbf{v} \cdot \hat{\mathbf{n}}\, \mathrm{d}S}{\pi\varepsilon^2} \\
&= \lim_{\varepsilon \to 0} \frac{\oint_{C_\varepsilon} \mathbf{v} \cdot \mathrm{d}\mathbf{r}}{\pi\varepsilon^2} \\
&= \lim_{\varepsilon \to 0} \frac{2\pi\varepsilon\, \overline{v_T}}{\pi\varepsilon^2} \\
&= \lim_{\varepsilon \to 0} \frac{2\pi\varepsilon\, (\Omega\varepsilon)}{\pi\varepsilon^2} \\
&= 2\Omega
\end{aligned}$$

so that

$$\Omega = \tfrac{1}{2} \boldsymbol{\nabla} \times \mathbf{v}(x_0, y_0, z_0) \cdot \hat{\mathbf{n}}$$

The component of $\boldsymbol{\nabla} \times \mathbf{v}(x_0, y_0, z_0)$ in any direction $\hat{\mathbf{n}}$ is twice the rate at which the paddle wheel turns when it is put into the fluid at (x_0, y_0, z_0) with its axle pointing in the direction $\hat{\mathbf{n}}$. The direction of $\boldsymbol{\nabla} \times \mathbf{v}(x_0, y_0, z_0)$ is the axle direction which gives maximum rate of rotation and the magnitude of $\boldsymbol{\nabla} \times \mathbf{v}(x_0, y_0, z_0)$ is twice that maximum rate of rotation. For this reason, $\boldsymbol{\nabla} \times \mathbf{v}$ is called the "vorticity".

4.4.2 ▸▸ Optional — An Application of Stokes' Theorem — Faraday's Law

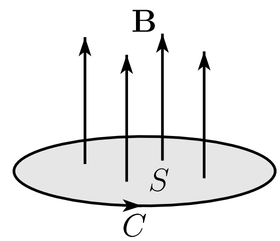
Magnetic induction refers to a physical process whereby an electric voltage is created ("induced") by a time varying magnetic field. This process is exploited in many applications, including electric generators, induction motors, induction cooking, induction welding and inductive charging. Michael Faraday[48] is generally credited with the discovery of

48 Michael Faraday (1791–1867) was an English physicist and chemist. He ended up being an extremely influential scientist despite having only the most basic of formal educations.

magnetic induction. Faraday's law is the following. Let S be an oriented surface with boundary C. Let $\mathbf{E}$ and $\mathbf{B}$ be the (time dependent) electric and magnetic fields and define

$$\oint_C \mathbf{E} \cdot \mathrm{d}\mathbf{r} = \text{voltage around } C$$

$$\iint_S \mathbf{B} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \text{magnetic flux through } S$$



Then the voltage around C is the negative of the rate of change of the magnetic flux through S. As an equation, Faraday's Law is

$$\oint_C \mathbf{E} \cdot \mathrm{d}\mathbf{r} = -\frac{\partial}{\partial t} \iint_S \mathbf{B} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$$

We can reformulate this as a partial differential equation. By Stokes' Theorem,

$$\oint_C \mathbf{E} \cdot \mathrm{d}\mathbf{r} = \iint_S (\nabla \times \mathbf{E}) \cdot \hat{\mathbf{n}}\, \mathrm{d}S$$

so Faraday's law becomes

$$\iint_S \left(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t}\right) \cdot \hat{\mathbf{n}}\, \mathrm{d}S = 0$$

This is true for all surfaces S. So the integrand, assuming that it is continuous, must be zero.

To see this, let $\mathbf{G} = \left(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t}\right)$. Suppose that $\mathbf{G}(\mathbf{x}_0) \neq 0$. Pick a unit vector $\hat{\mathbf{n}}$ in the direction of $\mathbf{G}(\mathbf{x}_0)$. Let S be a very small flat disk centered on $\mathbf{x}_0$ with normal $\hat{\mathbf{n}}$ (the vector we picked). Then $\mathbf{G}(\mathbf{x}_0) \cdot \hat{\mathbf{n}} > 0$ and, by continuity, $\mathbf{G}(\mathbf{x}) \cdot \hat{\mathbf{n}} > 0$ for all $\mathbf{x}$ on S, if we have picked S small enough. Then $\iint_S \left(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t}\right) \cdot \hat{\mathbf{n}}\, \mathrm{d}S > 0$, which is a contradiction. So $\mathbf{G} = \mathbf{0}$ everywhere and we conclude that

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

This is one of Maxwell's electromagnetic field equations[49].

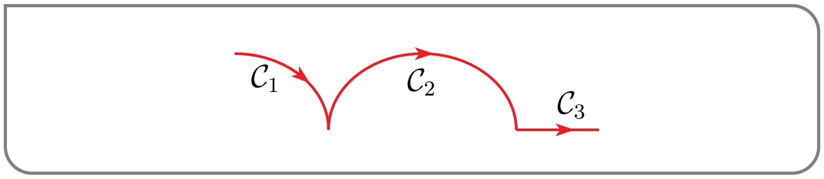
4.5▲ Optional — Which Vector Fields Obey $\nabla \times \mathbf{F} = 0$

We already know that if a vector field $\mathbf{F}$ passes the screening test $\nabla \times \mathbf{F} = 0$ on *all* of $\mathbb{R}^2$ or $\mathbb{R}^3$, then there is a function φ with $\mathbf{F} = \nabla\varphi$. That is, $\mathbf{F}$ is conservative. We are now going to take a first look at what happens[50] when $\mathbf{F}$ passes the screening test $\nabla \times \mathbf{F} = 0$ only on

49 For the others, see Example 4.1.2
50 Russell Crowe posed a related question in the movie *A Beautiful Mind*. The movie is based on the life of the American mathematician John Nash, who won a Nobel Prize in Economics.

some proper subset $\mathcal{D}$ of $\mathbb{R}^n$, $n = 2$ or 3. We will just scratch the surface of this topic — there is a whole subbranch of Mathematics (cohomology theory, which is part of algebraic topology) concerned with a general form of this question. We shall imagine that we are given a vector field $\mathbf{F}$ that is only defined on $\mathcal{D}$ and we shall assume

- that $\mathcal{D}$ is a connected, open subset of $\mathbb{R}^n$ with $n = 2$ or $n = 3$ (see Definition 4.5.1, below)
- that all first order derivatives of all vector fields and functions that we consider are continuous and
- that all curves we consider are piecewise smooth. A curve is piecewise smooth if it is a union of a finite number of smooth curves $\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_m$ with the end point of $\mathcal{C}_i$ being the beginning point of $\mathcal{C}_{i+1}$ for each $1 \leqslant i < m$. A curve is smooth[51] if it has a parametrization $\mathbf{r}(t)$, $a \leqslant t \leqslant b$, whose first derivative $\mathbf{r}'(t)$ exists, is continuous and is nonzero everywhere.



Definition 4.5.1.

Let $n \geqslant 1$ be an integer.

(a) Let $\mathbf{a} \in \mathbb{R}^n$ and $\varepsilon > 0$. The open ball of radius ε centred on $\mathbf{a}$ is

$$B_\varepsilon(\mathbf{a}) = \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; |\mathbf{x} - \mathbf{a}| < \varepsilon \right\}$$

Note the strict inequality in $|\mathbf{x} - \mathbf{a}| < \varepsilon$.

(b) A subset $\mathcal{O} \subset \mathbb{R}^n$ is said to be an "open subset of $\mathbb{R}^n$" if, for each point $\mathbf{a} \in \mathcal{O}$, there is an $\varepsilon > 0$ such that $B_\varepsilon(\mathbf{a}) \subset \mathcal{O}$. Equivalently, $\mathcal{O}$ is open if and only if it is a union of open balls.

(c) A subset $\mathcal{D} \subset \mathbb{R}^n$ is said to be (pathwise) connected if every pair of points in $\mathcal{D}$ can be joined by a piecewise smooth curve in $\mathcal{D}$.

Here are some examples to help explain this definition.

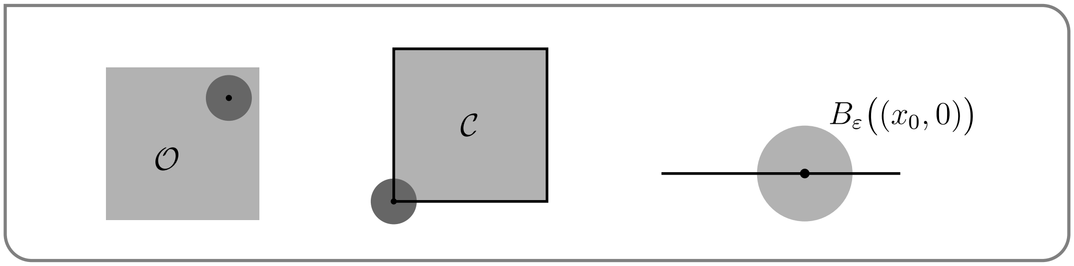
Example 4.5.2

(a) The "open rectangle" $\mathcal{O} = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; 0 < x < 1,\ 0 < y < 1 \right\}$ is an open subset of $\mathbb{R}^2$ because any point $\mathbf{a} = (x_0, y_0) \in \mathcal{O}$ is a nonzero distance, namely $d =$

51 The word "smooth" does not have a universal meaning in mathematics. It is used with different meanings in different contexts. We are here using one of the standard definitions. Another standard definition requires that all derivatives of all orders are continuous.

$\min\{x_0, 1-x_0, y_0, 1-y_0\}$ away from the boundary of $\mathcal{O}$. So the open ball $B_{d/2}(\mathbf{a})$ is contained in $\mathcal{O}$.

(b) The "closed rectangle" $\mathcal{C} = \{ (x,y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1 \}$ is *not* an open subset of $\mathbb{R}^2$. For example, $\mathbf{0} = (0,0)$ is a point in $\mathcal{C}$. No matter what $\varepsilon > 0$ we pick, the open ball $B_\varepsilon(\mathbf{0})$ is not contained in $\mathcal{C}$ because $B_\varepsilon(\mathbf{0})$ contains the point $(-\frac{\varepsilon}{2}, 0)$, which is not in $\mathcal{C}$.



(c) The x-axis, $\mathcal{X} = \{ (x,y) \in \mathbb{R}^2 \mid y = 0 \}$, in $\mathbb{R}^2$ is *not* an open subset of $\mathbb{R}^2$ because for any point $(x_0, 0) \in \mathcal{X}$ and any $\varepsilon > 0$, the ball $B_\varepsilon\big((x_0,0)\big)$ contains points with nonzero y-coordinates and so is not contained in $\mathcal{X}$.

(d) The union of open balls

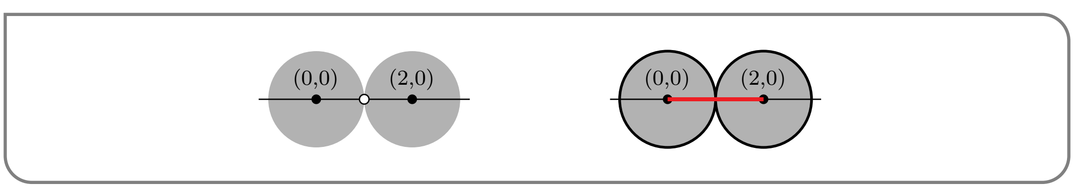
$$B_1\big((0,0)\big) \cup B_1\big((2,0)\big) = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1 \text{ or } (x-2)^2 + y^2 < 1 \}$$

is not connected, since any continuous path from, for example, $(2,0)$ to $(0,0)$ must leave the union. In the figure on the left below, an "empty disk" has been sketched at $(1,0)$ just to emphasise that the point $(1,0)$ is *not* in the union.

(e) On the other hand the union of "closed balls"

$$\{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leqslant 1 \text{ or } (x-2)^2 + y^2 \leqslant 1 \}$$

is connected. For example, the straight line segment from $(2,0)$ to $(0,0)$ remains in the union.



Example 4.5.2

Many, but not all, of the basic facts that we developed, in §2.4.1, about conservative fields in $\mathbb{R}^n$ also applies (with the same proofs) to fields on $\mathcal{D}$.

Theorem 4.5.3.

For a vector field $\mathbf{F}$ on $\mathcal{D} \subset \mathbb{R}^n$,

$$\begin{aligned}
\mathbf{F} \text{ is conservative on } \mathcal{D} &\iff \mathbf{F} = \nabla\varphi \text{ on } \mathcal{D}, \text{ for some function } \varphi \\
&\iff \text{for each } P_0, P_1 \in \mathcal{D}, \text{ the integral } \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \text{ takes} \\
&\qquad \text{the same value for all curves } \mathcal{C} \text{ from } P_0 \text{ to } P_1 \\
&\iff \oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = 0 \ \text{ for all closed curves } \mathcal{C} \text{ in } \mathcal{D} \\
&\implies \nabla \times \mathbf{F} = 0 \ \text{ on } \mathcal{D}
\end{aligned}$$

Note that the last line of this theorem contains only a one way implication.

Combining this with Stokes' Theorem 4.4.1 (when $n = 3$, or Green's Theorem 4.3.2 when $n = 2$) gives us the following two consequences.

Theorem 4.5.4.

(a) If $\mathcal{D}$ has the property that

$$\begin{array}{l}\text{every closed curve } \mathcal{C} \text{ in } \mathcal{D} \text{ is the boundary} \\ \text{of a bounded oriented surface, } \mathcal{S}, \text{ in } \mathcal{D}\end{array} \tag{H}$$

then

$$\mathbf{F} \text{ is conservative on } \mathcal{D} \iff \nabla \times \mathbf{F} = 0 \text{ on } \mathcal{D}$$

(b) For any $\mathcal{D}$, if $\nabla \times \mathbf{F} = 0$ on $\mathcal{D}$, then $\mathbf{F}$ is locally conservative. This means that for each point $\mathbf{x}_0 \in \mathcal{D}$, there is an $\varepsilon > 0$ and a function φ such that $\mathbf{F} = \nabla\varphi$ on $B_\varepsilon(\mathbf{x}_0)$.

Proof. (a) This is simply because if $\nabla \times \mathbf{F} = 0$ on $\mathcal{D}$ and if the curve $\mathcal{C} = \partial\mathcal{S}$, with $\mathcal{S}$ an oriented surface in $\mathcal{D}$, then Stokes' theorem gives

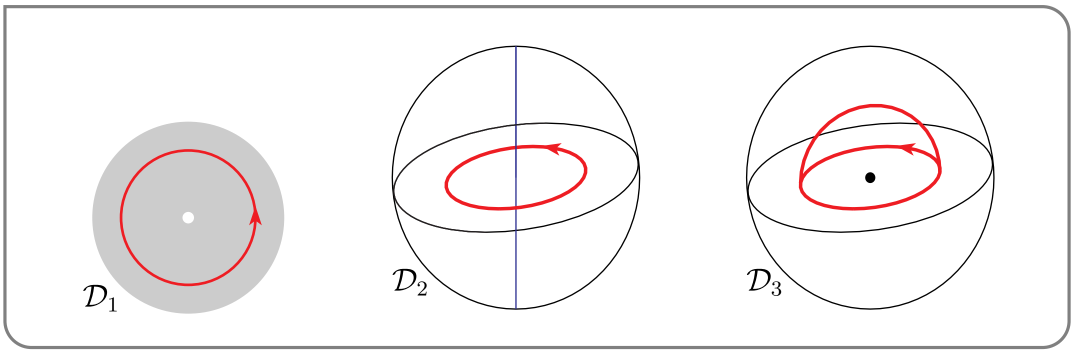
$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\partial\mathcal{S}} \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathcal{S}} \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = 0$$

So $\mathbf{F}$ is conservative by Theorem 4.5.3.
(b) This is true simply because $B_\varepsilon(\mathbf{x}_0)$ satisfies property (H). □

Example 4.5.5

Here are some examples of $\mathcal{D}$'s that violate (H).

- When $\mathcal{D} = \mathcal{D}_1 = \big\{ (x,y) \in \mathbb{R}^2 \ \big|\ 0 < |(x,y)| < 3 \big\}$ (an open ball with its centre removed), then the circle $x^2 + y^2 = 4$ is a curve in $\mathcal{D}$ that is not the boundary of a surface in $\mathcal{D}$. The circle $x^2 + y^2 = 4$ is the boundary of the disk $x^2 + y^2 < 4$, but the disk $x^2 + y^2 < 4$ is not contained in $\mathcal{D}$ because the point $(0,0)$ is in the disk and not in $\mathcal{D}$. See the figure on the left below.
- When $\mathcal{D} = \mathcal{D}_2 = \big\{ (x,y,z) \in \mathbb{R}^3 \ \big|\ |(x,y,z)| < 2, |(x,y)| > 0 \big\}$ (an open ball with the z-axis removed), then the circle $x^2 + y^2 = 1$, $z = 0$ is a curve in $\mathcal{D}$ that is not the boundary of a surface in $\mathcal{D}$. The circle is the boundary of many different surfaces in $\mathbb{R}^3$, but each contains a point on the z-axis and so is not contained in $\mathcal{D}$. See the figure in the centre below.



On the other hand, here is an example which does satisfy (H).

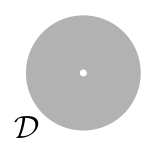
- Let $\mathcal{D} = \mathcal{D}_3 = \big\{ (x,y,z) \in \mathbb{R}^3 \ \big|\ 0 < |(x,y,z)| < 2 \big\}$ (an open ball with its centre removed). For example the circle $x^2 + y^2 = 1$, $z = 0$ is the boundary of the surface $\big\{ (x,y,z) \in \mathbb{R}^3 \ \big|\ x^2 + y^2 + z^2 = 1, z > 0 \big\} \subset \mathcal{D}$. See the figure on the right above.

Example 4.5.5

This leaves the question of what happens when (H) is violated. We'll just look at one example, which however gives the flavour of the general theory.

The punctured disk is

$$\mathcal{D} = \big\{ (x,y) \in \mathbb{R}^2 \ \big|\ 0 < |(x,y)| < 1 \big\}$$



We'll start by looking at one particular vector field, which passes the screening test, but which cannot possibly be conservative. The field, which we saw in Example 2.3.14, is

$$\mathbf{\Theta} = -\frac{y}{x^2+y^2}\hat{\boldsymbol{\imath}} + \frac{x}{x^2+y^2}\hat{\boldsymbol{\jmath}}$$

with domain of definition $\mathcal{D}$. We'll first check that it passes the screening test:

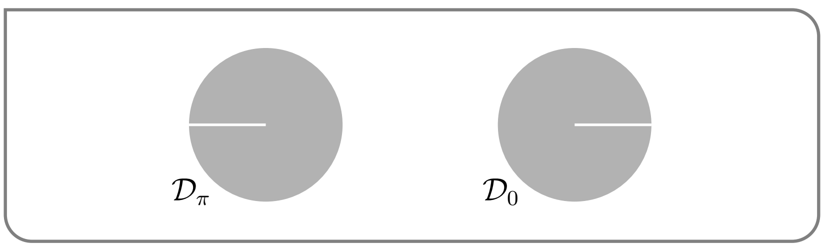
$$\begin{aligned}
\nabla \times \mathbf{\Theta} &= \Big\{ \frac{\partial}{\partial x}\Big(\frac{x}{x^2+y^2}\Big) - \frac{\partial}{\partial y}\Big(-\frac{y}{x^2+y^2}\Big) \Big\}\hat{\mathbf{k}} \\
&= \Big\{ \Big(\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}\Big) + \Big(\frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}\Big) \Big\}\hat{\mathbf{k}} \\
&= 0
\end{aligned}$$

Next we'll check that it cannot be conservative. Denote by $\mathcal{C}_\varepsilon$ the circle $x^2 + y^2 = \varepsilon^2$, with counterclockwise orientation. Parametrize $\mathcal{C}_\varepsilon$ by $\mathbf{r}(\theta) = \varepsilon \cos\theta\,\hat{\boldsymbol{\imath}} + \varepsilon \sin\theta\,\hat{\boldsymbol{\jmath}}$ with $0 \leqslant \theta \leqslant 2\pi$. Then

$$\begin{aligned}\int_{\mathcal{C}_\varepsilon} \boldsymbol{\Theta} \cdot \mathrm{d}\mathbf{r} &= \int_0^{2\pi} \boldsymbol{\Theta}\big(\mathbf{r}(\theta)\big) \cdot \frac{\mathrm{d}\mathbf{r}}{d\theta}(\theta)\, d\theta \\ &= \int_0^{2\pi} \Big(-\frac{1}{\varepsilon} \sin\theta\,\hat{\boldsymbol{\imath}} + \frac{1}{\varepsilon}\cos\theta\,\hat{\boldsymbol{\jmath}} \Big) \cdot \big(-\varepsilon \sin\theta\,\hat{\boldsymbol{\imath}} + \varepsilon\cos\theta\,\hat{\boldsymbol{\jmath}} \big)\, d\theta \\ &= \int_0^{2\pi} d\theta \\ &= 2\pi \end{aligned} \tag{E1}$$

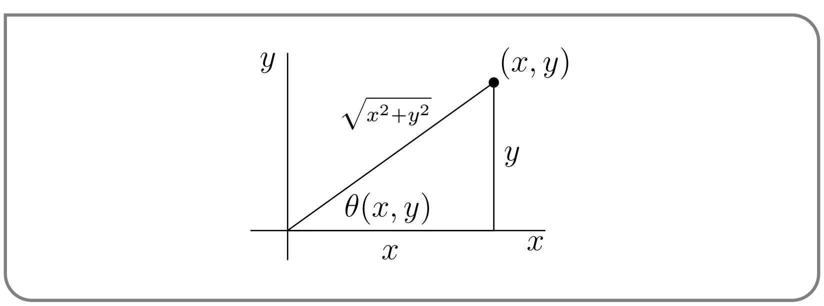
is not zero. By Theorem 4.5.3, $\boldsymbol{\Theta}$ cannot be conservative on the punctured disk since the integral $\int_{\mathcal{C}_\varepsilon} \boldsymbol{\Theta} \cdot \mathrm{d}\mathbf{r}$ around the closed curve $\mathcal{C}_\varepsilon$ is nonzero.

Next we'll check that it is locally conservative. That is, it can be written in the form $\nabla\theta(x, y)$ near any point (x_0, y_0) in its domain. Define $\theta(x, y)$ to be the polar angle of (x, y) with, for example, $-\pi < \theta < \pi$. This θ is defined on all of $\mathcal{D}$, *except* for the negative real axis. The domain of definition, $\mathcal{D}_\pi$, is sketched on the left below. If (x_0, y_0) happens to lie



on the negative real axis, just replace $-\pi < \theta < \pi$ by a different interval of length 2π, like $0 < \theta < 2\pi$. The domain of definition of θ would then change to the $\mathcal{D}_0$, sketched on the right above.

It's now a simple matter to check that $\nabla\theta(x, y) = \boldsymbol{\Theta}(x, y)$ on the domain of definition of θ. If $x \neq 0$, then, from the figure below,



we have that $\tan\theta(x,y) = \frac{y}{x}$, and $\cos\theta(x,y) = \frac{x}{\sqrt{x^2+y^2}}$, so that

$$\begin{aligned}
\frac{\partial}{\partial x}\tan\theta(x,y) = -\frac{y}{x^2} &\implies \left[\frac{\partial}{\partial x}\theta(x,y)\right]\sec^2\theta(x,y) = -\frac{y}{x^2}\\
&\implies \frac{\partial}{\partial x}\theta(x,y) = -\frac{y}{x^2}\cos^2\theta(x,y) = -\frac{y}{x^2}\,\frac{x^2}{x^2+y^2} = -\frac{y}{x^2+y^2}\\
\frac{\partial}{\partial y}\tan\theta(x,y) = \frac{1}{x} &\implies \left[\frac{\partial}{\partial y}\theta(x,y)\right]\sec^2\theta(x,y) = \frac{1}{x}\\
&\implies \frac{\partial}{\partial y}\theta(x,y) = \frac{1}{x}\cos^2\theta(x,y) = \frac{1}{x}\,\frac{x^2}{x^2+y^2} = \frac{x}{x^2+y^2}
\end{aligned}$$

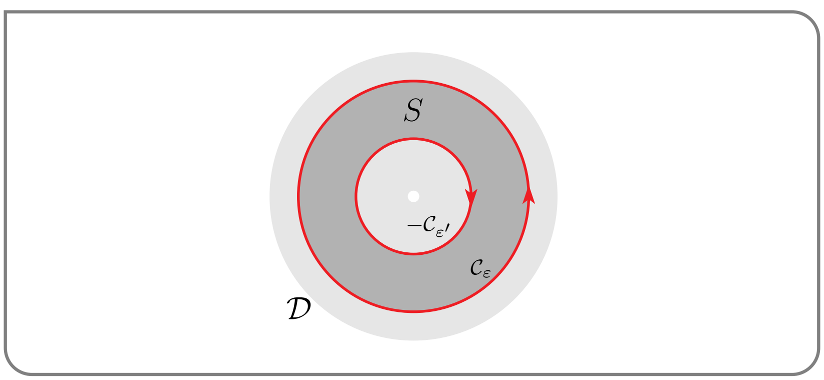
If $x = 0$, then we must have $y \neq 0$ (since $(0,0)$ is not in the domain of definition to θ), and we can use $\cot\theta(x,y) = \frac{x}{y}$ instead and arrive at the same result.

So far we have just looked at one vector field on $\mathcal{D}$. We are now ready to consider any vector field $\mathbf{F}$ on $\mathcal{D}$ that passes the screening test $\nabla \times \mathbf{F} = 0$ on $\mathcal{D}$. We claim that there is a function φ on $\mathcal{D}$ such that

$$\mathbf{F} = \alpha_{\mathbf{F}}\,\boldsymbol{\Theta} + \nabla\varphi \qquad \text{where} \qquad \alpha_{\mathbf{F}} = \frac{1}{2\pi}\oint_{\mathcal{C}_\varepsilon} \mathbf{F}\cdot \mathrm{d}\mathbf{r} \tag{E2}$$

The significance of this claim is that it says that if a vector field on $\mathcal{D}$ passes the screening test on $\mathcal{D}$, then, either it is conservative (that's the case if and only if $\alpha_{\mathbf{F}} = 0$) or, if it fails to be conservative, then it differs from a conservative field (namely $\nabla\varphi$) only by a constant (namely $\alpha_{\mathbf{F}}$) times the fixed vector field $\boldsymbol{\Theta}$. That is, there is only one nonconservative vector field on $\mathcal{D}$ that passes the screening test, up to multiplication by constants and addition of conservative fields. This is a nice simple surprise.

Observe that in the definition of $\alpha_{\mathbf{F}}$, we did not specify the radius ε of the circle $\mathcal{C}_\varepsilon$ to be used for the integration curve. That's because the answer to the integral does not depend on the choice of ε. To see this, take any $0 < \varepsilon' < \varepsilon < 1$ and consider the surface $S = \{\,(x,y) \in \mathbb{R}^2 \mid \varepsilon' < |(x,y)| < \varepsilon\}$. It is completely contained in $\mathcal{D}$. The boundary

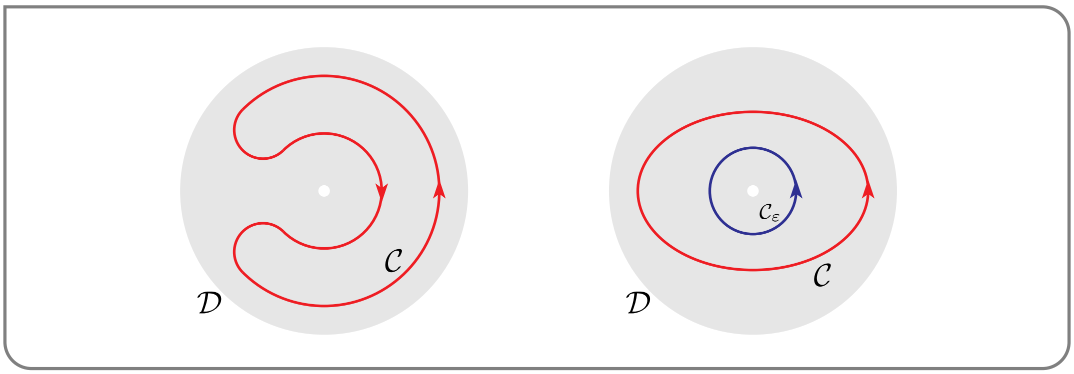


of S consists of two parts. The outside part is $\mathcal{C}_\varepsilon$, oriented counterclockwise as usual. The inside part is $\mathcal{C}_{\varepsilon'}$, but oriented clockwise. It is usually denoted $-\mathcal{C}_{\varepsilon'}$. So, by Stokes' theorem,

$$\oint_{\mathcal{C}_\varepsilon}\mathbf{F}\cdot\mathrm{d}\mathbf{r} - \oint_{\mathcal{C}_{\varepsilon'}}\mathbf{F}\cdot\mathrm{d}\mathbf{r} = \oint_{\mathcal{C}_\varepsilon}\mathbf{F}\cdot\mathrm{d}\mathbf{r} + \oint_{-\mathcal{C}_{\varepsilon'}}\mathbf{F}\cdot\mathrm{d}\mathbf{r} = \oint_{\partial S}\mathbf{F}\cdot\mathrm{d}\mathbf{r} = \iint_S \nabla\times\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = 0$$

Finally to verify the claim (E2), we check that the vector field $\mathbf{G} = \mathbf{F} - \alpha_{\mathbf{F}}\mathbf{\Theta}$ is conservative on $\mathcal{D}$. To do so, it suffices to check that $\oint_{\mathcal{C}} \mathbf{G} \cdot d\mathbf{r} = 0$ for any closed curve $\mathcal{C}$ in $\mathcal{D}$. In fact we can restrict our attention to curves $\mathcal{C}$ that are simple, closed, counterclockwise oriented curves on $\mathcal{D}$. A curve is called simple if it does not cross itself. Closed curves which are not simple can be split up into simple closed subcurves. And changing the orientation of $\mathcal{C}$ just changes the sign of $\oint_{\mathcal{C}} \mathbf{G} \cdot d\mathbf{r} = 0$, which does not affect whether it is zero or not.

So let $\mathcal{C}$ be a simple, closed, counterclockwise oriented curve in $\mathcal{D}$. We need to verify that $\oint_{\mathcal{C}} \mathbf{G} \cdot d\mathbf{r} = 0$. Any simple closed curve in $\mathbb{R}^2$ divides $\mathbb{R}^2$ into three mutually disjoint subsets[52] — $\mathcal{C}$ itself, the set of points inside $\mathcal{C}$ and the set of points outside $\mathcal{C}$. Since $(0, 0)$ is not on $\mathcal{C}$, it must be either outside $\mathcal{C}$, as in the figure of the left below, or inside $\mathcal{C}$ as in the figure on the right below.



- *Case 1:* $(0, 0)$ *outside* $\mathcal{C}$. In this case $\mathcal{C}$ is the boundary of a set, S, which is completely contained in $\mathcal{D}$, namely all of the points inside $\mathcal{C}$. So, by Stokes' theorem,

$$\oint_{\mathcal{C}} \mathbf{G} \cdot d\mathbf{r} = \oint_{\partial S} \big(\mathbf{F} - \alpha_{\mathbf{F}}\mathbf{\Theta}\big) \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, dS - \alpha_{\mathbf{F}} \iint_S \nabla \times \mathbf{\Theta} \cdot \hat{\mathbf{n}}\, dS = 0 - \alpha_{\mathbf{F}} 0$$
$$= 0$$

- *Case 2:* $(0, 0)$ *inside* $\mathcal{C}$. Since $(0, 0)$ is not on $\mathcal{C}$, we can choose ε small enough that the circle $\mathcal{C}_\varepsilon$ lies completely inside $\mathcal{C}$. Then the curve $\mathcal{C} - \mathcal{C}_\varepsilon$ is the boundary of a set, S, which is completely contained in $\mathcal{D}$, namely the part of $\mathcal{D}$ that is between $\mathcal{C}_\varepsilon$ and $\mathcal{C}$. So, by Stokes' theorem,

$$\oint_{\mathcal{C}} \mathbf{G} \cdot d\mathbf{r} - \oint_{\mathcal{C}_\varepsilon} \mathbf{G} \cdot d\mathbf{r} = \oint_{\mathcal{C} - \mathcal{C}_\varepsilon} \mathbf{G} \cdot d\mathbf{r} = \oint_{\partial S} \mathbf{G} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{G} \cdot \hat{\mathbf{n}}\, dS = 0$$

since $\nabla \times \mathbf{G} = \nabla \times \mathbf{F} - \alpha_{\mathbf{F}} \nabla \times \mathbf{\Theta} = 0$ on $\mathcal{D}$. Hence

$$\oint_{\mathcal{C}} \mathbf{G} \cdot d\mathbf{r} = \oint_{\mathcal{C}_\varepsilon} \mathbf{G} \cdot d\mathbf{r} = \oint_{\mathcal{C}_\varepsilon} \mathbf{F} \cdot d\mathbf{r} - \alpha_{\mathbf{F}} \oint_{\mathcal{C}_\varepsilon} \mathbf{\Theta} \cdot d\mathbf{r} = 2\pi\alpha_{\mathbf{F}} - \alpha_{\mathbf{F}}(2\pi) = 0$$

by the definition, (E2), of $\alpha_{\mathbf{F}}$ and (E1).

52 This, intuitively obvious, but hard to prove, result is called the Jordan curve theorem. It is named after the French mathematician Camille Jordan (1838–1922), who first proved it.

So $\mathbf{G}$ is conservative on $\mathcal{D}$ and $\mathbf{F}$ is of the form (E2) on $\mathcal{D}$.

The ideas that we have explored here can be generalised quite a bit. For example, if we had a disk with $n > 1$ punctures, we could use arguments like those above to show that any vector field $\mathbf{F}$ that passes the screening test has to be of the form

$$\mathbf{F} = \nabla\varphi + \sum_{\ell=1}^{n} \alpha_\ell \mathbf{\Theta}_\ell$$

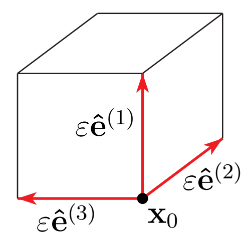
with $\mathbf{\Theta}_\ell$ simply being the above $\mathbf{\Theta}$ translated so as to be centered on the ℓ^{th} puncture.

4.6▲ Really Optional — More Interpretation of Div and Curl

We are now going to determine, in much more detail than before[53], what the divergence and curl of a vector field tells us about the flow of that vector field.

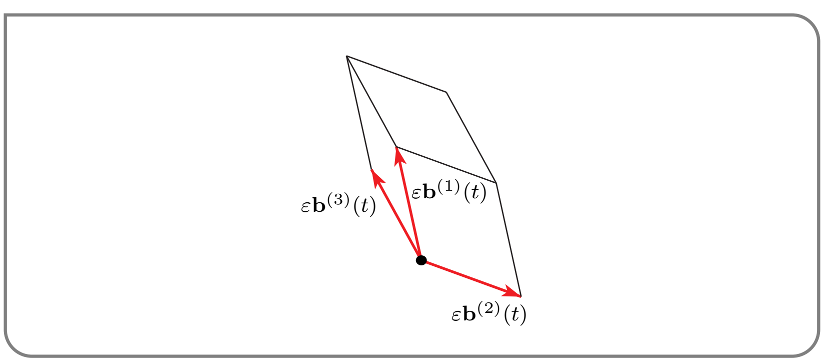
Consider a (possibly compressible) fluid with velocity field $\mathbf{v}(\mathbf{x}, t)$. Pick any time t_0 and a really tiny piece of the fluid; assume that, at time t_0, it is a cube with corners at

$$\left\{\ \mathbf{x}_0 + n_1\varepsilon\hat{\mathbf{e}}^{(1)} + n_2\varepsilon\hat{\mathbf{e}}^{(2)} + n_3\varepsilon\hat{\mathbf{e}}^{(3)}\ \middle|\ n_1, n_2, n_3 \in \{0,1\}\ \right\}$$



Here $\varepsilon > 0$ is the length of each edge of the cube and is assumed to be really small. The vectors $\hat{\mathbf{e}}^{(1)}$, $\hat{\mathbf{e}}^{(2)}$ and $\hat{\mathbf{e}}^{(3)}$ are three mutually perpendicular unit vectors that give the orientation of the edges of the cube. The vectors from the corner $\mathbf{x}_0$ to its three nearest neighbour corners are $\varepsilon\hat{\mathbf{e}}^{(1)}$, $\varepsilon\hat{\mathbf{e}}^{(2)}$ and $\varepsilon\hat{\mathbf{e}}^{(3)}$.

As time progresses, the chunk of fluid moves. In particular, the corners move. Let us denote by $\varepsilon\mathbf{b}^{(1)}(t)$ the vector, at time t, joining the $n_1 = n_2 = n_3 = 0$ corner to the $n_1 = 1$, $n_2 = n_3 = 0$ corner. Define $\varepsilon\mathbf{b}^{(2)}(t)$ and $\varepsilon\mathbf{b}^{(3)}(t)$ similarly. For times very close to t_0 we can think of our chunk of fluid as being essentially a parallelepiped with edges $\varepsilon\mathbf{b}^{(k)}(t)$.



53 We'll also use some more mathematics than before. In this section, we'll use matrix eigenvalues and eigenvectors and solve some simple systems of ordinary differential equations. We'll also need to use a lot of subscripts and superscripts. It only looks intimidating.

By concentrating on the edges $\varepsilon\mathbf{b}^{(k)}(t)$ of the chunk of fluid, rather than the corners, we are ignoring any translations that the chunk of fluid might have undergone. We want, instead, to determine how the size and orientation of the parallelepiped changes as t increases.

At time t_0, $\mathbf{b}^{(k)} = \hat{\mathbf{e}}^{(k)}$. The velocities of the corners of the chunk of fluid at time t_0 are

$$\mathbf{v}\big(\mathbf{x}_0 + n_1\varepsilon\hat{\mathbf{e}}^{(1)} + n_2\varepsilon\hat{\mathbf{e}}^{(2)} + n_3\varepsilon\hat{\mathbf{e}}^{(3)}, t_0\big)$$

In particular, at time t_0, the tail of $\varepsilon\mathbf{b}^{(k)}$ has velocity $\mathbf{v}(\mathbf{x}_0, t_0)$ and the head of $\varepsilon\mathbf{b}^{(k)}$ has velocity $\mathbf{v}(\mathbf{x}_0 + \varepsilon\hat{\mathbf{e}}^{(k)}, t_0)$. Consequently (using a Taylor approximation),

$$\varepsilon\frac{\mathrm{d}\mathbf{b}^{(k)}}{\mathrm{d}t}(t_0) = \mathbf{v}\big(\mathbf{x}_0 + \varepsilon\hat{\mathbf{e}}^{(k)}, t_0\big) - \mathbf{v}\big(\mathbf{x}_0, t_0\big) = \sum_{j=1}^{3}\varepsilon\frac{\partial\mathbf{v}}{\partial x_j}\big(\mathbf{x}_0, t_0\big)\hat{\mathbf{e}}_j^{(k)} + O(\varepsilon^2)$$

and so

$$\frac{\mathrm{d}\mathbf{b}^{(k)}}{\mathrm{d}t}(t_0) = \sum_{j=1}^{3}\frac{\partial\mathbf{v}}{\partial x_j}\big(\mathbf{x}_0, t_0\big)\hat{\mathbf{e}}_j^{(k)} + O(\varepsilon)$$

The notation $O(\varepsilon^n)$ represents a function that is bounded by a constant times ε^n for all sufficiently small ε. That is, we are saying that $\frac{\mathrm{d}\mathbf{b}^{(k)}}{\mathrm{d}t}(t_0)$ is $\sum_{j=1}^{3}\frac{\partial\mathbf{v}}{\partial x_j}\big(\mathbf{x}_0, t_0\big)\hat{\mathbf{e}}_j^{(k)}$ plus a small error that is bounded by a constant time ε. The notation $\hat{\mathbf{e}}_j^{(k)}$ just refers to the j^{th} component of the vector $\hat{\mathbf{e}}^{(k)}$.

Denote by $\mathcal{V}$ the 3×3 matrix whose (i, j) matrix element is

$$\mathcal{V}_{i,j} = \frac{\partial\mathbf{v}_i}{\partial x_j}\big(\mathbf{x}_0, t_0\big) \qquad 1 \leqslant i, j \leqslant 3 \tag{M}$$

Then we can write the above more compactly:

$$\frac{\mathrm{d}\mathbf{b}^{(k)}}{\mathrm{d}t}(t_0) = \mathcal{V}\mathbf{b}^{(k)}(t_0) + O(\varepsilon)$$

Here $\mathcal{V}\mathbf{b}^{(k)}(t_0)$ is the product of the 3×3 matrix $\mathcal{V}$ and the 3×1 column vector $\mathbf{b}^{(k)}(t_0)$. We study the behaviour of $\mathbf{b}^{(k)}(t)$ for small ε and t close to t_0, by studying the behaviour of the solutions to the initial value problems

$$\frac{\mathrm{d}\mathbf{b}^{(k)}}{\mathrm{d}t}(t) = \mathcal{V}\mathbf{b}^{(k)}(t) \qquad \mathbf{b}^{(k)}(t_0) = \hat{\mathbf{e}}^{(k)} \tag{IVP}$$

To warm up, we first look at two two-dimensional examples. In both examples, the velocity field $\mathbf{v}(x, y)$ is linear in (x, y). Consequently, in these examples, $\mathbf{v}\big(\mathbf{x}_0 + \varepsilon\hat{\mathbf{e}}^{(k)}, t_0\big) - \mathbf{v}\big(\mathbf{x}_0, t_0\big)$ is exactly $\sum_{j=1}^{3}\varepsilon\frac{\partial\mathbf{v}}{\partial x_j}\big(\mathbf{x}_0, t_0\big)\hat{\mathbf{e}}_j^{(k)}$ and the solution to (IVP) coincides with the exact $\mathbf{b}^{(k)}(t)$. Following each example, we discuss a broad class of $\mathcal{V}$'s that generate behaviour similar to that example.

Example 4.6.1 ($\mathbf{v}(x, y) = 2x\hat{\boldsymbol{\imath}} + 3y\hat{\boldsymbol{\jmath}}$)

In this example

$$\mathcal{V} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

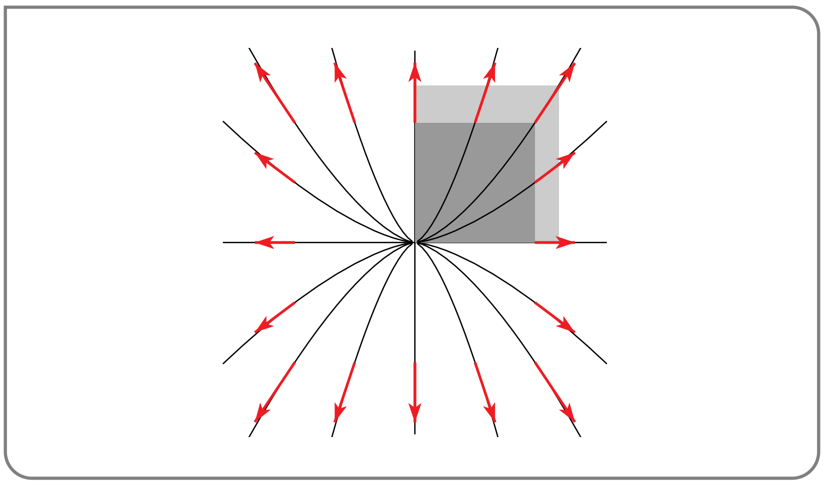
The solution to the initial value problem

$$\mathbf{b}'(t) = \mathcal{V}\mathbf{b}(t) \quad \mathbf{b}(0) = \begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix} \quad \text{or equivalently} \quad \begin{array}{ll} \mathbf{b}_1'(t) = 2\mathbf{b}_1(t) & \mathbf{b}_1(0) = \beta_1 \\ \mathbf{b}_2'(t) = 3\mathbf{b}_2(t) & \mathbf{b}_2(0) = \beta_2 \end{array}$$

is

$$\begin{array}{l} \mathbf{b}_1(t) = e^{2t}\beta_1 \\ \mathbf{b}_2(t) = e^{3t}\beta_2 \end{array} \quad \text{or equivalently} \quad \mathbf{b}(t) = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t}\end{bmatrix}\mathbf{b}(0)$$

If one chooses $\hat{\mathbf{e}}^{(1)} = \hat{\imath}$ and $\hat{\mathbf{e}}^{(2)} = \hat{\jmath}$, the edges, $\mathbf{b}^{(1)}(t) = e^{2t}\hat{\mathbf{e}}^{(1)}$ and $\mathbf{b}^{(2)}(t) = e^{3t}\hat{\mathbf{e}}^{(2)}$, of the chunk of fluid never change direction. But their lengths do change. The relative rate of change of length per unit time, $|\frac{\mathrm{d}\mathbf{b}^{(k)}}{\mathrm{d}t}(t)|/|\mathbf{b}^{(k)}(t)|$, is 2 for $\mathbf{b}^{(1)}$ and 3 for $\mathbf{b}^{(2)}$. In the figure below, the darker rectangle is the initial square. That is, the square with edges $\mathbf{b}^{(k)}(t_0) = \hat{\mathbf{e}}^{(k)}$. The lighter rectangle is that with edges $\mathbf{b}^{(k)}(t)$ for some t a bit bigger than t_0.



Example 4.6.1

As time increases the initial cube becomes a larger and larger rectangle.

Example 4.6.2 (Example 4.6.1, generalized.)

The behaviour of Example 4.6.1 is typical of $\mathcal{V}$'s that are symmetric matrices, i.e. that obey[54] $\mathcal{V}_{i,j} = \mathcal{V}_{j,i}$ for all i, j. Any $d \times d$ symmetric matrix[55] (with real entries)

- has d real eigenvalues
- has d mutually orthogonal real unit eigenvectors.

Denote by λ_k, $1 \leqslant k \leqslant d$, the eigenvalues of $\mathcal{V}$ and choose d mutually perpendicular real unit vectors, $\hat{\mathbf{e}}^{(k)}$, that obey $\mathcal{V}\hat{\mathbf{e}}^{(k)} = \lambda_k\hat{\mathbf{e}}^{(k)}$ for all $1 \leqslant k \leqslant d$. Then

$$\mathbf{b}^{(k)}(t) = e^{\lambda_k(t-t_0)}\ \hat{\mathbf{e}}^{(k)}$$

54 In terms of our original vector field, this condition is that $\frac{\partial \mathbf{v}_i}{\partial x_j}(\mathbf{x}_0, t_0) = \frac{\partial \mathbf{v}_j}{\partial x_i}(\mathbf{x}_0, t_0)$. So, in three dimensions, it comes down to the requirement that $\nabla \times \mathbf{v}$ be zero at the point $(\mathbf{x}_0, t_0)$.

55 This was proven by the French mathematician and physicist Augustin-Louis Cauchy (1789–1857) in 1829.

obeys

$$\frac{\mathrm{d}\mathbf{b}^{(k)}}{\mathrm{d}t}(t) = \lambda_k e^{\lambda_k(t-t_0)}\,\hat{\mathbf{e}}^{(k)} = e^{\lambda_k(t-t_0)}\,\mathcal{V}\hat{\mathbf{e}}^{(k)} = \mathcal{V}\mathbf{b}^{(k)}(t) \qquad \text{and} \qquad \mathbf{b}^{(k)}(t_0) = \hat{\mathbf{e}}^{(k)}$$

So $\mathbf{b}^{(k)}(t) = e^{\lambda_k(t-t_0)}\,\hat{\mathbf{e}}^{(k)}$ satisfies (IVP) for all t and $1 \leqslant k \leqslant d$.

If we start, at time t_0, with a cube whose edges, $\hat{\mathbf{e}}^{(k)}$, are eigenvectors of $\mathcal{V}$, then as time progresses the edges, $\mathbf{b}^{(k)}(t)$, of the chunk of fluid never change direction. But their lengths change with the relative rate of change of length per unit time being λ_k for edge number k. This rate of change may be positive (the edge grows with time) or negative (the edge shrinks in time) depending on the sign of λ_k.

The volume of the chunk of fluid at time t is $V(t) = e^{\lambda_1(t-t_0)} \cdots e^{\lambda_d(t-t_0)}$. The relative rate of change of volume per unit time is $V'(t)/V(t) = \lambda_1 \cdots + \lambda_d$, the sum of the d eigenvalues. The sum of the eigenvalues of any $d \times d$ matrix $\mathcal{V}$ is given by its trace $\sum_{i=1}^{d} \mathcal{V}_{i,i}$. For the matrix (M)

$$\frac{V'(t_0)}{V(t_0)} = \sum_{i=1}^{d} \frac{\partial v_i}{\partial x_i}(\mathbf{x}_0, t_0) = \boldsymbol{\nabla} \cdot \mathbf{v}(\mathbf{x}_0, t_0)$$

So, at least when the matrix $\mathcal{V}$ defined in (M) is symmetric, the divergence $\boldsymbol{\nabla} \cdot \mathbf{v}(\mathbf{x}_0, t_0)$ gives the relative rate of change of volume per unit time for our tiny chunk of fluid at time t_0 and position $\mathbf{x}_0$. Thus when $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$ the volume is fixed. In particular, this is the case when the fluid is incompressible.

Example 4.6.2

In fact we can relax the symmetry condition.

Example 4.6.3 (Example 4.6.1, generalized yet again.)

For any $d \times d$ matrix $\mathcal{V}$, the solution of

$$\mathbf{b}'(t) = \mathcal{V}\mathbf{b}(t) \quad \mathbf{b}(t_0) = \mathbf{e}$$

is

$$\mathbf{b}(t) = e^{\mathcal{V}(t-t_0)}\mathbf{e}$$

where the exponential of a $d \times d$ matrix B is defined by the power series

$$e^B = \mathbb{1} + B + \tfrac{1}{2}B^2 + \frac{1}{3!}B^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}B^n$$

with $\mathbb{1}$ denoting the $d \times d$ identity matrix. This sum converges[56] for all $d \times d$ matrices B. Furthermore it easy to check, using the power series, that $e^{\mathcal{V}(t-t_0)}$ obeys $\frac{\mathrm{d}}{\mathrm{d}t}e^{\mathcal{V}(t-t_0)} = \mathcal{V}e^{\mathcal{V}(t-t_0)}$ and is the identity matrix when $t = t_0$. So $\mathbf{b}(t) = e^{\mathcal{V}(t-t_0)}\mathbf{e}$ really does obey $\mathbf{b}'(t) = \mathcal{V}\mathbf{b}(t)$ and $\mathbf{b}(t_0) = \mathbf{e}$.

Pick any d vectors $\mathbf{e}^{(k)}$, $1 \leqslant k \leqslant d$, and define $\mathbf{b}^{(k)}(t) = e^{\mathcal{V}(t-t_0)}\mathbf{e}^{(k)}$. Also let E be the $d \times d$ matrix whose k^{th} column is $\mathbf{e}^{(k)}$ and $E(t)$ be the $d \times d$ matrix whose k^{th} column is

56 The proof is not so hard, though we'll only outline it. Just denote by β the magnitude of the largest matrix element of B. Then use the definition of the matrix product to prove that the largest matrix element of B^n has magnitude at most $(d\beta)^n$.

$\mathbf{b}^{(k)}(t)$. Then the volume of the parallelepiped with edges $\mathbf{e}^{(k)}$, $1 \leqslant k \leqslant d$, is $V(t_0) = \det E$ and the volume of the parallelepiped with edges $\mathbf{b}^{(k)}(t)$, $1 \leqslant k \leqslant d$, is

$$V(t) = \det E(t) = \det\big(e^{\mathcal{V}(t-t_0)}E\big) = \det\big(e^{\mathcal{V}(t-t_0)}\big)\det E = \det\big(e^{\mathcal{V}(t-t_0)}\big)V(t_0)$$

Of course now we have to compute the determinant of the exponential of a matrix. Luckily, there is an easy way to do this. For any $d \times d$ matrix B, we have[57] $\det e^B = e^{\operatorname{tr} B}$, where $\operatorname{tr} B$, called the trace of the matrix B, is the sum of the diagonal matrix elements of B. So

$$V(t) = e^{(t-t_0)\operatorname{tr}\mathcal{V}}V(t_0) \qquad \Rightarrow \qquad \frac{V'(t_0)}{V(t_0)} = \operatorname{tr}\mathcal{V} = \sum_{i=1}^{d}\mathcal{V}_{i,i}$$

So, for any matrix $\mathcal{V}$ defined as in (M) and any choice of $\hat{\mathbf{e}}^{(k)}$, $1 \leqslant k \leqslant d$, the divergence $\boldsymbol{\nabla}\cdot\mathbf{v}\big(\mathbf{x}_0, t_0\big)$ gives the relative rate of change of volume per unit time for our tiny chunk of fluid at time t_0 and position $\mathbf{x}_0$.

Example 4.6.3

Example 4.6.4 ($\mathbf{v}(x,y) = -y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}}$)

In this example

$$\mathcal{V} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

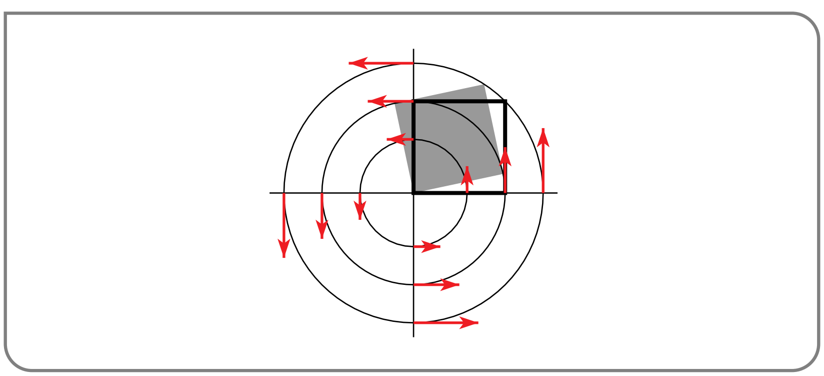
The solution[58] to

$$\mathbf{b}'(t) = \mathcal{V}\mathbf{b}(t) \quad \mathbf{b}(0) = \begin{bmatrix}\beta_1 \\ \beta_2\end{bmatrix} \quad \text{or equivalently} \quad \begin{matrix} b_1'(t) = -b_2(t) & b_1(0) = \beta_1 \\ b_2'(t) = b_1(t) & b_2(0) = \beta_2 \end{matrix}$$

is

$$\begin{matrix} b_1(t) = \beta_1\cos t - \beta_2\sin t \\ b_2(t) = \beta_1\sin t + \beta_2\cos t \end{matrix} \quad \text{or equivalently} \quad \mathbf{b}(t) = \begin{bmatrix}\cos t & -\sin t \\ \sin t & \cos t\end{bmatrix}\mathbf{b}(0)$$

Consequently the vector $\mathbf{b}(t)$ has the same length as $\mathbf{b}(0)$. The angle between $\mathbf{b}(t)$ and $\mathbf{b}(0)$ is just t radians. So, in this example, no matter what direction vectors $\hat{\mathbf{e}}^{(k)}$ we pick, the chunk of fluid just rotates at one radian per unit time. In the figure below, the outlined rectangle is the initial square. That is, the square with edges $\mathbf{b}^{(k)}(t_0) = \hat{\mathbf{e}}^{(k)}$. The shaded rectangle is that with edges $\mathbf{b}^{(k)}(t)$ for some t a bit bigger than t_0.



57 Again, we won't prove this. But for a diagonal matrix, it is easy — just compute both sides. So for a diagonalizable matrix it is also easy — diagonalize.

58 You can find the solution either by guessing, or by using eigenvalues and eigenvectors.

Example 4.6.4

Example 4.6.5 (Example 4.6.4, generalized.)

The behaviour of Example 4.6.4 is typical of $\mathcal{V}$'s that are antisymmetric matrices, i.e. that obey $\mathcal{V}_{i,j} = -\mathcal{V}_{j,i}$ for all i, j. As we have already observed, for any $d \times d$ matrix $\mathcal{V}$, the solution of $\mathbf{b}'(t) = \mathcal{V}\mathbf{b}(t)$, $\mathbf{b}(0) = \mathbf{e}$ is $\mathbf{b}(t) = e^{\mathcal{V}t}\mathbf{e}$. We now show that if $\mathcal{V}$ is a 3×3 antisymmetric matrix, then $e^{\mathcal{V}t}$ is a rotation.

Assuming that $\mathcal{V}$ is not the zero matrix (in which case $e^{\mathcal{V}t}$ is the identity matrix for all t), we can find a number $\Omega > 0$ and a unit vector $\hat{\mathbf{k}} = (k_1, k_2, k_3)$ (not necessarily the standard unit vector parallel to the z-axis) such that

$$\mathcal{V} = \begin{bmatrix} 0 & -\Omega k_3 & \Omega k_2 \\ \Omega k_3 & 0 & -\Omega k_1 \\ -\Omega k_2 & \Omega k_1 & 0 \end{bmatrix} \tag{R}$$

This is easy. Because $\mathcal{V}$ is antisymmetric, all of the entries on its diagonal must be zero. Define Ω to be $\sqrt{\mathcal{V}_{1,2}^2 + \mathcal{V}_{1,3}^2 + \mathcal{V}_{2,3}^2}$ and $k_1 = -\mathcal{V}_{2,3}/\Omega$, $k_2 = \mathcal{V}_{1,3}/\Omega$, $k_3 = -\mathcal{V}_{1,2}/\Omega$. Also, let $\hat{\boldsymbol{\imath}}$ be any unit vector orthogonal to $\hat{\mathbf{k}}$ (again, not necessarily the standard one) and $\hat{\boldsymbol{\jmath}} = \hat{\mathbf{k}} \times \hat{\boldsymbol{\imath}}$. So $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, $\hat{\mathbf{k}}$ is a right-handed system of three mutually perpendicular unit vectors.

Observe that, for any vector $\mathbf{e} = (e_1, e_2, e_3)$

$$\mathcal{V}\mathbf{e} = \begin{bmatrix} 0 & -\Omega k_3 & \Omega k_2 \\ \Omega k_3 & 0 & -\Omega k_1 \\ -\Omega k_2 & \Omega k_1 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \Omega \begin{bmatrix} k_2 e_3 - k_3 e_2 \\ k_3 e_1 - k_1 e_3 \\ k_1 e_2 - k_2 e_1 \end{bmatrix} = \Omega \hat{\mathbf{k}} \times \mathbf{e}$$

In particular,

$$\begin{array}{lll}
\mathcal{V}\hat{\boldsymbol{\imath}} = \Omega\hat{\mathbf{k}} \times \hat{\boldsymbol{\imath}} = \Omega\hat{\boldsymbol{\jmath}} & \mathcal{V}\hat{\boldsymbol{\jmath}} = \Omega\hat{\mathbf{k}} \times \hat{\boldsymbol{\jmath}} = -\Omega\hat{\boldsymbol{\imath}} & \mathcal{V}\hat{\mathbf{k}} = \Omega\hat{\mathbf{k}} \times \hat{\mathbf{k}} = \mathbf{0} \\
\mathcal{V}^2\hat{\boldsymbol{\imath}} = \Omega\mathcal{V}\hat{\boldsymbol{\jmath}} = -\Omega^2\hat{\boldsymbol{\imath}} & \mathcal{V}^2\hat{\boldsymbol{\jmath}} = -\Omega\mathcal{V}\hat{\boldsymbol{\imath}} = -\Omega^2\hat{\boldsymbol{\jmath}} & \mathcal{V}^2\hat{\mathbf{k}} = \mathcal{V}\mathbf{0} = \mathbf{0} \\
\mathcal{V}^3\hat{\boldsymbol{\imath}} = \Omega\mathcal{V}^2\hat{\boldsymbol{\jmath}} = -\Omega^3\hat{\boldsymbol{\jmath}} & \mathcal{V}^3\hat{\boldsymbol{\jmath}} = -\Omega\mathcal{V}^2\hat{\boldsymbol{\imath}} = \Omega^3\hat{\boldsymbol{\imath}} & \mathcal{V}^3\hat{\mathbf{k}} = \mathcal{V}^2\mathbf{0} = \mathbf{0} \\
\mathcal{V}^4\hat{\boldsymbol{\imath}} = \Omega\mathcal{V}^3\hat{\boldsymbol{\jmath}} = \Omega^4\hat{\boldsymbol{\imath}} & \mathcal{V}^4\hat{\boldsymbol{\jmath}} = -\Omega\mathcal{V}^3\hat{\boldsymbol{\imath}} = \Omega^4\hat{\boldsymbol{\jmath}} & \mathcal{V}^4\hat{\mathbf{k}} = \mathcal{V}^3\mathbf{0} = \mathbf{0}
\end{array}$$

and so on. For all odd $n \geqslant 1$,

$$\mathcal{V}^n\hat{\boldsymbol{\imath}} = (-1)^{(n-1)/2}\Omega^n\hat{\boldsymbol{\jmath}} \qquad \mathcal{V}^n\hat{\boldsymbol{\jmath}} = -(-1)^{(n-1)/2}\Omega^n\hat{\boldsymbol{\imath}} \qquad \mathcal{V}^n\hat{\mathbf{k}} = \mathbf{0}$$

and all even $n \geqslant 2$,

$$\mathcal{V}^n\hat{\boldsymbol{\imath}} = (-1)^{n/2}\Omega^n\hat{\boldsymbol{\imath}} \qquad \mathcal{V}^n\hat{\boldsymbol{\jmath}} = (-1)^{n/2}\Omega^n\hat{\boldsymbol{\jmath}} \qquad \mathcal{V}^n\hat{\mathbf{k}} = \mathbf{0}$$

Hence we can write

$$\begin{aligned}
e^{\mathcal{V}t}\hat{\boldsymbol{\imath}} &= \sum_{n=0}^{\infty}\frac{1}{n!}(\mathcal{V}t)^n\hat{\boldsymbol{\imath}} \;=\; \sum_{n\text{ even}}\frac{(-1)^{n/2}}{n!}(\Omega t)^n\hat{\boldsymbol{\imath}} \;+\; \sum_{n\text{ odd}}\frac{(-1)^{(n-1)/2}}{n!}(\Omega t)^n\hat{\boldsymbol{\jmath}}\\
&= \cos(\Omega t)\,\hat{\boldsymbol{\imath}}+\sin(\Omega t)\,\hat{\boldsymbol{\jmath}}\\
e^{\mathcal{V}t}\hat{\boldsymbol{\jmath}} &= \sum_{n=0}^{\infty}\frac{1}{n!}(\mathcal{V}t)^n\hat{\boldsymbol{\jmath}} \;=\; \sum_{n\text{ even}}\frac{(-1)^{n/2}}{n!}(\Omega t)^n\hat{\boldsymbol{\jmath}} \;-\; \sum_{n\text{ odd}}\frac{(-1)^{(n-1)/2}}{n!}(\Omega t)^n\hat{\boldsymbol{\imath}}\\
&= -\sin(\Omega t)\,\hat{\boldsymbol{\imath}}+\cos(\Omega t)\,\hat{\boldsymbol{\jmath}}\\
e^{\mathcal{V}t}\hat{\mathbf{k}} &= \sum_{n=0}^{\infty}\frac{1}{n!}(\mathcal{V}t)^n\hat{\mathbf{k}} = \hat{\mathbf{k}}
\end{aligned}$$

So $e^{\mathcal{V}t}$ is rotation by an angle Ωt about the axis $\hat{\mathbf{k}}$.

Example 4.6.5

Example 4.6.6 (Example 4.6.5, continued.)

Whether or not the matrix $\mathcal{V}$ defined in (M) is antisymmetric, the related matrix with entries

$$A_{i,j} = \tfrac{1}{2}\big(\mathcal{V}_{i,j} - \mathcal{V}_{j,i}\big)$$

is. When $\mathcal{V}$ is antisymmetric, A and $\mathcal{V}$ coincide. The matrix A is (to write it out explicitly)

$$A=\frac{1}{2}\begin{bmatrix} 0 & \frac{\partial \mathbf{v}_1}{\partial x_2}(\mathbf{x}_0,t_0)-\frac{\partial \mathbf{v}_2}{\partial x_1}(\mathbf{x}_0,t_0) & \frac{\partial \mathbf{v}_1}{\partial x_3}(\mathbf{x}_0,t_0)-\frac{\partial \mathbf{v}_3}{\partial x_1}(\mathbf{x}_0,t_0)\\ -\frac{\partial \mathbf{v}_1}{\partial x_2}(\mathbf{x}_0,t_0)+\frac{\partial \mathbf{v}_2}{\partial x_1}(\mathbf{x}_0,t_0) & 0 & \frac{\partial \mathbf{v}_2}{\partial x_3}(\mathbf{x}_0,t_0)-\frac{\partial \mathbf{v}_3}{\partial x_2}(\mathbf{x}_0,t_0)\\ -\frac{\partial \mathbf{v}_1}{\partial x_3}(\mathbf{x}_0,t_0)+\frac{\partial \mathbf{v}_3}{\partial x_1}(\mathbf{x}_0,t_0) & -\frac{\partial \mathbf{v}_2}{\partial x_3}(\mathbf{x}_0,t_0)+\frac{\partial \mathbf{v}_3}{\partial x_2}(\mathbf{x}_0,t_0) & 0 \end{bmatrix}$$

Comparing this with (R), we see that

$$\Omega\hat{\mathbf{k}} = \tfrac{1}{2}\nabla\times\mathbf{v}\big(\mathbf{x}_0,t_0\big)$$

So, at least when the matrix $\mathcal{V}$ defined in (M) is antisymmetric, our tiny cube rotates about the axis with $\nabla\times\mathbf{v}\big(\mathbf{x}_0,t_0\big)$ at rate $\frac{1}{2}\big|\nabla\times\mathbf{v}\big(\mathbf{x}_0,t_0\big)\big|$.

Example 4.6.6

Remark:

In the generalization, Example 4.6.5, of Example 4.6.4, we only considered dimension 3. It is a nice exercise in eigenvalues and eigenvectors to handle general dimension. Here are the main facts about antisymmetric matrices with real entries that are used.

- All eigenvalues of antisymmetric matrices are either zero or pure imaginary.
- For antisymmetric matrices with real entries, the nonzero eigenvalues come in complex conjugate pairs. The corresponding eigenvectors may also be chosen to be complex conjugates.

Choose as basis vectors (like $\hat{\boldsymbol{\imath}},\ \hat{\boldsymbol{\jmath}},\ \hat{\mathbf{k}}$ above)

- the eigenvectors of eigenvalue 0 (they act like $\hat{\mathbf{k}}$ above)
- the real and imaginary parts of each complex conjugate pair of eigenvectors (they act like $\hat{\imath}$, $\hat{\jmath}$ above)

Resumé so far:
We have now seen that

- when the matrix $\mathcal{V}$ defined in (M) is symmetric and the direction vectors $\hat{\mathbf{e}}^{(k)}$ of the cube are eigenvectors of $\mathcal{V}$, then, at time t_0, the chunk of fluid is not changing orientation but is changing volume at instantaneous relative rate $\nabla \cdot \mathbf{v}(\mathbf{x}_0, t_0)$ and
- when the matrix $\mathcal{V}$ defined in (M) is antisymmetric, then, at time t_0, the chunk of fluid is not changing shape or size but is rotating about the axis $\nabla \times \mathbf{v}(\mathbf{x}_0, t_0)$ at rate $\frac{1}{2}|\nabla \times \mathbf{v}(\mathbf{x}_0, t_0)|$. For this reason, $\nabla \times \mathbf{v}$ is often referred to as a "vorticity" meter.

These agree with our earlier interpretations of divergence and curl.

The general case:
Now consider a general matrix $\mathcal{V}$. It can always be written as the sum

$$\mathcal{V} = S + A$$

of a symmetric matrix S and an antisymmetric matrix A. Just define

$$S_{i,j} = \tfrac{1}{2}(\mathcal{V}_{i,j} + \mathcal{V}_{j,i}) \qquad A_{i,j} = \tfrac{1}{2}(\mathcal{V}_{i,j} - \mathcal{V}_{j,i})$$

As we have already observed, the solution of

$$\mathbf{b}'(t) = \mathcal{V}\mathbf{b}(t) \quad \mathbf{b}(0) = \mathbf{e}$$

is

$$\mathbf{b}(t) = e^{\mathcal{V}t}\mathbf{e} = e^{(A+S)t}\mathbf{e}$$

If S and A were ordinary numbers, we would have $e^{(A+S)t} = e^{At}e^{St}$. But for matrices this need not be the case, unless S and A happen to commute[59]. For arbitrary matrices, it is still true that

$$e^{(A+S)t} = \lim_{n\to\infty} \left[e^{At/n}e^{St/n}\right]^n$$

This is called the Lie[60] product formula. It shows that our tiny chunk of fluid mixes together the behaviours of A and S, scaling a bit, then rotating a bit, then scaling a bit and so on.

Example 4.6.7 ($\mathbf{v}(x, y) = 2y\hat{\imath}$.)

In this example

$$\mathcal{V} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = S + A \qquad \text{with} \qquad S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

59 By definition, the matrices S and A commute when $AS = SA$.
60 This formula is named after the Norwegian mathematician Marius Sophus Lie (1842–1899). In 1870, he was arrested and held in prison in France for a month, because he was suspected of being a German spy. His mathematics notes were thought to be top secret coded messages.

The solution to the full flow

$$\mathbf{b}'(t) = \mathcal{V}\mathbf{b}(t) \quad \mathbf{b}(0) = \begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix} \quad \text{or equivalently} \quad \begin{matrix} b_1'(t) = 2b_2(t) & b_1(0) = \beta_1 \\ b_2'(t) = 0 & b_2(0) = \beta_2 \end{matrix}$$

is

$$\begin{matrix} b_1(t) = \beta_1 + 2\beta_2 t \\ b_2(t) = \beta_2 \end{matrix} \quad \text{or equivalently} \quad \mathbf{b}(t) = \begin{bmatrix}1 & 2t\\ 0 & 1\end{bmatrix}\mathbf{b}(0)$$

The solution to the S part of the flow

$$\mathbf{b}'(t) = S\mathbf{b}(t) \quad \mathbf{b}(0) = \begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix} \quad \text{or equivalently} \quad \begin{matrix} b_1'(t) = b_2(t) & b_1(0) = \beta_1 \\ b_2'(t) = b_1(t) & b_2(0) = \beta_2 \end{matrix}$$

is[61]

$$\begin{matrix} b_1(t) = \beta_1 \cosh t + \beta_2 \sinh t \\ b_2(t) = \beta_1 \sinh t + \beta_2 \cosh t \end{matrix} \quad \text{or equivalently} \quad \mathbf{b}(t) = \begin{bmatrix}\cosh t & \sinh t\\ \sinh t & \cosh t\end{bmatrix}\mathbf{b}(0)$$

The eigenvectors of S are

$$\hat{\mathbf{e}}^{(1)} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix} \qquad \hat{\mathbf{e}}^{(2)} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix}$$

The corresponding eigenvalues are $+1$ and -1. The eigenvectors obey

$$e^{St}\hat{\mathbf{e}}^{(1)} = \begin{bmatrix}\cosh t & \sinh t\\ \sinh t & \cosh t\end{bmatrix}\hat{\mathbf{e}}^{(1)} = e^{t}\hat{\mathbf{e}}^{(1)} \qquad e^{St}\hat{\mathbf{e}}^{(2)} = \begin{bmatrix}\cosh t & \sinh t\\ \sinh t & \cosh t\end{bmatrix}\hat{\mathbf{e}}^{(2)} = e^{-t}\hat{\mathbf{e}}^{(2)}$$

Under the S part of the flow $\hat{\mathbf{e}}^{(1)}$ scales by a factor of e^t, which is bigger than one for $t > 0$ and $\hat{\mathbf{e}}^{(2)}$ scales by a factor of e^{-t}, which is smaller than one for $t > 0$.

The solution to the A part of the flow

$$\mathbf{b}'(t) = A\mathbf{b}(t) \quad \mathbf{b}(0) = \begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix} \quad \text{or equivalently} \quad \begin{matrix} b_1'(t) = b_2(t) & b_1(0) = \beta_1 \\ b_2'(t) = -b_1(t) & b_2(0) = \beta_2 \end{matrix}$$

is

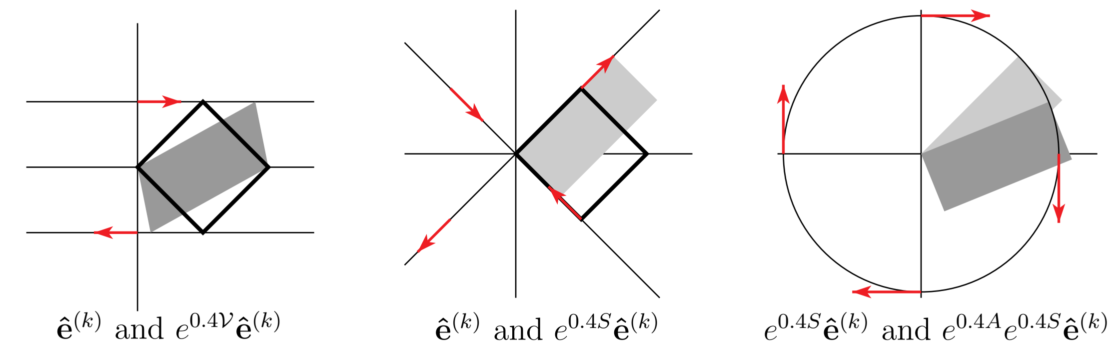
$$\begin{matrix} b_1(t) = \beta_1 \cos t + \beta_2 \sin t \\ b_2(t) = -\beta_1 \sin t + \beta_2 \cos t \end{matrix} \quad \text{or equivalently} \quad \mathbf{b}(t) = \begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}\mathbf{b}(0)$$

The A part of the flow rotates clockwise about the origin at one radian per unit time.

Here are some figures to help us visualize this.

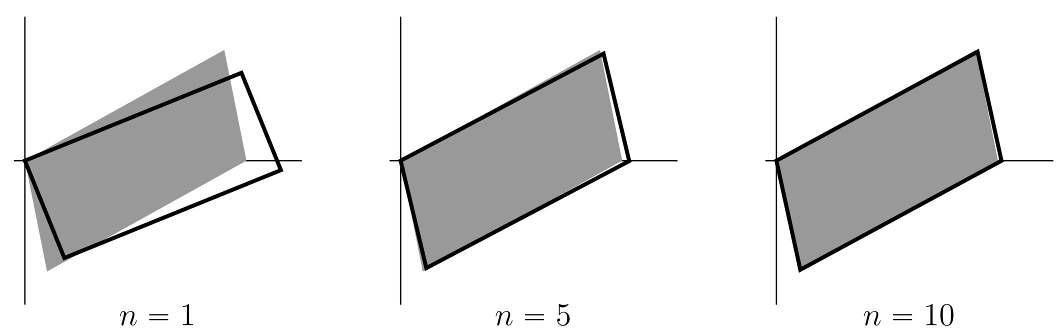
- The first shows a square with edges $\hat{\mathbf{e}}^{(1)}$, $\hat{\mathbf{e}}^{(2)}$ and its image under the full flow $t = 0.4$ later. Under this full flow the vector $\hat{\mathbf{e}}^{(k)} \to e^{0.4\mathcal{V}}\hat{\mathbf{e}}^{(k)}$. The darkly shaded parallelogram has edges $e^{0.4\mathcal{V}}\hat{\mathbf{e}}^{(k)}$.
- The second shows its image under 0.4 time units of the S-flow (that is, $\hat{\mathbf{e}}^{(k)} \to e^{0.4S}\hat{\mathbf{e}}^{(k)}$). The lightly shaded rectangle has edges $e^{0.4S}\hat{\mathbf{e}}^{(k)}$.
- The third applies 0.4 time units of the A-flow to the shaded rectangle of the middle figure. So the lightly shaded rectangle of the third figure has edges $e^{0.4S}\hat{\mathbf{e}}^{(k)}$ and the darkly shaded rectangle has edges $e^{0.4A}e^{0.4S}\hat{\mathbf{e}}^{(k)}$.

61 Recall that $\sinh t = \frac{1}{2}(e^t - e^{-t})$ and $\cosh t = \frac{1}{2}(e^t + e^{-t})$.



Of course $e^{0.4A}e^{0.4S}\hat{\mathbf{e}}^{(k)}$ (as in the darly shaded rectangle of the right hand figure) is not a very good approximation for $e^{0.4(A+S)}\hat{\mathbf{e}}^{(k)}$ (as in the darkly shaded parallelogram of the left hand figure). It is much better to take $\big[e^{0.4A/n}e^{0.4S/n}\big]^n\hat{\mathbf{e}}^{(k)}$ with n large.

Each of the following figures shows two parallelograms. In each, the shaded region has edges $e^{0.4\mathcal{V}}\hat{\mathbf{e}}^{(k)} = e^{0.4(A+S)}\hat{\mathbf{e}}^{(k)}$ and the outlined region has edges $\big[e^{0.4A/n}e^{0.4S/n}\big]^n\hat{\mathbf{e}}^{(k)}$.



So we can see that, as n increases, $\big[e^{0.4A/n}e^{0.4S/n}\big]^n\hat{\mathbf{e}}^{(k)}$ becomes a better and better approximation to $e^{0.4(A+S)}\hat{\mathbf{e}}^{(k)}$.

Example 4.6.7

4.7▲ Optional — A Generalized Stokes' Theorem

As we have seen, the fundamental theorem of calculus, the divergence theorem, Greens' theorem and Stokes' theorem share a number of common features. There is in fact a single framework which encompasses and generalizes all of them, and there is a single theorem of which they are all special cases. We now give a bare bones introduction to this framework and theorem. A proper treatment typically takes up a good part of a full course. Here is an outline of what we shall do:

- First, we will define "differential forms". To try and keep things as simple and concrete as possible, we'll only define[62] differential forms on $\mathbb{R}^3$ — all of our functions will be defined on $\mathbb{R}^3$. Very roughly speaking, a k-form is what you write after the integral sign of an integral over a k dimensional object. Here k is one of 0, 1, 2, 3. As a example, a 1-form is an expression of the form $F_1(x,y,z)\,\mathrm{d}x + F_2(x,y,z)\,\mathrm{d}y + F_3(x,y,z)\,\mathrm{d}z$. For $k = 0$, think of a point as a zero dimensional object and think of evaluating a function at a point as "integrating the function over the point".
- Then we will define some operations on differential forms, so that we can add them, multiply them, differentiate them and, eventually, integrate them. The derivative of a k-form ω is a $(k+1)$-form that is denoted $\mathrm{d}\omega$. It will turn out that
  - differentiating a 0-form amounts to taking a gradient,
  - differentiating a 1-form amounts to taking a curl, and
  - differentiating a 2-form amounts to taking a divergence.
- Finally we will get to the generalized Stokes' theorem which says that, if ω is a k-form (with $k = 0, 1, 2$) and D is a $(k+1)$-dimensional domain of integration, then

$$\int_D d\omega = \int_{\partial D} \omega$$

  It will turn out that
  - when $k = 0$, this is just the fundamental theorem of calculus and
  - when $k = 1$, this is both Green's theorem and our Stokes' theorem, and
  - when $k = 2$, this is the divergence theorem.

Now let's get to work. For simplicity, we will assume throughout this section that all derivatives of all functions exist and are continuous. Our first task to define differential forms.

As we said above we will define a 1-form as an expression of the form $F_1(x,y,z)\,\mathrm{d}x + F_2(x,y,z)\,\mathrm{d}y + F_3(x,y,z)\,\mathrm{d}z$. When you learned the definition of the integral the symbol "$\mathrm{d}x$" was not given any mathematical meaning by itself. A meaning was given only to the collections of symbols "$\int f(x)\,\mathrm{d}x$" and "$\int_a^b f(x)\,\mathrm{d}x$". Later in this section, we will give a meaning to $\mathrm{d}x$. We will, in Definition 4.7.11, define a differentiation operator that we will call d. Then $\mathrm{d}x$ will be that differentiation operator applied to the function $f(x) = x$. However, until then we will have to treat $\mathrm{d}x$ and $\mathrm{d}y$ and $\mathrm{d}z$ just as symbols. Their sole role in $F_1(x,y,z)\,\mathrm{d}x + F_2(x,y,z)\,\mathrm{d}y + F_3(x,y,z)\,\mathrm{d}z$ is to allow us to distinguish[63] $F_1(x,y,z)$, $F_2(x,y,z)$ and $F_3(x,y,z)$.

Similarly, we will define a 2-form as an expression of the form $F_1(x,y,z)\,\mathrm{d}y \wedge \mathrm{d}z + F_2(x,y,z)\,\mathrm{d}z \wedge \mathrm{d}x + F_3(x,y,z)\,\mathrm{d}x \wedge \mathrm{d}y$. Once again there is a symbol, namely "$\wedge$", that we have not yet given a meaning to. We will, in Definition 4.7.3, define a product, called the wedge product, with $\wedge$ as the multiplication symbol. Then $\mathrm{d}x \wedge \mathrm{d}y$ will be the wedge

62 In general, a differential form is defined on a "manifold", which is an abstract generalization of a multi-dimensional surface, like a sphere or a torus.

63 We could also define, for example, a 1-form as an ordered list $\big(F_1(x,y,z)\,,\, F_2(x,y,z)\,,\, F_3(x,y,z)\big)$ of three functions and just view $F_1(x,y,z)\,\mathrm{d}x + F_2(x,y,z)\,\mathrm{d}y + F_3(x,y,z)\,\mathrm{d}z$ as another notation for $\big(F_1(x,y,z)\,,\, F_2(x,y,z)\,,\, F_3(x,y,z)\big)$.

product of $\mathrm{d}x$ and $\mathrm{d}y$. Until then we will have to treat $\mathrm{d}y \wedge \mathrm{d}z$, $\mathrm{d}z \wedge \mathrm{d}x$ and $\mathrm{d}x \wedge \mathrm{d}y$ just as three more meaningless symbols.

Finally here is the definition.

Definition 4.7.1.

(a) A 0-form is a function $f(x, y, z)$.

(b) A 1-form is an expression of the form

$$F_1(x, y, z)\,\mathrm{d}x + F_2(x, y, z)\,\mathrm{d}y + F_3(x, y, z)\,\mathrm{d}z$$

with $F_1(x, y, z)$, $F_2(x, y, z)$ and $F_3(x, y, z)$ being functions of three variables.

(c) A 2-form is an expression of the form

$$F_1(x, y, z)\,\mathrm{d}y \wedge \mathrm{d}z + F_2(x, y, z)\,\mathrm{d}z \wedge \mathrm{d}x + F_3(x, y, z)\,\mathrm{d}x \wedge \mathrm{d}y$$

with $F_1(x, y, z)$, $F_2(x, y, z)$ and $F_3(x, y, z)$ being functions of three variables.

(d) A 3-form is an expression of the form $f(x, y, z)\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$, with $f(x, y, z)$ being a function of three variables.

At this stage (there'll be more later), just think of "$\mathrm{d}x$", "$\mathrm{d}y$", "$\mathrm{d}z$", "$\mathrm{d}x \wedge \mathrm{d}y$", and so on, as symbols. Do not yet attempt to attach any significance to them.

There are four operations involving differential forms — addition, multiplication ($\wedge$), differentiation (d) and integration. Here are their definitions. First, addition is defined, and works, just the way that you would expect it to.

Definition 4.7.2 (Addition of differential forms).

(a) The sum of the 0-forms f and g is the 0-form $f + g$.

(b) The sum of two 1-forms is the 1-form

$$\begin{aligned}
&\big[F_1\,\mathrm{d}x + F_2\,\mathrm{d}y + F_3\,\mathrm{d}z\big] \\
+&\big[G_1\,\mathrm{d}x + G_2\,\mathrm{d}y + G_3\,\mathrm{d}z\big] \\
=&(F_1 + G_1)\,\mathrm{d}x + (F_2 + G_2)\,\mathrm{d}y + (F_3 + G_3)\,\mathrm{d}z
\end{aligned}$$

(c) The sum of two 2-forms is the 2-form

$$\begin{aligned}
&\big[F_1\,\mathrm{d}y \wedge \mathrm{d}z + F_2\,\mathrm{d}z \wedge \mathrm{d}x + F_3\,\mathrm{d}x \wedge \mathrm{d}y\big] \\
+&\big[G_1\,\mathrm{d}y \wedge \mathrm{d}z + G_2\,\mathrm{d}z \wedge \mathrm{d}x + G_3\,\mathrm{d}x \wedge \mathrm{d}y\big] \\
=&(F_1 + G_1)\,\mathrm{d}y \wedge \mathrm{d}z + (F_2 + G_2)\,\mathrm{d}z \wedge \mathrm{d}x + (F_3 + G_3)\,\mathrm{d}x \wedge \mathrm{d}y
\end{aligned}$$

Definition 4.7.2 (continued)**.**

(d) The sum of two 3-forms is the 3-form

$$f\,\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z \;+\; g\,\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z \;=\; \big(f+g\big)\,\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z$$

There is one wrinkle in multiplication. It is not commutative, meaning that $\alpha\wedge\beta$ need not be the same as $\beta\wedge\alpha$. You have already seen some noncommutative products. If **a** and **b** are two vectors in $\mathbb{R}^3$, then $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$. Also, if A and B are two $n\times n$ matrices, the matrix product AB need not be the same as BA.

Definition 4.7.3 (Multiplication of differential forms)**.**

We now define a multiplication rule for differential forms. If ω is a k-form and ω' is a k'-form then the product will be a $(k+k')$-form and will be denoted $\omega\wedge\omega'$ (read "omega wedge omega prime"). It is determined by the following properties.

(a) If f is a function (i.e. a 0-form), then

$$\begin{aligned} f\big[F_1\,\mathrm{d}x+F_2\,\mathrm{d}y+F_3\,\mathrm{d}z\big] &= (fF_1)\,\mathrm{d}x+(fF_2)\,\mathrm{d}y+(fF_3)\,\mathrm{d}z\\ f\big[F_1\,\mathrm{d}y\wedge\mathrm{d}z+F_2\,\mathrm{d}z\wedge\mathrm{d}x+F_3\,\mathrm{d}x\wedge\mathrm{d}y\big] &= (fF_1)\,\mathrm{d}y\wedge\mathrm{d}z+(fF_2)\,\mathrm{d}z\wedge\mathrm{d}x\\ &\qquad +(fF_3)\,\mathrm{d}x\wedge\mathrm{d}y\\ f\big[g\,\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z\big] &= (fg)\,\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z \end{aligned}$$

Traditionally, the $\wedge$ is not written when multiplying a differential form by a function (i.e. a 0-form).

(b) $\omega\wedge\omega'$ is linear in ω and in ω'. This means that if $\omega=f_1\omega_1+f_2\omega_2$, where f_1,f_2 are functions and ω_1,ω_2 are forms, then

$$\big(f_1\omega_1+f_2\omega_2\big)\wedge\omega' = f_1(\omega_1\wedge\omega')+f_2(\omega_2\wedge\omega')$$

Similarly,

$$\omega\wedge\big(f_1\omega_1'+f_2\omega_2'\big) = f_1(\omega\wedge\omega_1')+f_2(\omega\wedge\omega_2')$$

Definition 4.7.3 (continued).

(c) If ω is a k-form and ω' is a k'-form then

$$\omega \wedge \omega' = (-1)^{kk'} \omega' \wedge \omega$$

That is, if at least one of k and k' is even, then

$$\omega \wedge \omega' = \omega' \wedge \omega$$

(so that the wedge product is commutative) and if k and k' are *both odd* then

$$\omega \wedge \omega' = -\omega' \wedge \omega$$

(so that the wedge product is anticommutative). In particular, if ω is a d-form with d *odd*

$$\omega \wedge \omega = 0$$

(d) The wedge product is associative. This means that

$$(\omega \wedge \omega') \wedge \omega'' = \omega \wedge \big(\omega' \wedge \omega''\big)$$

So the wedge product obeys most of the usual multiplication rules, with the one big exception that if ω is k-form and ω' is a k'-form with k and k' *both odd* then $\omega \wedge \omega' = -\omega' \wedge \omega$.

The best way to get a handle on the wedge product is to work through some examples, like these.

Example 4.7.4

Let $\omega = F_1\,\mathrm{d}x + F_2\,\mathrm{d}y + F_3\,\mathrm{d}z$ and $\omega' = G_1\,\mathrm{d}x + G_2\,\mathrm{d}y + G_3\,\mathrm{d}z$ be any two 1-forms. Their product is

$$\begin{aligned}
\omega \wedge \omega' &= \big[F_1\,\mathrm{d}x + F_2\,\mathrm{d}y + F_3\,\mathrm{d}z\big] \wedge \big[G_1\,\mathrm{d}x + G_2\,\mathrm{d}y + G_3\,\mathrm{d}z\big]\\
&= \ \big(F_1\,\mathrm{d}x\big) \wedge \big(G_1\,\mathrm{d}x\big) + \big(F_1\,\mathrm{d}x\big) \wedge \big(G_2\,\mathrm{d}y\big) + \big(F_1\,\mathrm{d}x\big) \wedge \big(G_3\,\mathrm{d}z\big)\\
&\quad + \big(F_2\,\mathrm{d}y\big) \wedge \big(G_1\,\mathrm{d}x\big) + \big(F_2\,\mathrm{d}y\big) \wedge \big(G_2\,\mathrm{d}y\big) + \big(F_2\,\mathrm{d}y\big) \wedge \big(G_3\,\mathrm{d}z\big)\\
&\quad + \big(F_3\,\mathrm{d}z\big) \wedge \big(G_1\,\mathrm{d}x\big) + \big(F_3\,\mathrm{d}z\big) \wedge \big(G_2\,\mathrm{d}y\big) + \big(F_3\,\mathrm{d}z\big) \wedge \big(G_3\,\mathrm{d}z\big)\\
&\qquad\quad \text{(by linearity, i.e. by part (b) of Definition 4.7.3)}\\
&= \ F_1G_1\,\mathrm{d}x \wedge \mathrm{d}x + F_1G_2\,\mathrm{d}x \wedge \mathrm{d}y + F_1G_3\,\mathrm{d}x \wedge \mathrm{d}z\\
&\quad + F_2G_1\,\mathrm{d}y \wedge \mathrm{d}x + F_2G_2\,\mathrm{d}y \wedge \mathrm{d}y + F_2G_3\,\mathrm{d}y \wedge \mathrm{d}z\\
&\quad + F_3G_1\,\mathrm{d}z \wedge \mathrm{d}x + F_3G_2\,\mathrm{d}z \wedge \mathrm{d}y + F_3G_3\,\mathrm{d}z \wedge \mathrm{d}z\\
&= \big(F_1G_2 - F_2G_1\big)\,\mathrm{d}x \wedge \mathrm{d}y + \big(F_3G_1 - F_1G_3\big)\,\mathrm{d}z \wedge \mathrm{d}x + \big(F_2G_3 - F_3G_2\big)\,\mathrm{d}y \wedge \mathrm{d}z
\end{aligned}$$

because

$$\mathrm{d}x \wedge \mathrm{d}x = \mathrm{d}y \wedge \mathrm{d}y = \mathrm{d}z \wedge \mathrm{d}z = 0$$

and

$$\mathrm{d}x \wedge \mathrm{d}y = -\mathrm{d}y \wedge \mathrm{d}x \qquad \mathrm{d}x \wedge \mathrm{d}z = -\mathrm{d}z \wedge \mathrm{d}x \qquad \mathrm{d}z \wedge \mathrm{d}y = -\mathrm{d}y \wedge \mathrm{d}z$$

Note that, if we view $\mathbf{F} = (F_1, F_2, F_3)$ and $\mathbf{G} = (G_1, G_2, G_3)$ as vectors, we can write the product simply as

Equation 4.7.5.

$$\begin{aligned}&\big[F_1\,\mathrm{d}x + F_2\,\mathrm{d}y + F_3\,\mathrm{d}z\big] \wedge \big[G_1\,\mathrm{d}x + G_2\,\mathrm{d}y + G_3\,\mathrm{d}z\big]\\ &\qquad = (\mathbf{F}\times\mathbf{G})_1\,\mathrm{d}y \wedge \mathrm{d}z + (\mathbf{F}\times\mathbf{G})_2\,\mathrm{d}z \wedge \mathrm{d}x + (\mathbf{F}\times\mathbf{G})_3\,\mathrm{d}x \wedge \mathrm{d}y\end{aligned}$$

where we are using $(\mathbf{F}\times\mathbf{G})_\ell$ to denote the ℓ^{th} component of the cross product $\mathbf{F}\times\mathbf{G}$. In the special case that $F_3 = G_3 = 0$, we have

Equation 4.7.6.

$$\big[F_1\,\mathrm{d}x + F_2\,\mathrm{d}y\big] \wedge \big[G_1\,\mathrm{d}x + G_2\,\mathrm{d}y\big] = \big(F_1G_2 - F_2G_1\big)\,\mathrm{d}x \wedge \mathrm{d}y = \det\begin{bmatrix} F_1 & F_2 \\ G_1 & G_2\end{bmatrix}\mathrm{d}x \wedge \mathrm{d}y$$

We can now see why in the Definition 4.7.1.c of 2-forms

- there were no $\mathrm{d}x \wedge \mathrm{d}x$ or $\mathrm{d}y \wedge \mathrm{d}y$ or $\mathrm{d}z \wedge \mathrm{d}z$ terms — they are all zero and
- there were no $\mathrm{d}y \wedge \mathrm{d}x$ or $\mathrm{d}z \wedge \mathrm{d}y$ or $\mathrm{d}x \wedge \mathrm{d}z$ terms — they can all be rewritten using $\mathrm{d}x \wedge \mathrm{d}y$, $\mathrm{d}y \wedge \mathrm{d}z$ and $\mathrm{d}z \wedge \mathrm{d}x$ terms (or vice versa).

The reason that we chose to write the Definition 4.7.1.c as

$$F_1\,\mathrm{d}y \wedge \mathrm{d}z + F_2\,\mathrm{d}z \wedge \mathrm{d}x + F_3\,\mathrm{d}x \wedge \mathrm{d}y$$

as opposed to in the form, for example,

$$f_1\,\mathrm{d}x \wedge \mathrm{d}y + f_2\,\mathrm{d}x \wedge \mathrm{d}z + f_3\,\mathrm{d}y \wedge \mathrm{d}z$$

was to make formulae like (4.7.5) work. The easy way to remember

$$F_1\,\mathrm{d}y \wedge \mathrm{d}z + F_2\,\mathrm{d}z \wedge \mathrm{d}x + F_3\,\mathrm{d}x \wedge \mathrm{d}y$$

is to rename (in your head) x, y, z to x_1, x_2, x_3. Then the subscripts in the three terms of

$$F_1\,\mathrm{d}x_2 \wedge \mathrm{d}x_3 + F_2\,\mathrm{d}x_3 \wedge \mathrm{d}x_1 + F_3\,\mathrm{d}x_1 \wedge \mathrm{d}x_2$$

are just $1, 2, 3$ and $2, 3, 1$ and $3, 1, 2$ — the three cyclic permutations of $1, 2, 3$.

Example 4.7.4

Example 4.7.7

The product of the (general) 1-form $\omega = F_1\,\mathrm{d}x + F_2\,\mathrm{d}y + F_3\,\mathrm{d}z$ and the (general) 2-form

$\omega' = \big[G_1\,\mathrm{d}y\wedge\mathrm{d}z + G_2\,\mathrm{d}z\wedge\mathrm{d}x + G_3\,\mathrm{d}x\wedge\mathrm{d}y\big]$ (again note the numbering of the coefficients in the 2-form) is

$$\begin{aligned}\omega\wedge\omega' &= \big[F_1\,\mathrm{d}x + F_2\,\mathrm{d}y + F_3\,\mathrm{d}z\big]\wedge\big[G_1\,\mathrm{d}y\wedge\mathrm{d}z + G_2\,\mathrm{d}z\wedge\mathrm{d}x + G_3\,\mathrm{d}x\wedge\mathrm{d}y\big]\\ &= F_1G_1\,\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z + F_2G_2\,\mathrm{d}y\wedge\mathrm{d}z\wedge\mathrm{d}x + F_3G_3\,\mathrm{d}z\wedge\mathrm{d}x\wedge\mathrm{d}y\\ &= (F_1G_1 + F_2G_2 + F_3G_3)\,\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z\end{aligned}$$

Here we have used that, for 1-forms, $\alpha\wedge\beta = -\beta\wedge\alpha$, so that

$$\begin{aligned}\mathrm{d}y\wedge\mathrm{d}z\wedge\mathrm{d}x &= -\mathrm{d}y\wedge\mathrm{d}x\wedge\mathrm{d}z = \mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z\\ \mathrm{d}z\wedge\mathrm{d}x\wedge\mathrm{d}y &= -\mathrm{d}x\wedge\mathrm{d}z\wedge\mathrm{d}y = \mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z\end{aligned}$$

We have also used that any wedge product of three d$\{x$ or y or $z\}$'s with at least two of the coordinates being the same is zero. For example

$$\mathrm{d}x\wedge\mathrm{d}z\wedge\mathrm{d}x = -\mathrm{d}x\wedge\mathrm{d}x\wedge\mathrm{d}z = 0$$

So

Equation 4.7.8.

$$\big[F_1\,\mathrm{d}x + F_2\,\mathrm{d}y + F_3\,\mathrm{d}z\big]\wedge\big[G_1\,\mathrm{d}y\wedge\mathrm{d}z + G_2\,\mathrm{d}z\wedge\mathrm{d}x + G_3\,\mathrm{d}x\wedge\mathrm{d}y\big] = \mathbf{F}\cdot\mathbf{G}\,\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z$$

Example 4.7.7

Example 4.7.9

Combining Examples 4.7.4 and 4.7.7, we have the wedge product of any three (general) 1-forms $F_1\,\mathrm{d}x + F_2\,\mathrm{d}y + F_3\,\mathrm{d}z$ and $G_1\,\mathrm{d}x + G_2\,\mathrm{d}y + G_3\,\mathrm{d}z$ and $H_1\,\mathrm{d}x + H_2\,\mathrm{d}y + H_3\,\mathrm{d}z$ is

$$\begin{aligned}&\big[F_1\,\mathrm{d}x + F_2\,\mathrm{d}y + F_3\,\mathrm{d}z\big]\wedge\big[G_1\,\mathrm{d}x + G_2\,\mathrm{d}y + G_3\,\mathrm{d}z\big]\wedge\big[H_1\,\mathrm{d}x + H_2\,\mathrm{d}y + H_3\,\mathrm{d}z\big]\\ &\quad= \big[F_1\,\mathrm{d}x + F_2\,\mathrm{d}y + F_3\,\mathrm{d}z\big]\wedge\big[(\mathbf{G}\times\mathbf{H})_1\,\mathrm{d}y\wedge\mathrm{d}z + (\mathbf{G}\times\mathbf{H})_2\,\mathrm{d}z\wedge\mathrm{d}x + (\mathbf{G}\times\mathbf{H})_3\,\mathrm{d}x\wedge\mathrm{d}y\big]\\ &\quad= \big\{F_1(\mathbf{G}\times\mathbf{H})_1 + F_2(\mathbf{G}\times\mathbf{H})_2 + F_3(\mathbf{G}\times\mathbf{H})_3\big\}\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z\\ &\quad= \big\{F_1(G_2H_3 - G_3H_2) + F_2(G_3H_1 - G_1H_3) + F_3(G_1H_2 - G_2H_1\big\}\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z\end{aligned}$$

This can be expressed cleanly in terms of determinants. Recalling the rule for expanding a determinant along its top row

Equation 4.7.10.

$$\big[F_1\,\mathrm{d}x + F_2\,\mathrm{d}y + F_3\,\mathrm{d}z\big]\wedge\big[G_1\,\mathrm{d}x + G_2\,\mathrm{d}y + G_3\,\mathrm{d}z\big]\wedge\big[H_1\,\mathrm{d}x + H_2\,\mathrm{d}y + H_3\,\mathrm{d}z\big] = \det\begin{bmatrix}F_1 & F_2 & F_3\\ G_1 & G_2 & G_3\\ H_1 & H_2 & H_3\end{bmatrix}\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z$$

Example 4.7.9

Our next operation is a differential operator which unifies and generalizes gradient, curl and divergence.

Definition 4.7.11 (Differentiation of differential forms).

If ω is a k-form, then $\mathrm{d}\omega$ is a $k+1$-form, with d being the unique[64] such operator that obeys

(a) d is linear. That is, if ω_1, ω_2 are k-forms and $a_1, a_2 \in \mathbb{R}$, then

$$\mathrm{d}\big(a_1\omega_1 + a_2\omega_2\big) = a_1\mathrm{d}\omega_1 + a_2\mathrm{d}\omega_2$$

(b) d obeys a "graded product rule". Precisely, if $\omega^{(k)}$ is a k-form and $\omega^{(\ell)}$ is an ℓ-form, then

$$\mathrm{d}\big(\omega^{(k)} \wedge \omega^{(\ell)}\big) = \big(d\omega^{(k)}\big) \wedge \omega^{(\ell)} + (-1)^k\omega^{(k)} \wedge \big(\mathrm{d}\omega^{(\ell)}\big)$$

(c) If $f(x,y,z)$ is a 0-form, then

$$\begin{aligned}\mathrm{d}f &= \frac{\partial f}{\partial x}(x,y,z)\ \mathrm{d}x + \frac{\partial f}{\partial y}(x,y,z)\ \mathrm{d}y + \frac{\partial f}{\partial z}(x,y,z)\ \mathrm{d}z \\ &= \boldsymbol{\nabla} f(x,y,z)\cdot \mathrm{d}\mathbf{r} \qquad \text{where } \mathrm{d}\mathbf{r} = \mathrm{d}x\,\hat{\boldsymbol{\imath}} + \mathrm{d}y\,\hat{\boldsymbol{\jmath}} + \mathrm{d}z\,\hat{\mathbf{k}}\end{aligned}$$

(d) For any differential form ω,

$$\mathrm{d}\big(\mathrm{d}\omega\big) = 0$$

Example 4.7.12

(a) If $f(x,y,z) = x$, then

$$\mathrm{d}f = \frac{\partial x}{\partial x}(x,y,z)\ \mathrm{d}x + \frac{\partial x}{\partial y}(x,y,z)\ \mathrm{d}y + \frac{\partial x}{\partial z}(x,y,z)\ \mathrm{d}z = \mathrm{d}x$$

That is, $\mathrm{d}x$ really is the operator d applied to the function x. Similarly, $\mathrm{d}y$ really is the operator d applied to the function y and $\mathrm{d}z$ really is the operator d applied to the function z.

(b) For any k-form ω

$$\begin{aligned}\mathrm{d}\big[\omega \wedge \mathrm{d}x\big] &= \mathrm{d}\omega \wedge \mathrm{d}x + (-1)^k\omega \wedge \mathrm{d}\big(\mathrm{d}x\big) \\ &= \mathrm{d}\omega \wedge \mathrm{d}x\end{aligned}$$

64 That d is unique just means that the action of d on *any* differential form is completely determined by the four rules (a), (b), (c), (d). We will see in Example 4.7.12.c,d,e, that this is indeed the case.

Similarly

$$\mathrm{d}\big[\omega \wedge \mathrm{d}y\big] = \mathrm{d}\omega \wedge \mathrm{d}y \qquad \mathrm{d}\big[\omega \wedge \mathrm{d}z\big] = \mathrm{d}\omega \wedge \mathrm{d}z$$

(c) For any 1-form

$$\begin{aligned}
\mathrm{d}\big[F_1\mathrm{d}x + F_2\mathrm{d}y + F_3\mathrm{d}z\big] &= \mathrm{d}F_1 \wedge \mathrm{d}x + \mathrm{d}F_2 \wedge \mathrm{d}y + \mathrm{d}F_3 \wedge \mathrm{d}z \\
&= \Big(\frac{\partial F_1}{\partial x}\,\mathrm{d}x + \frac{\partial F_1}{\partial y}\,\mathrm{d}y + \frac{\partial F_1}{\partial z}\,\mathrm{d}z\Big) \wedge \mathrm{d}x + \Big(\frac{\partial F_2}{\partial x}\,\mathrm{d}x + \frac{\partial F_2}{\partial y}\,\mathrm{d}y + \frac{\partial F_2}{\partial z}\,\mathrm{d}z\Big) \wedge \mathrm{d}y \\
&\qquad + \Big(\frac{\partial F_3}{\partial x}\,\mathrm{d}x + \frac{\partial F_3}{\partial y}\,\mathrm{d}y + \frac{\partial F_3}{\partial z}\,\mathrm{d}z\Big) \wedge \mathrm{d}z \\
&= \Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)\,\mathrm{d}y \wedge \mathrm{d}z + \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big)\,\mathrm{d}z \wedge \mathrm{d}x + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\,\mathrm{d}x \wedge \mathrm{d}y \\
&= (\boldsymbol{\nabla} \times \mathbf{F})_1\,\mathrm{d}y \wedge \mathrm{d}z + (\boldsymbol{\nabla} \times \mathbf{F})_2\,\mathrm{d}z \wedge \mathrm{d}x + (\boldsymbol{\nabla} \times \mathbf{F})_3\,\mathrm{d}x \wedge \mathrm{d}y
\end{aligned}$$

(d) For any 2-form

$$\begin{aligned}
&\mathrm{d}\big[F_1\,\mathrm{d}y \wedge \mathrm{d}z + F_2\,\mathrm{d}z \wedge \mathrm{d}x + F_3\,\mathrm{d}x \wedge \mathrm{d}y\big] \\
&\qquad = \mathrm{d}F_1 \wedge \mathrm{d}y \wedge \mathrm{d}z + \mathrm{d}F_2 \wedge \mathrm{d}z \wedge \mathrm{d}x + \mathrm{d}F_3 \wedge \mathrm{d}x \wedge \mathrm{d}y \\
&\qquad = \Big(\frac{\partial F_1}{\partial x}\,\mathrm{d}x + \frac{\partial F_1}{\partial y}\,\mathrm{d}y + \frac{\partial F_1}{\partial z}\,\mathrm{d}z\Big) \wedge \mathrm{d}y \wedge \mathrm{d}z \\
&\qquad\qquad + \Big(\frac{\partial F_2}{\partial x}\,\mathrm{d}x + \frac{\partial F_2}{\partial y}\,\mathrm{d}y + \frac{\partial F_2}{\partial z}\,\mathrm{d}z\Big) \wedge \mathrm{d}z \wedge \mathrm{d}x \\
&\qquad\qquad + \Big(\frac{\partial F_3}{\partial x}\,\mathrm{d}x + \frac{\partial F_3}{\partial y}\,\mathrm{d}y + \frac{\partial F_3}{\partial z}\,\mathrm{d}z\Big) \wedge \mathrm{d}x \wedge \mathrm{d}y \\
&\qquad = \Big(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\Big)\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\
&\qquad = \boldsymbol{\nabla} \cdot \mathbf{F}\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z
\end{aligned}$$

(e) For any 3-form

$$\begin{aligned}
\mathrm{d}\big[f\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z\big] &= \Big(\frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z\Big) \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\
&= 0
\end{aligned}$$

Example 4.7.12

Example 4.7.13

In Definition 4.7.11.c, we defined, for any function $f(x,y,z)$ of three variables

$$\mathrm{d}f = \frac{\partial f}{\partial x}(x,y,z)\,\mathrm{d}x + \frac{\partial f}{\partial y}(x,y,z)\,\mathrm{d}y + \frac{\partial f}{\partial z}(x,y,z)\,\mathrm{d}z$$

The analogous formulae[65] for functions of one or two variables also apply.

$$\begin{aligned} \mathrm{d}f(t) &= \frac{\mathrm{d}f}{\mathrm{d}t}(t)\,\mathrm{d}t \\ \mathrm{d}f(u,v) &= \frac{\partial f}{\partial u}(u,v)\,\mathrm{d}u + \frac{\partial f}{\partial v}(u,v)\,\mathrm{d}v \end{aligned}$$

(a) Let $F_1(x,y,z)\,\mathrm{d}x + F_2(x,y,z)\,\mathrm{d}y + F_3(x,y,z)\,\mathrm{d}z$ be a 1-form. Suppose that we substitute $x = x(t)$, $y = y(t)$ and $z = z(t)$, so that we are restricting our 1-form to a parametrized curve. Then, writing $\mathbf{r}(t) = \big(x(t), y(t), z(t)\big)$,

$$\begin{aligned} &F_1\big(x(t),y(t),z(t)\big)\,\mathrm{d}x(t) + F_2\big(x(t),y(t),z(t)\big)\,\mathrm{d}y(t) + F_3\big(x(t),y(t),z(t)\big)\,\mathrm{d}z(t) \\ &= F_1\big(\mathbf{r}(t)\big)\frac{\mathrm{d}x}{\mathrm{d}t}(t)\,\mathrm{d}t + F_2\big(\mathbf{r}(t)\big)\frac{\mathrm{d}y}{\mathrm{d}t}(t)\,\mathrm{d}t + F_3\big(\mathbf{r}(t)\big)\frac{\mathrm{d}z}{\mathrm{d}t}(t)\,\mathrm{d}t \\ &= \mathbf{F}\big(\mathbf{r}(t)\big)\cdot\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\,\mathrm{d}t \end{aligned}$$

(b) Let $F_1(x,y,z)\,\mathrm{d}y\wedge\mathrm{d}z + F_2(x,y,z)\,\mathrm{d}z\wedge\mathrm{d}x + F_3(x,y,z)\,\mathrm{d}x\wedge\mathrm{d}y$ be a 2-form. Suppose that we substitute $x = x(u,v)$, $y = y(u,v)$ and $z = z(u,v)$, so that we are restricting our 2-form to a parametrized surface. Then, writing $\mathbf{r}(u,v) = \big(x(u,v), y(u,v), z(u,v)\big)$,

$$\begin{aligned} &F_1\big(x(u,v),y(u,v),z(u,v)\big)\,\mathrm{d}y(u,v)\wedge\mathrm{d}z(u,v) \\ &\qquad + F_2\big(x(u,v),y(u,v),z(u,v)\big)\,\mathrm{d}z(u,v)\wedge\mathrm{d}x(u,v) \\ &\qquad + F_3\big(x(u,v),y(u,v),z(u,v)\big)\,\mathrm{d}x(u,v)\wedge\mathrm{d}y(u,v) \\ &= F_1\big(\mathbf{r}(u,v)\big)\left(\frac{\partial y}{\partial u}\mathrm{d}u + \frac{\partial y}{\partial v}\mathrm{d}v\right)\wedge\left(\frac{\partial z}{\partial u}\mathrm{d}u + \frac{\partial z}{\partial v}\mathrm{d}v\right) \\ &\qquad + F_2\big(\mathbf{r}(u,v)\big)\left(\frac{\partial z}{\partial u}\mathrm{d}u + \frac{\partial z}{\partial v}\mathrm{d}v\right)\wedge\left(\frac{\partial x}{\partial u}\mathrm{d}u + \frac{\partial x}{\partial v}\mathrm{d}v\right) \\ &\qquad + F_3\big(\mathbf{r}(u,v)\big)\left(\frac{\partial x}{\partial u}\mathrm{d}u + \frac{\partial x}{\partial v}\mathrm{d}v\right)\wedge\left(\frac{\partial y}{\partial u}\mathrm{d}u + \frac{\partial y}{\partial v}\mathrm{d}v\right) \\ &= \Big[F_1\big(\mathbf{r}(u,v)\big)\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right) + F_2\big(\mathbf{r}(u,v)\big)\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u}\right) \\ &\qquad + F_3\big(\mathbf{r}(u,v)\big)\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)\Big]\mathrm{d}u\wedge\mathrm{d}v \\ &= \Big[\mathbf{F}\big(\mathbf{r}(u,v)\big)\cdot\frac{\partial\mathbf{r}}{\partial u}(u,v)\times\frac{\partial\mathbf{r}}{\partial v}(u,v)\Big]\mathrm{d}u\wedge\mathrm{d}v \end{aligned}$$

Example 4.7.13

Let us summarize what we have seen in the Example 4.7.12.

65 Indeed, you can view $f(t)$ as a function of three variables that happens to be independent of two of the three variables. Similarly you can view $f(u,v)$ as a function of three variables that happens to be independent of one of the three variables.

Lemma 4.7.14.

(a) For any 0-form
$$\mathrm{d}f = \nabla f(x,y,z) \cdot \mathrm{d}\mathbf{r}$$

(b) For any 1-form
$$\mathrm{d}\big[F_1\mathrm{d}x + F_2\mathrm{d}y + F_3\mathrm{d}z\big]$$
$$= (\nabla \times \mathbf{F})_1\,\mathrm{d}y \wedge \mathrm{d}z + (\nabla \times \mathbf{F})_2\,\mathrm{d}z \wedge \mathrm{d}x + (\nabla \times \mathbf{F})_3\,\mathrm{d}x \wedge \mathrm{d}y$$

(c) For any 2-form
$$\mathrm{d}\big[F_1\,\mathrm{d}y \wedge \mathrm{d}z + F_2\,\mathrm{d}z \wedge \mathrm{d}x + F_3\,\mathrm{d}x \wedge \mathrm{d}y\big] = \nabla \cdot \mathbf{F}\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$$

(d) For any 3-form
$$\mathrm{d}\big[f\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z\big] = 0$$

Our final operation is integration of differential forms.

Definition 4.7.15 (Integration of differential forms)**.**

(a) Let $f(x,y,z)$ be a 0-form and $P = (x_0, y_0, z_0) \in \mathbb{R}^3$ be a point. Then
$$\int_P f = f(x_0, y_0, z_0)$$

More generally if, for each $1 \leqslant i \leqslant \ell$, $P_i = (x_i, y_i, z_i) \in \mathbb{R}^3$ is a point and n_i is an integer, then
$$\int_{\sum_{i=1}^{\ell} n_i P_i} f = \sum_{i=1}^{\ell} n_i f(x_i, y_i, z_i)$$

(b) Let $\omega = \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = F_1(x,y,z)\,\mathrm{d}x + F_2(x,y,z)\,\mathrm{d}y + F_3(x,y,z)\,\mathrm{d}z$ be a 1-form. Let $\mathcal{C}$ be a curve that is parametrized by $\mathbf{r}(t) = \big(x(t)\,,\, y(t)\,,\, z(t)\big)$, $a \leqslant t \leqslant b$. Then, motivated by Example 4.7.13.a above,
$$\int_{\mathcal{C}} \omega = \int_a^b \mathbf{F}\big(\mathbf{r}(t)\big) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\,\mathrm{d}t = \int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

Definition 4.7.15 (continued).

(c) Let $\omega = F_1(x,y,z)\,\mathrm{d}y \wedge \mathrm{d}z + F_2(x,y,z)\,\mathrm{d}z \wedge \mathrm{d}x + F_3(x,y,z)\,\mathrm{d}x \wedge \mathrm{d}y$ be a 2-form. Let S be an oriented surface that is parametrized by $\mathbf{r}(u,v) = \big(x(u,v)\,,\,y(u,v)\,,\,z(u,v)\big)$, with (u,v) running over a region R in the uv-plane. Assume that $\mathbf{r}(u,v)$ is orientation preserving in the sense that $\hat{\mathbf{n}}\,\mathrm{d}S = +\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\,\mathrm{d}u\,\mathrm{d}v$.Then, motivated by Example 4.7.13.b above,

$$\int_S \omega = \iint_R \Big[\mathbf{F}\big(\mathbf{r}(u,v)\big) \cdot \frac{\partial \mathbf{r}}{\partial u}(u,v) \times \frac{\partial \mathbf{r}}{\partial v}(u,v)\Big]\mathrm{d}u \wedge \mathrm{d}v = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S$$

(d) Let $\omega = f(x,y,z)\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ be a 3-form. Let V be a solid in $\mathbb{R}^3$. Then

$$\int_V \omega = \iiint_V f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

Finally, after all of these definitions, we have a very compact theorem that simultaneously covers the fundamental theorem of calculus, Green's theorem. Stokes' theorem and the divergence theorem. Had we given all of our definitions in n dimensions, rather than just three dimensions, it would cover a lot more. This general theorem is also called Stokes' theorem.

Theorem 4.7.16 (Stokes' Theorem).

If ω is a k-form (with $k = 0,1,2$) and D is a $(k+1)$-dimensional domain of integration, then

$$\int_D d\omega = \int_{\partial D} \omega$$

Here ∂D is the boundary of D (suitably oriented).

To see the connection between the general Stokes' theorem 4.7.16 and the Stokes' and divergence theorems of the earlier part of this chapter, here are the $k = 1$ and $k = 2$ cases of Theorem 4.7.16 again.

- Let $\omega = F_1\mathrm{d}x + F_2\mathrm{d}y + F_3\mathrm{d}z$ be a 1-form and let S be a piecewise smooth oriented surface as in (our original) Stokes' theorem 4.4.1. Then, by Lemma 4.7.14.b,

$$d\omega = (\boldsymbol{\nabla} \times \mathbf{F})_1\,\mathrm{d}y \wedge \mathrm{d}z + (\boldsymbol{\nabla} \times \mathbf{F})_2\,\mathrm{d}z \wedge \mathrm{d}x + (\boldsymbol{\nabla} \times \mathbf{F})_3\,\mathrm{d}x \wedge \mathrm{d}y$$

  So, by parts (c) (but with $\mathbf{F}$ replaced by $\boldsymbol{\nabla} \times \mathbf{F}$) and (b) of Definition 4.7.15, the conclusion $\int_D d\omega = \int_{\partial D} \omega$ of (the general) Stokes' theorem 4.7.16 is

$$\iint_S \boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S = \int_S d\omega = \int_{\partial S} \omega = \int_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

  which is the conclusion of (our original) Stokes' theorem 4.4.1.

- $\omega = F_1(x,y,z)\,\mathrm{d}y \wedge \mathrm{d}z + F_2(x,y,z)\,\mathrm{d}z \wedge \mathrm{d}x + F_3(x,y,z)\,\mathrm{d}x \wedge \mathrm{d}y$ be a 2-form and let V be a solid as in the divergence theorem 4.2.2. Then, by Lemma 4.7.14.c,

$$d\omega = \nabla \cdot \mathbf{F}\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$$

So, by parts (d) (with $f = \nabla \cdot \mathbf{F}$) and (c) of Definition 4.7.15, the conclusion $\int_D d\omega = \int_{\partial D} \omega$ of (the general) Stokes' theorem 4.7.16 is

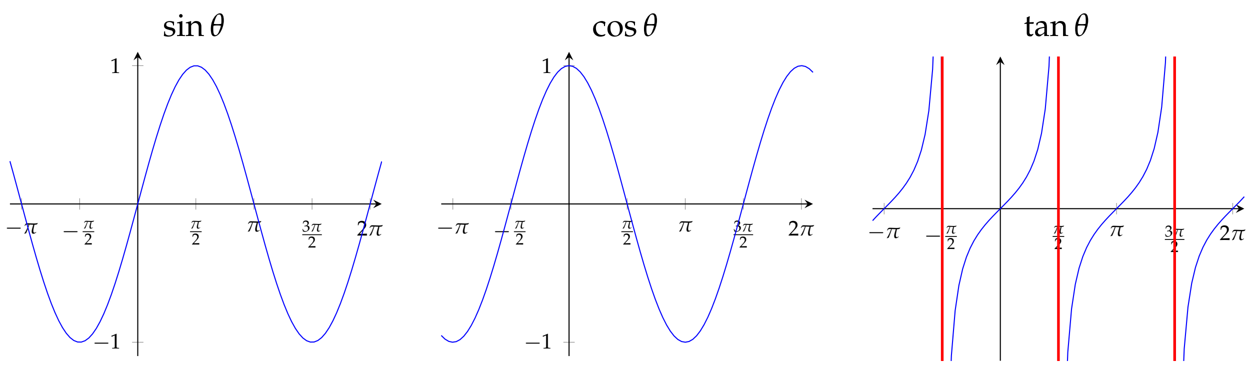
$$\iiint_V \nabla \cdot \mathbf{F}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_V d\omega = \int_{\partial V} \omega = \iint_{\partial V} \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S$$

which is the conclusion of the divergence theorem 4.2.2.

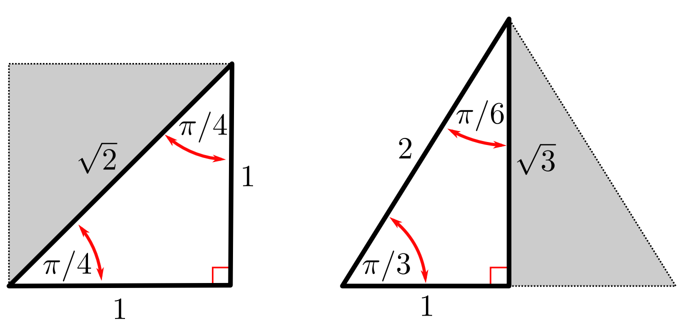
Appendix A

Trigonometry

A.1 Trigonometry — Graphs



A.2 Trigonometry — Special Triangles



From the above pair of special triangles we have

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \sin\frac{\pi}{6} = \frac{1}{2} \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$
$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{3} = \frac{1}{2}$$
$$\tan\frac{\pi}{4} = 1 \qquad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \qquad \tan\frac{\pi}{3} = \sqrt{3}$$

A.3▲ Trigonometry — Simple Identities

- Periodicity

$$\sin(\theta + 2\pi) = \sin(\theta) \qquad \cos(\theta + 2\pi) = \cos(\theta)$$

- Reflection

$$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta)$$

- Reflection around $\pi/4$

$$\sin\left(\tfrac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\tfrac{\pi}{2} - \theta\right) = \sin\theta$$

- Reflection around $\pi/2$

$$\sin(\pi - \theta) = \sin\theta \qquad \cos(\pi - \theta) = -\cos\theta$$

- Rotation by π

$$\sin(\theta + \pi) = -\sin\theta \qquad \cos(\theta + \pi) = -\cos\theta$$

- Pythagoras

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

- sin and cos building blocks

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$$

A.4▲ Trigonometry — Add and Subtract Angles

- Sine

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$$

- Cosine

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$$

- Tangent

$$\begin{aligned}
\tan(\alpha + \beta) &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \\
\tan(\alpha - \beta) &= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}
\end{aligned}$$

- Double angle

$$\begin{aligned}
\sin(2\theta) &= 2\sin(\theta)\cos(\theta) \\
\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\
&= 2\cos^2(\theta) - 1 \\
&= 1 - 2\sin^2(\theta) \\
\tan(2\theta) &= \frac{2\tan(\theta)}{1 - \tan^2\theta} \\
\cos^2\theta &= \frac{1 + \cos(2\theta)}{2} \\
\sin^2\theta &= \frac{1 - \cos(2\theta)}{2} \\
\tan^2\theta &= \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}
\end{aligned}$$

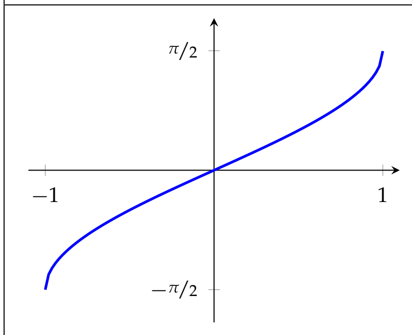
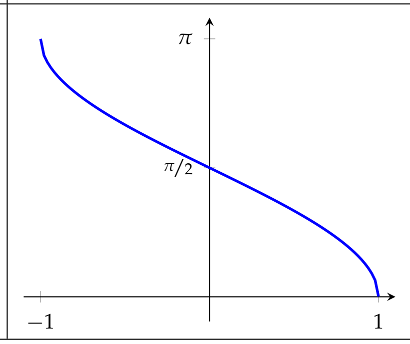
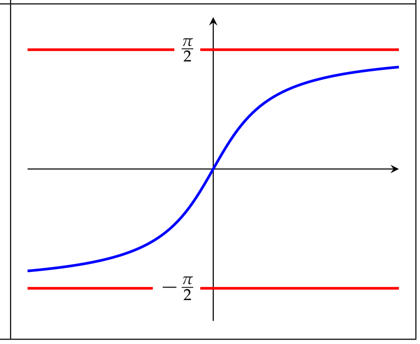
- Products to sums

$$\begin{aligned}
\sin(\alpha)\cos(\beta) &= \frac{\sin(\alpha+\beta) + \sin(\alpha-\beta)}{2} \\
\sin(\alpha)\sin(\beta) &= \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{2} \\
\cos(\alpha)\cos(\beta) &= \frac{\cos(\alpha-\beta) + \cos(\alpha+\beta)}{2}
\end{aligned}$$

- Sums to products

$$\begin{aligned}
\sin\alpha + \sin\beta &= 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\
\sin\alpha - \sin\beta &= 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \\
\cos\alpha + \cos\beta &= 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\
\cos\alpha - \cos\beta &= -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}
\end{aligned}$$

A.5▲ Inverse Trigonometric Functions

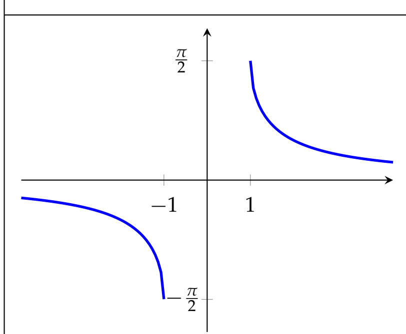
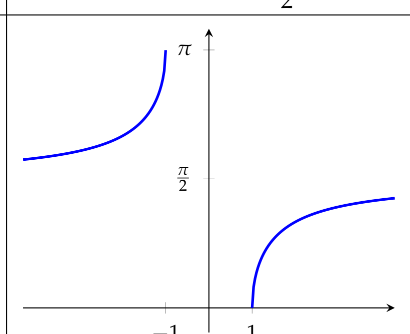
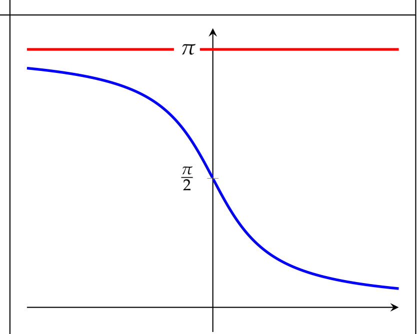
$\arcsin x$	$\arccos x$	$\arctan x$
Domain: $-1 \leqslant x \leqslant 1$ Range: $-\frac{\pi}{2} \leqslant \arcsin x \leqslant \frac{\pi}{2}$	Domain: $-1 \leqslant x \leqslant 1$ Range: $0 \leqslant \arccos x \leqslant \pi$	Domain: all real numbers Range: $-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$
		

Since these functions are inverses of each other we have

$$
\begin{aligned}
\arcsin(\sin\theta) &= \theta & -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2} \\
\arccos(\cos\theta) &= \theta & 0 \leqslant \theta \leqslant \pi \\
\arctan(\tan\theta) &= \theta & -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}
\end{aligned}
$$

and also

$$
\begin{aligned}
\sin(\arcsin x) &= x & -1 \leqslant x \leqslant 1 \\
\cos(\arccos x) &= x & -1 \leqslant x \leqslant 1 \\
\tan(\arctan x) &= x & \text{any real } x
\end{aligned}
$$

$\operatorname{arccsc} x$	$\operatorname{arcsec} x$	$\operatorname{arccot} x$
Domain: $\lvert x \rvert \geqslant 1$ Range: $-\frac{\pi}{2} \leqslant \operatorname{arccsc} x \leqslant \frac{\pi}{2}$ $\operatorname{arccsc} x \neq 0$	Domain: $\lvert x \rvert \geqslant 1$ Range: $0 \leqslant \operatorname{arcsec} x \leqslant \pi$ $\operatorname{arcsec} x \neq \frac{\pi}{2}$	Domain: all real numbers Range: $0 < \operatorname{arccot} x < \pi$
		

Again

$$\begin{aligned}
\operatorname{arccsc}(\csc\theta) &= \theta & -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2},\ \theta \neq 0 \\
\operatorname{arcsec}(\sec\theta) &= \theta & 0 \leqslant \theta \leqslant \pi,\ \theta \neq \frac{\pi}{2} \\
\operatorname{arccot}(\cot\theta) &= \theta & 0 < \theta < \pi
\end{aligned}$$

and

$$\begin{aligned}
\csc(\operatorname{arccsc} x) &= x & |x| \geqslant 1 \\
\sec(\operatorname{arcsec} x) &= x & |x| \geqslant 1 \\
\cot(\operatorname{arccot} x) &= x & \text{any real } x
\end{aligned}$$

Appendix B

Powers and Logarithms

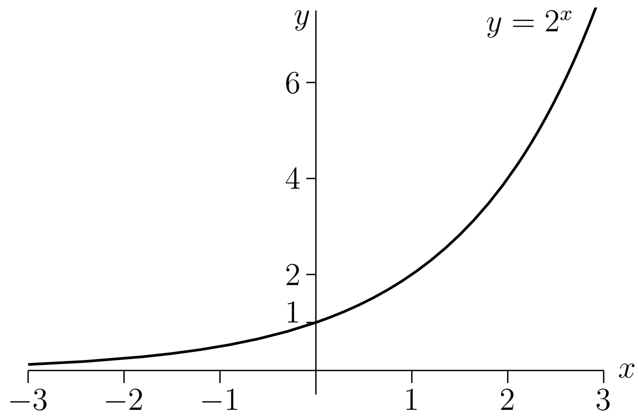
B.1 Powers

In the following, x and y are arbitrary real numbers, q is an arbitrary constant that is strictly bigger than zero and e is 2.7182818284, to ten decimal places.

- $e^0 = 1, \quad q^0 = 1$
- $e^{x+y} = e^x e^y, \quad e^{x-y} = \frac{e^x}{e^y}, \quad q^{x+y} = q^x q^y, \quad q^{x-y} = \frac{q^x}{q^y}$
- $e^{-x} = \frac{1}{e^x}, \quad q^{-x} = \frac{1}{q^x}$
- $\left(e^x\right)^y = e^{xy}, \quad \left(q^x\right)^y = q^{xy}$
- $\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x, \quad \frac{\mathrm{d}}{\mathrm{d}x} e^{g(x)} = g'(x) e^{g(x)}, \quad \frac{\mathrm{d}}{\mathrm{d}x} q^x = (\ln q)\, q^x$
- $\int e^x \,\mathrm{d}x = e^x + C, \quad \int e^{ax} \,\mathrm{d}x = \frac{1}{a} e^{ax} + C$ if $a \neq 0$
- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
- $\lim_{x\to\infty} e^x = \infty, \quad \lim_{x\to-\infty} e^x = 0$

  $\lim_{x\to\infty} q^x = \infty, \quad \lim_{x\to-\infty} q^x = 0$ if $q > 1$

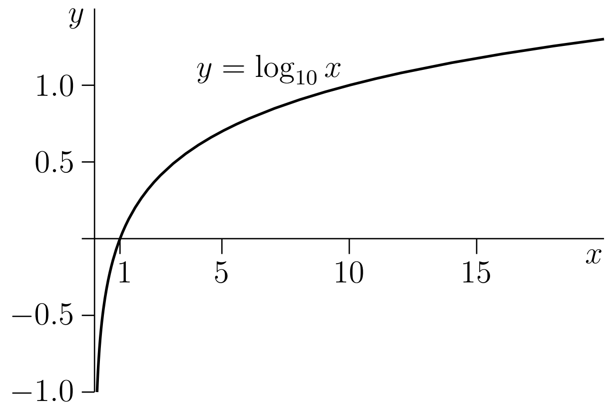
  $\lim_{x\to\infty} q^x = 0, \quad \lim_{x\to-\infty} q^x = \infty$ if $0 < q < 1$
- The graph of 2^x is given below. The graph of q^x, for any $q > 1$, is similar.



B.2▲ Logarithms

In the following, x and y are arbitrary real numbers that are strictly bigger than 0 (except where otherwise specified), p and q are arbitrary constants that are strictly bigger than one, and e is 2.7182818284, to ten decimal places. The notation $\ln x$ means $\log_e x$. Some people use $\log x$ to mean $\log_{10} x$, others use it to mean $\log_e x$ and still others use it to mean $\log_2 x$.

- $e^{\ln x} = x, \quad q^{\log_q x} = x$
- $\ln\left(e^x\right) = x, \quad \log_q\left(q^x\right) = x \quad$ for all $-\infty < x < \infty$
- $\log_q x = \frac{\ln x}{\ln q}, \quad \ln x = \frac{\log_p x}{\log_p e}, \quad \log_q x = \frac{\log_p x}{\log_p q}$
- $\ln 1 = 0, \quad \ln e = 1$
  $\log_q 1 = 0, \quad \log_q q = 1$
- $\ln(xy) = \ln x + \ln y, \quad \log_q(xy) = \log_q x + \log_q y$
- $\ln\left(\frac{x}{y}\right) = \ln x - \ln y, \quad \log_q\left(\frac{x}{y}\right) = \log_q x - \log_q y$
- $\ln\left(\frac{1}{y}\right) = -\ln y, \quad \log_q\left(\frac{1}{y}\right) = -\log_q y$
- $\ln(x^y) = y\ln x, \quad \log_q(x^y) = y\log_q x$
- $\frac{d}{dx}\ln x = \frac{1}{x}, \quad \frac{d}{dx}\log_q x = \frac{1}{x\ln q}$
- $\int \ln x\, dx = x\ln x - x + C, \quad \int \log_q x\, dx = x\log_q x - \frac{x}{\ln q} + C,$
- $\lim_{x\to\infty} \ln x = \infty, \quad \lim_{x\to 0+} \ln x = -\infty$
  $\lim_{x\to\infty} \log_q x = \infty, \quad \lim_{x\to 0+} \log_q x = -\infty$
- The graph of $\log_{10} x$ is given below. The graph of $\log_q x$, for any $q > 1$, is similar.


y
$y = \log_{10} x$
1.0
0.5
−0.5
−1.0
1
5
10
15
x

Appendix C

TABLE OF DERIVATIVES

Throughout this table, a and b are constants, independent of x.

$F(x)$	$F'(x) = \frac{dF}{dx}$
$af(x) + bg(x)$	$af'(x) + bg'(x)$
$f(x) + g(x)$	$f'(x) + g'(x)$
$f(x) - g(x)$	$f'(x) - g'(x)$
$af(x)$	$af'(x)$
$f(x)g(x)$	$f'(x)g(x) + f(x)g'(x)$
$f(x)g(x)h(x)$	$f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$
$\frac{f(x)}{g(x)}$	$\frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$
$\frac{1}{g(x)}$	$-\frac{g'(x)}{g(x)^2}$
$f\big(g(x)\big)$	$f'\big(g(x)\big)g'(x)$

$F(x)$	$F'(x) = \frac{dF}{dx}$
a	0
x^a	ax^{a-1}
$g(x)^a$	$ag(x)^{a-1}g'(x)$
$\sin x$	$\cos x$
$\sin g(x)$	$g'(x)\cos g(x)$
$\cos x$	$-\sin x$
$\cos g(x)$	$-g'(x)\sin g(x)$
$\tan x$	$\sec^2 x$
$\csc x$	$-\csc x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\csc^2 x$
e^x	e^x
$e^{g(x)}$	$g'(x)e^{g(x)}$
a^x	$(\ln a)\, a^x$

$F(x)$	$F'(x) = \frac{dF}{dx}$
$\ln x$	$\frac{1}{x}$
$\ln g(x)$	$\frac{g'(x)}{g(x)}$
$\log_a x$	$\frac{1}{x \ln a}$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arcsin g(x)$	$\frac{g'(x)}{\sqrt{1-g(x)^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$
$\arctan g(x)$	$\frac{g'(x)}{1+g(x)^2}$
$\operatorname{arccsc} x$	$-\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\operatorname{arcsec} x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\operatorname{arccot} x$	$-\frac{1}{1+x^2}$

Appendix D

TABLE OF INTEGRALS

Throughout this table, a and b are given constants, independent of x and C is an arbitrary constant.

$f(x)$	$F(x) = \int f(x)\,\mathrm{d}x$
$af(x) + bg(x)$	$a\int f(x)\,\mathrm{d}x + b\int g(x)\,\mathrm{d}x \;+\; C$
$f(x) + g(x)$	$\int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x \;+\; C$
$f(x) - g(x)$	$\int f(x)\,\mathrm{d}x - \int g(x)\,\mathrm{d}x \;+\; C$
$af(x)$	$a\int f(x)\,\mathrm{d}x \;+\; C$
$u(x)v'(x)$	$u(x)v(x) - \int u'(x)v(x)\,\mathrm{d}x \;+\; C$
$f\big(y(x)\big)y'(x)$	$F\big(y(x)\big)$ where $F(y) = \int f(y)\,\mathrm{d}y$
a	$ax + C$
x^a	$\frac{x^{a+1}}{a+1} + C$ if $a \neq -1$
$\frac{1}{x}$	$\ln\lvert x\rvert + C$
$g(x)^a g'(x)$	$\frac{g(x)^{a+1}}{a+1} + C$ if $a \neq -1$

$f(x)$	$F(x) = \int f(x)\, dx$
$\sin x$	$-\cos x + C$
$g'(x) \sin g(x)$	$-\cos g(x) + C$
$\cos x$	$\sin x + C$
$\tan x$	$\ln \lvert \sec x \rvert + C$
$\csc x$	$\ln \lvert \csc x - \cot x \rvert + C$
$\sec x$	$\ln \lvert \sec x + \tan x \rvert + C$
$\cot x$	$\ln \lvert \sin x \rvert + C$
$\sec^2 x$	$\tan x + C$
$\csc^2 x$	$-\cot x + C$
$\sec x \tan x$	$\sec x + C$
$\csc x \cot x$	$-\csc x + C$

$f(x)$	$F(x) = \int f(x)\, dx$
e^x	$e^x + C$
$e^{g(x)} g'(x)$	$e^{g(x)} + C$
e^{ax}	$\frac{1}{a}\, e^{ax} + C$
a^x	$\frac{1}{\ln a}\, a^x + C$
$\ln x$	$x \ln x - x + C$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + C$
$\frac{g'(x)}{\sqrt{1-g(x)^2}}$	$\arcsin g(x) + C$
$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin \frac{x}{a} + C$
$\frac{1}{1+x^2}$	$\arctan x + C$
$\frac{g'(x)}{1+g(x)^2}$	$\arctan g(x) + C$
$\frac{1}{a^2+x^2}$	$\frac{1}{a} \arctan \frac{x}{a} + C$
$\frac{1}{x\sqrt{x^2-1}}$	$\operatorname{arcsec} x + C \quad (x > 1)$

Appendix E

TABLE OF TAYLOR EXPANSIONS

Let $n \geqslant$ be an integer. Then if the function f has $n+1$ derivatives on an interval that contains both x_0 and x, we have the Taylor expansion

$$f(x) = f(x_0) + f'(x_0)\,(x-x_0) + \frac{1}{2!}f''(x_0)\,(x-x_0)^2 + \cdots + \frac{1}{n!}f^{(n)}(x_0)\,(x-x_0)^n$$
$$+ \frac{1}{(n+1)!}f^{(n+1)}(c)\,(x-x_0)^{n+1} \qquad \text{for some } c \text{ between } x_0 \text{ and } x$$

The limit as $n \to \infty$ gives the Taylor series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

for f. When $x_0 = 0$ this is also called the Maclaurin series for f. Here are Taylor series expansions of some important functions.

$$\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{1}{n!}x^n && \text{for } -\infty < x < \infty \\
&= 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots \\
\sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} && \text{for } -\infty < x < \infty \\
&= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} + \cdots \\
\cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} && \text{for } -\infty < x < \infty \\
&= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots + \frac{(-1)^n}{(2n)!}x^{2n} + \cdots \\
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n && \text{for } -1 \leqslant x < 1 \\
&= 1 + x + x^2 + x^3 + \cdots + x^n + \cdots
\end{aligned}$$

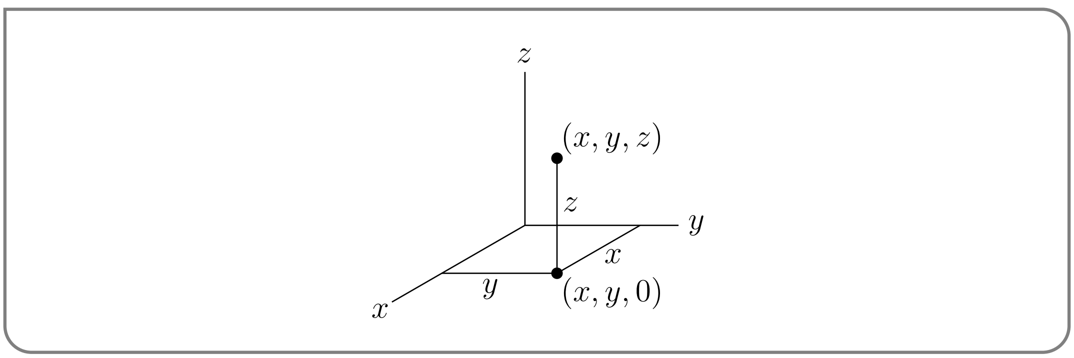
$$
\begin{aligned}
\frac{1}{1+x} &= \sum_{n=0}^{\infty} (-1)^n x^n && \text{for } -1 < x \leqslant 1 \\
&= 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots \\
\ln(1-x) &= -\sum_{n=1}^{\infty} \frac{1}{n} x^n && \text{for } -1 \leqslant x < 1 \\
&= -x - \tfrac{1}{2}x^2 - \frac{1}{3}x^3 - \cdots - \frac{1}{n}x^n - \cdots \\
\ln(1+x) &= -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n && \text{for } -1 < x \leqslant 1 \\
&= x - \tfrac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots - \frac{(-1)^n}{n}x^n - \cdots \\
(1+x)^p &= 1 + px + \frac{p(p-1)}{2}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots \\
&\qquad + \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}x^n + \cdots
\end{aligned}
$$

Appendix F

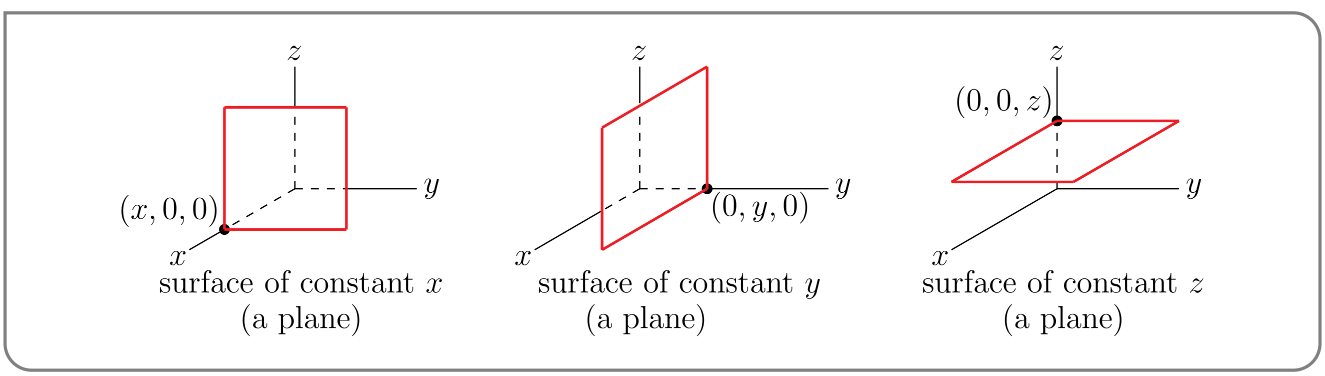
3D Coordinate Systems

F.1 Cartesian Coordinates

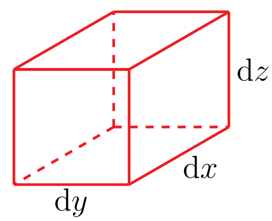
Here is a figure showing the definitions of the three Cartesian coordinates (x, y, z)



and here are three figures showing a surface of constant x, a surface of constant x, and a surface of constant z.



Finally here is a figure showing the volume element $\mathrm{d}V$ in cartesian coordinates.

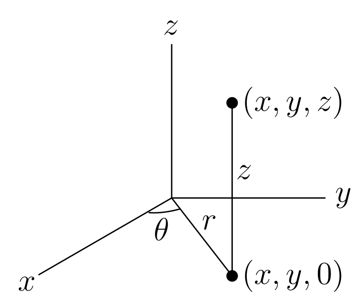


volume element $dV = dx\,dy\,dz$

F.2▲ Cylindrical Coordinates

Here is a figure showing the definitions of the three cylindrical coordinates

$$\begin{aligned} r &= \text{distance from } (0,0,0) \text{ to } (x,y,0) \\ \theta &= \text{angle between the } x \text{ axis and the line joining } (x,y,0) \text{ to } (0,0,0) \\ z &= \text{signed distance from } (x,y,z) \text{ to the } xy\text{-plane} \end{aligned}$$

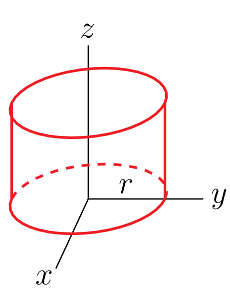


The cartesian and cylindrical coordinates are related by

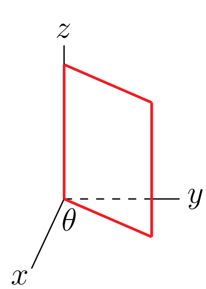
$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

$$r = \sqrt{x^2+y^2} \qquad \theta = \arctan\frac{y}{x} \qquad z = z$$

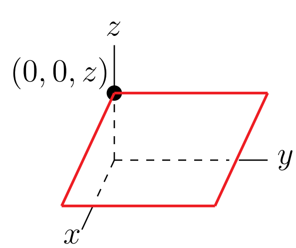
Here are three figures showing a surface of constant r, a surface of constant θ, and a surface of constant z.



surface of constant r
(a cylindrical shell)

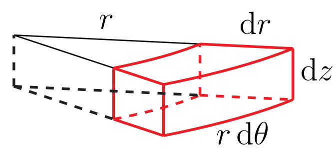


surface of constant θ
(a plane)



surface of constant z
(a plane)

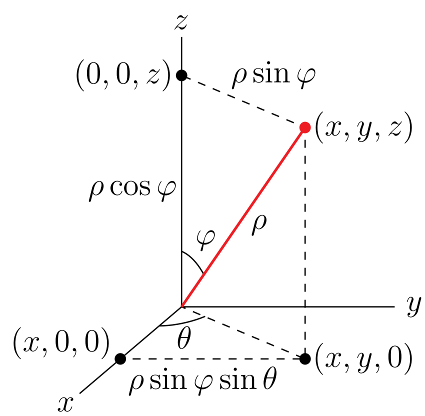
Finally here is a figure showing the volume element dV in cylindrical coordinates.



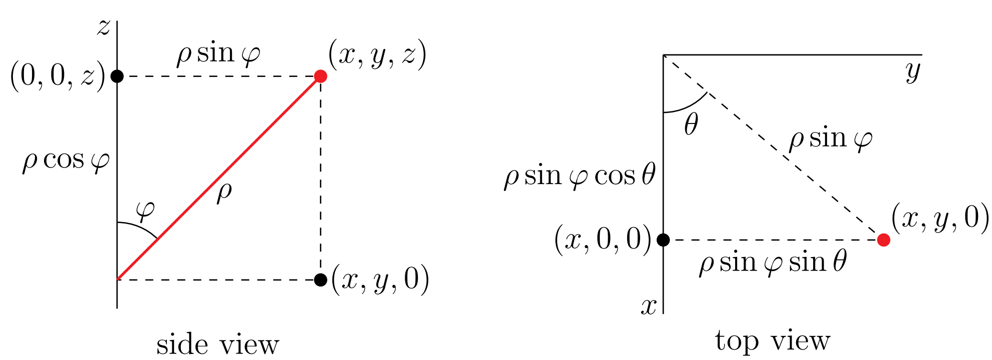
F.3▲ Spherical Coordinates

Here is a figure showing the definitions of the three spherical coordinates

$$\begin{aligned}\rho &= \text{ distance from } (0,0,0) \text{ to } (x,y,z)\\ \varphi &= \text{ angle between the } z \text{ axis and the line joining } (x,y,z) \text{ to } (0,0,0)\\ \theta &= \text{ angle between the } x \text{ axis and the line joining } (x,y,0) \text{ to } (0,0,0)\end{aligned}$$



and here are two more figures giving the side and top views of the previous figure.

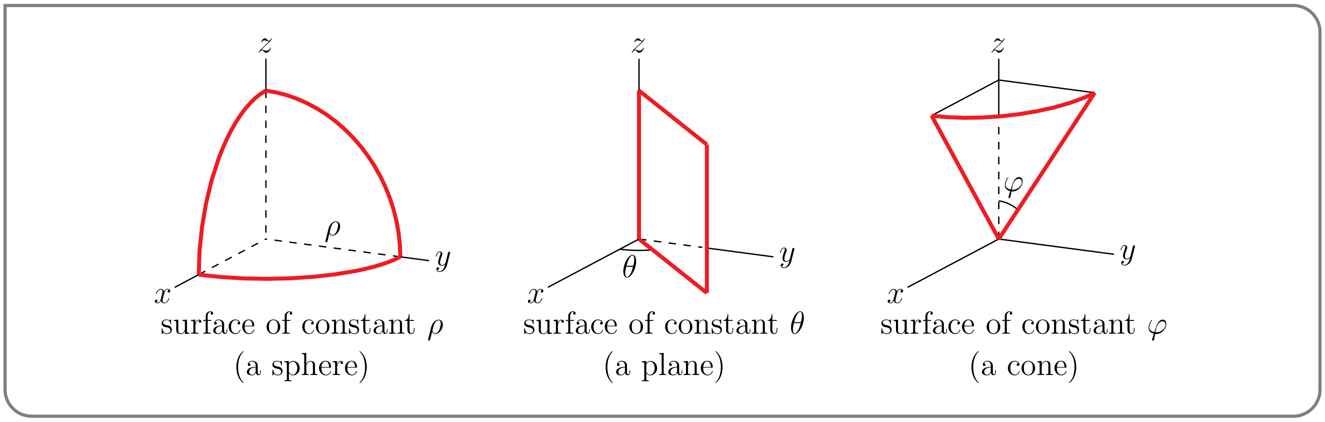


The cartesian and spherical coordinates are related by

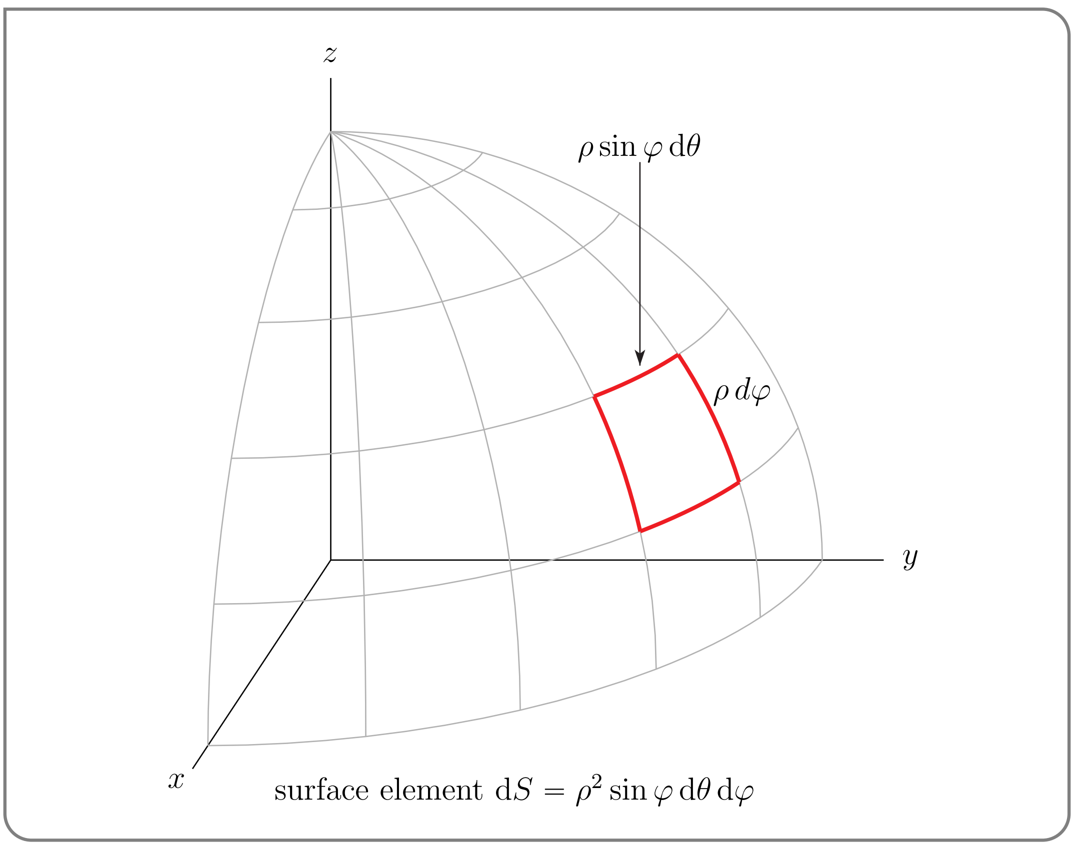
$$x = \rho \sin\varphi \cos\theta \qquad y = \rho \sin\varphi \sin\theta \qquad z = \rho \cos\varphi$$

$$\rho = \sqrt{x^2 + y^2 + z^2} \qquad \theta = \arctan\frac{y}{x} \qquad \varphi = \arctan\frac{\sqrt{x^2+y^2}}{z}$$

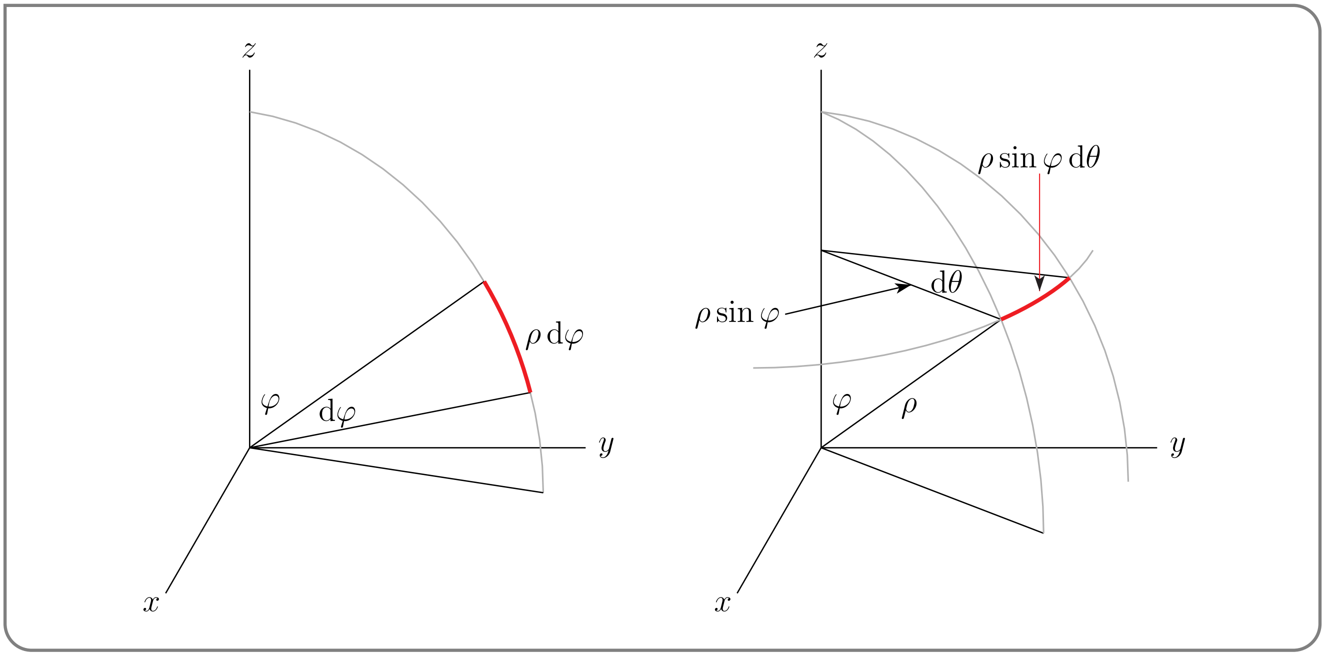
Here are three figures showing a surface of constant ρ, a surface of constant θ, and a surface of constant φ.



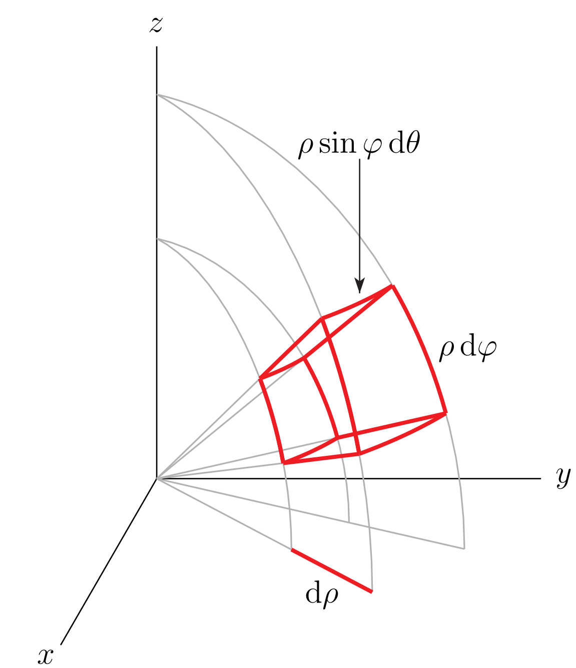
Here is a figure showing the surface element $\mathrm{d}S$ in spherical coordinates



and two extracts of the above figure to make it easier to see how the factors $\rho\ \mathrm{d}\varphi$ and $\rho \sin\varphi\ \mathrm{d}\theta$ arise.



Finally, here is a figure showing the volume element dV in spherical coordinates



volume element $dV = \rho^2 \sin\varphi \; d\rho\, d\theta\, d\varphi$

Appendix G

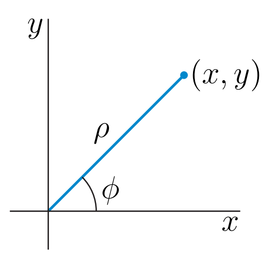
ISO Coordinate System Notation

In this text we have chosen symbols for the various polar, cylindrical and spherical coordinates that are standard for mathematics. There is another, different, set of symbols that are commonly used in the physical sciences and engineering. Indeed, there is an international convention, called ISO 80000-2, that specifies those symbols[1]. In this appendix, we summarize the definitions and standard properties of the polar, cylindrical and spherical coordinate systems using the ISO symbols.

G.1▲ Polar Coordinates

In the ISO convention the symbols ρ and ϕ are used (instead of r and θ) for polar coordinates.

$$\rho = \text{the distance from } (0,0) \text{ to } (x,y)$$
$$\phi = \text{the (counter-clockwise) angle between the } x \text{ axis and the line joining } (x,y) \text{ to } (0,0)$$

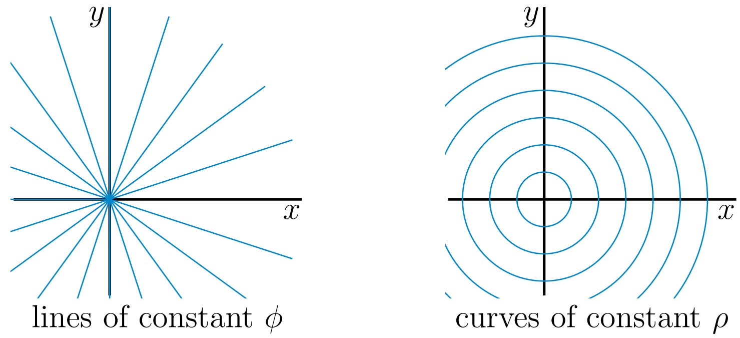


Cartesian and polar coordinates are related by

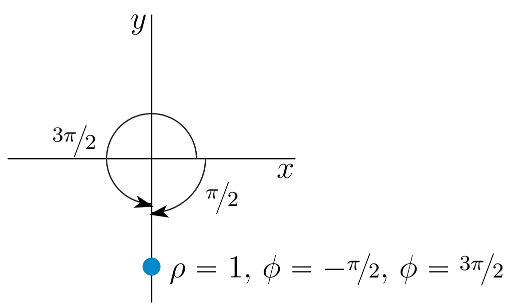
$$x = \rho\cos\phi \qquad\qquad y = \rho\sin\phi$$
$$\rho = \sqrt{x^2 + y^2} \qquad\qquad \phi = \arctan\frac{y}{x}$$

1 It specifies more than just those symbols. See https://en.wikipedia.org/wiki/ISO_31-11 and https://en.wikipedia.org/wiki/ISO/IEC_80000. The full ISO 80000-2 is available at https://www.iso.org/standard/64973.html — for $$.

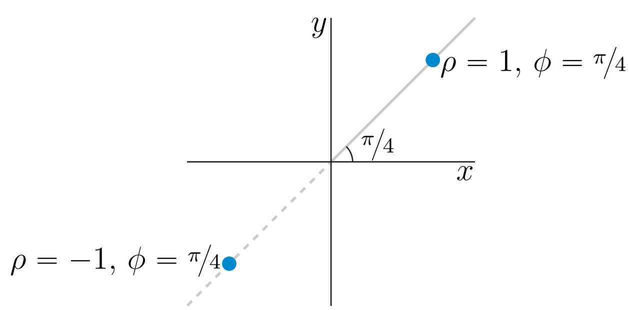
The following two figures show a number of lines of constant ϕ, on the left, and curves of constant ρ, on the right.



Note that the polar angle ϕ is only defined up to integer multiples of 2π. For example, the point $(1, 0)$ on the x-axis could have $\phi = 0$, but could also have $\phi = 2\pi$ or $\phi = 4\pi$. It is sometimes convenient to assign ϕ negative values. When $\phi < 0$, the counter-clockwise angle ϕ refers to the clockwise angle $|\phi|$. For example, the point $(0, -1)$ on the negative y-axis can have $\phi = -\frac{\pi}{2}$ and can also have $\phi = \frac{3\pi}{2}$.



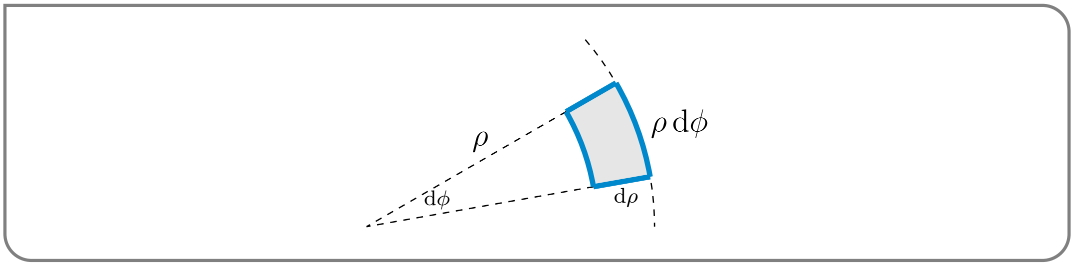
It is also sometimes convenient to extend the above definitions by saying that $x = \rho\cos\phi$ and $y = \rho\sin\phi$ even when ρ is negative. For example, the following figure shows (x, y) for $\rho = 1$, $\phi = {}^{\pi}/_4$ and for $\rho = -1$, $\phi = {}^{\pi}/_4$. Both points lie on the line through



the origin that makes an angle of 45° with the x-axis and both are a distance one from the origin. But they are on opposite sides of the origin.

The area element in polar coordinates is

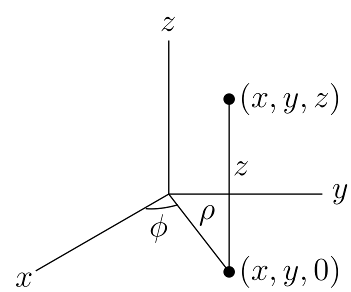
$$\mathrm{d}A = \rho\,\mathrm{d}\rho\,\mathrm{d}\phi$$



G.2▲ Cylindrical Coordinates

In the ISO convention the symbols ρ, ϕ and z are used (instead of r, θ and z) for cylindrical coordinates.

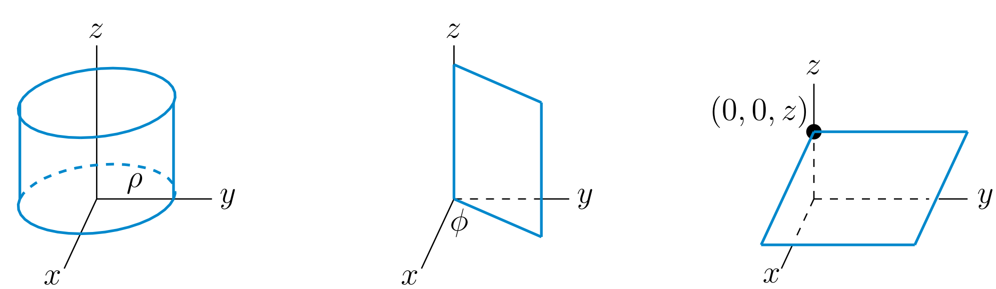
$$\begin{aligned}\rho &= \text{distance from } (0,0,0) \text{ to } (x,y,0)\\ \phi &= \text{angle between the } x \text{ axis and the line joining } (x,y,0) \text{ to } (0,0,0)\\ z &= \text{signed distance from } (x,y,z) \text{ to the } xy\text{-plane}\end{aligned}$$



The cartesian and cylindrical coordinates are related by

$$x = \rho\cos\phi \qquad y = \rho\sin\phi \qquad z = z$$
$$\rho = \sqrt{x^2 + y^2} \qquad \phi = \arctan\frac{y}{x} \qquad z = z$$

Here are three figures showing a surface of constant ρ, a surface of constant ϕ, and a surface of constant z.

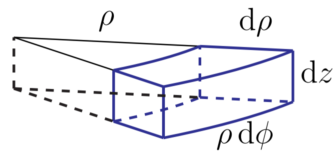


surface of constant ρ (a cylindrical shell)

surface of constant ϕ (half a plane)

surface of constant z (a plane)

Finally here is a figure showing the volume element $\mathrm{d}V$ in cylindrical coordinates.

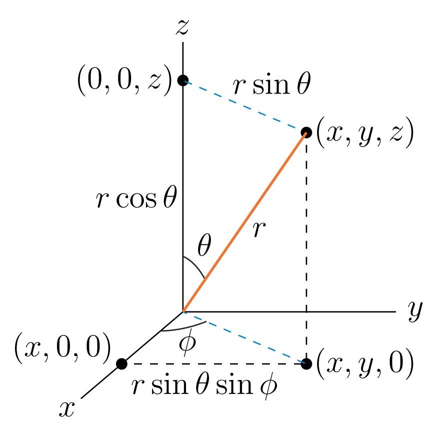


volume element $\mathrm{d}V = \rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}z$

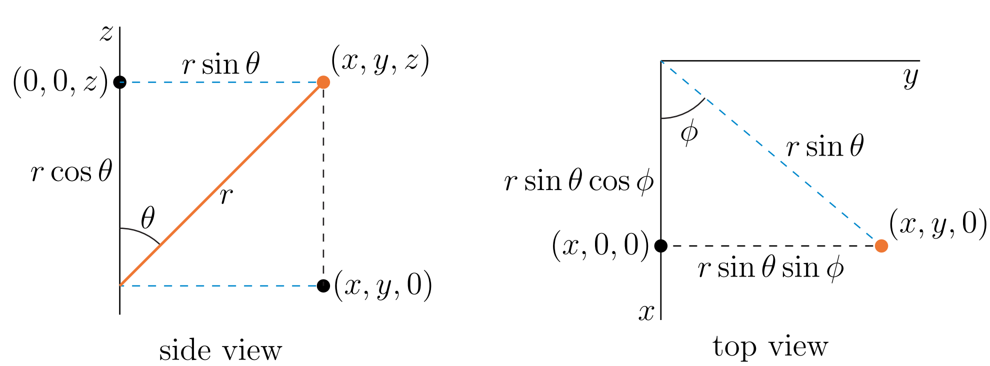
G.3▲ Spherical Coordinates

In the ISO convention the symbols r (instead of ρ), ϕ (instead of θ) and θ (instead of ϕ) are used for spherical coordinates.

$$\begin{aligned} r &= \text{distance from } (0,0,0) \text{ to } (x,y,z) \\ \theta &= \text{angle between the } z \text{ axis and the line joining } (x,y,z) \text{ to } (0,0,0) \\ \phi &= \text{angle between the } x \text{ axis and the line joining } (x,y,0) \text{ to } (0,0,0) \end{aligned}$$



Here are two more figures giving the side and top views of the previous figure.

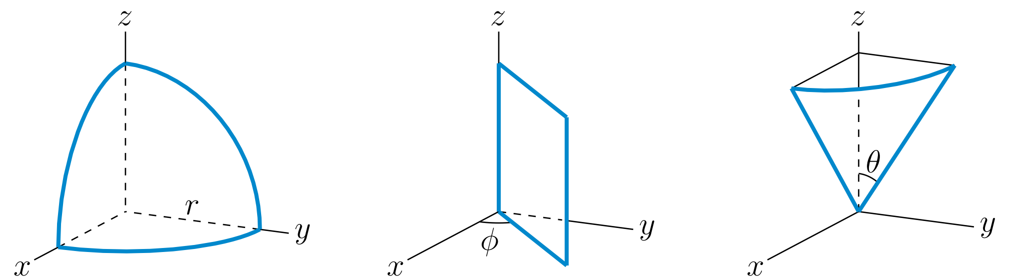


The cartesian and spherical coordinates are related by

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$$

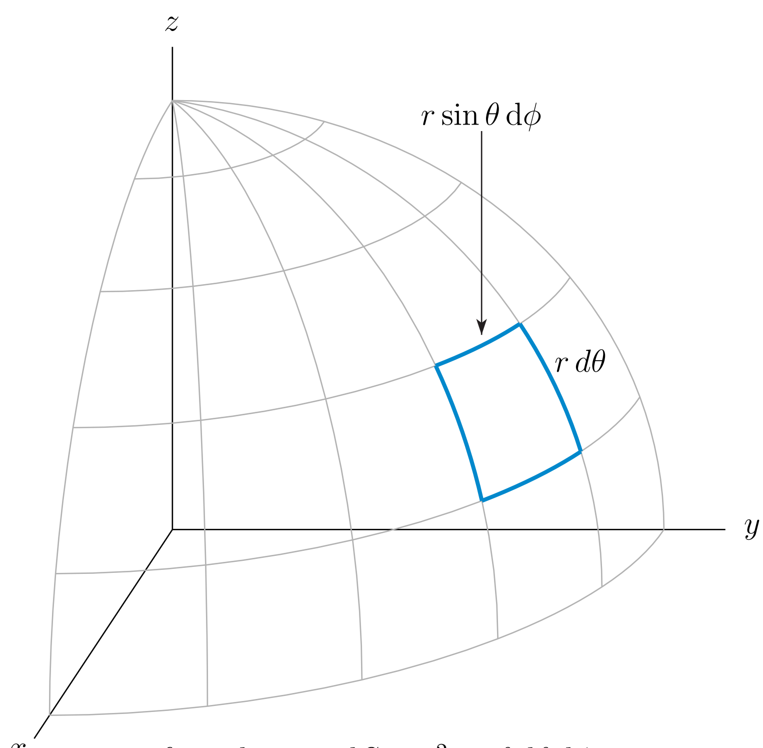
$$r = \sqrt{x^2+y^2+z^2} \qquad \phi = \arctan\frac{y}{x} \qquad \theta = \arctan\frac{\sqrt{x^2+y^2}}{z}$$

Here are three figures showing a surface of constant r, a surface of constant ϕ, and a surface of constant θ.



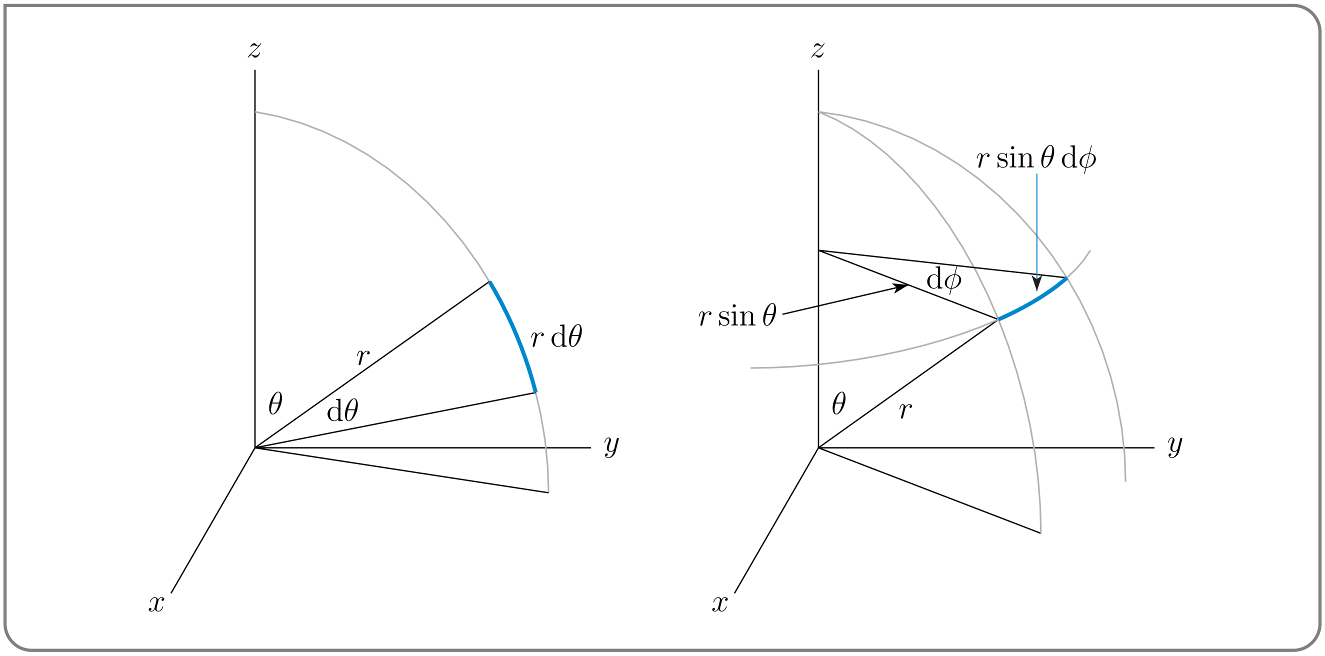
surface of constant r (a sphere) — surface of constant ϕ (half a plane) — surface of constant θ (a cone)

Here is a figure showing the surface element dS in spherical coordinates

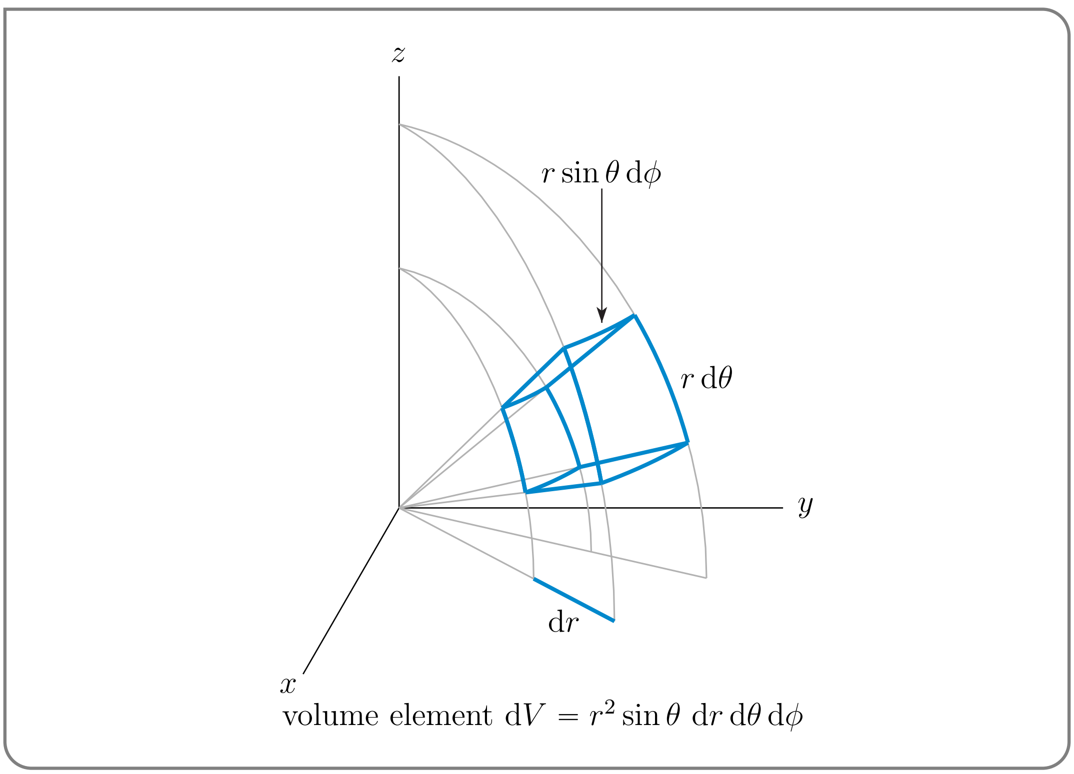


surface element $\mathrm{d}S = r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$

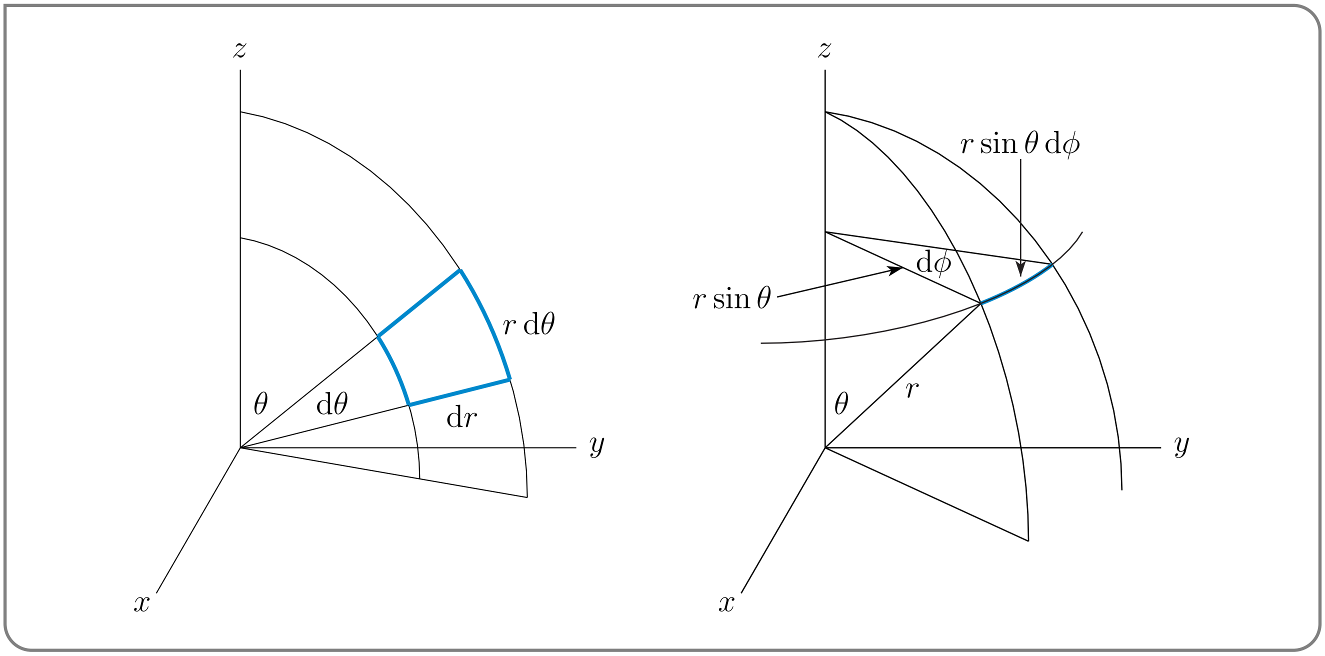
and two extracts of the above figure to make it easier to see how the factors r dθ and $r\sin\theta$ dϕ arise.



Finally, here is a figure showing the volume element dV in spherical coordinates



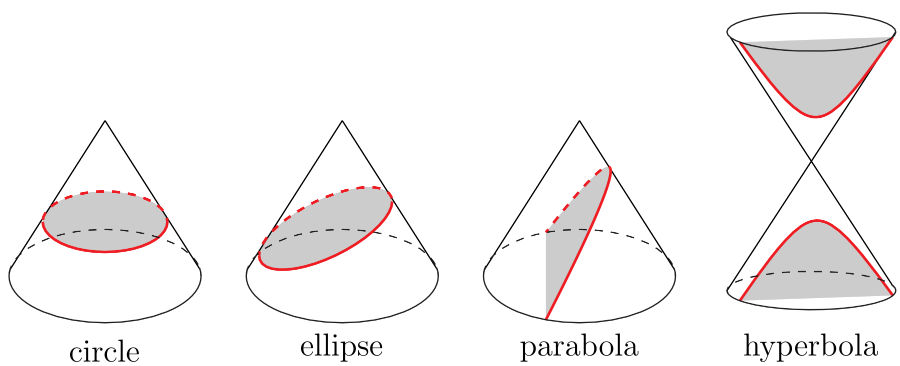
and two extracts of the above figure to make it easier to see how r dθ and $r \sin\theta$ dϕ arise.


z
θ
dθ
r dθ
dr
y
x
z
r sin θ dϕ
dϕ
r sin θ
θ
r
y
x

Appendix H

Conic Sections and Quadric Surfaces

A conic section is the curve of intersection of a cone and a plane that does not pass through the vertex of the cone. This is illustrated in the figures below. An equivalent[1] (and often



used) definition is that a conic section is the set of all points in the xy-plane that obey $Q(x,y) = 0$ with

$$Q(x,y) = Ax^2 + By^2 + Cxy + Dx + Ey + F = 0$$

being a polynomial of degree two[2]. By rotating and translating our coordinate system the equation of the conic section can be brought into one of the forms[3]

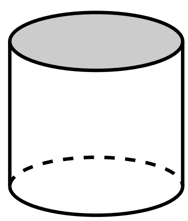
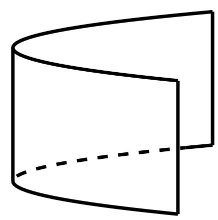
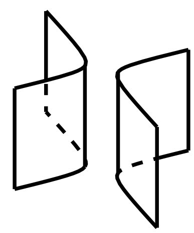
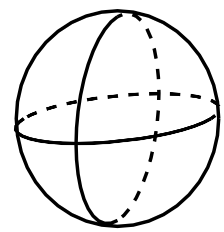
- $\alpha x^2 + \beta y^2 = \gamma$ with $\alpha, \beta, \gamma > 0$, which is an ellipse (or a circle),
- $\alpha x^2 - \beta y^2 = \gamma$ with $\alpha, \beta > 0$, $\gamma \neq 0$, which is a hyperbola,
- $x^2 = \delta y$, with $\delta \neq 0$ which is a parabola.

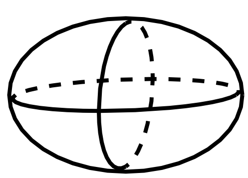
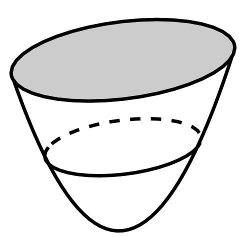
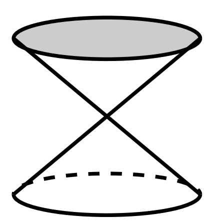
1 It is outside our scope to prove this equivalence.

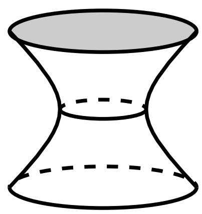
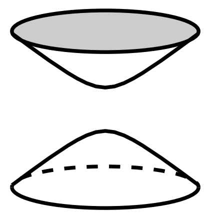
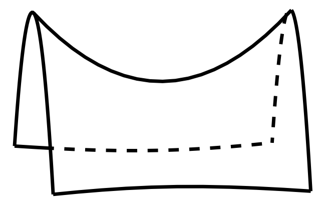
2 Technically, we should also require that the constants A, B, C, D, E, F, are real numbers, that A, B, C are not all zero, that $Q(x,y) = 0$ has more than one real solution, and that the polynomial can't be factored into the product of two polynomials of degree one.

3 This statement can be justified using a linear algebra eigenvalue/eigenvector analysis. It is beyond what we can cover here, but is not too difficult for a standard linear algeba course.

The three dimensional analogs of conic sections, surfaces in three dimensions given by quadratic equations, are called quadrics. An example is the sphere $x^2 + y^2 + z^2 = 1$. Here are some tables giving all of the quadric surfaces.

name	elliptic cylinder	parabolic cylinder	hyperbolic cylinder	sphere
equation in standard form	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$y = ax^2$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$x^2 + y^2 + z^2 = r^2$
$x =$ constant cross-section	two lines	one line	two lines	circle
$y =$ constant cross-section	two lines	two lines	two lines	circle
$z =$ constant cross-section	ellipse	parabola	hyperbola	circle
sketch				

name	ellipsoid	elliptic paraboloid	elliptic cone
equation in standard form	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$
$x =$ constant cross-section	ellipse	parabola	two lines if $x = 0$ hyperbola if $x \neq 0$
$y =$ constant cross-section	ellipse	parabola	two lines if $y = 0$ hyperbola if $y \neq 0$
$z =$ constant cross-section	ellipse	ellipse	ellipse
sketch			

name	hyperboloid of one sheet	hyperboloid of two sheets	hyperbolic paraboloid
equation in standard form	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$	$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}$
$x =$ constant cross-section	hyperbola	hyperbola	parabola
$y =$ constant cross-section	hyperbola	hyperbola	parabola
$z =$ constant cross-section	ellipse	ellipse	two lines if $z = 0$ hyperbola if $z \neq 0$
sketch			

Appendix I

Review of Linear Ordinary Differential Equations

Definition I.1.

(a) A *differential equation* is an equation for an unknown function that contains the derivatives of that unknown function. For example $y''(t) + y(t) = 0$ is a differential equation for the unknown function $y(t)$.

(b) A differential equation is called an *ordinary differential equation* (often shortened to "ODE") if only ordinary derivatives appear. That is, if the unknown function has only a single independent variable. A differential equation is called a *partial differential equation* (often shortened to "PDE") if partial derivatives appear. That is, if the unknown function has more than one independent variable. For example $y''(t) + y(t) = 0$ is an ODE while $\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t)$ is a PDE.

(c) The *order* of a differential equation is the order of the highest derivative that appears. For example $y''(t) + y(t) = 0$ is a second order ODE.

(d) An ordinary differential equation that is of the form

$$a_0(t)y^{(n)}(t) + a_1(t)y^{(n-1)}(t) + \cdots + a_{n-1}(t)y'(t) + a_n(t)y(t) = F(t) \tag{I.1}$$

with given coefficient functions $a_0(t)$, $\cdots$, $a_n(t)$ and $F(t)$ is said to be *linear*. Otherwise, the ODE is said to be *nonlinear*. For example, $y'(t)^2 + y(t) = 0$, $y'(t)y''(t) + y(t) = 0$ and $y'(t) = e^{y(t)}$ are all nonlinear.

(e) The ODE (I.1) is said to have *constant coefficients* if the coefficients $a_0(t)$, $a_1(t)$, $\cdots$, $a_n(t)$ are all constants. Otherwise, it is said to have *variable coefficients*. For example, the ODE $y''(t) + 7y(t) = \sin t$ is constant coefficient, while $y''(t) + ty(t) = \sin t$ is variable coefficient.

Definition I.1 (continued).

(f) The ODE (I.1) is said to be *homogeneous* if $F(t)$ is identically zero. Otherwise, it is said to be *inhomogeneous* or *nonhomogeneous*. For example, the ODE $y''(t) + 7y(t) = 0$ is homogeneous, while $y''(t) + 7y(t) = \sin t$ is inhomogeneous. A homogeneous ODE always has the trivial solution $y(t) = 0$.

(g) An *initial value problem* is a problem in which one is to find an unknown function $y(t)$ that satisfies both a given ODE and given initial conditions, like $y(t_0) = 1$, $y'(t_0) = 0$. Note that all of the conditions involve the function $y(t)$ (or its derivatives) evaluated at a single time $t = t_0$.

(h) A *boundary value problem* is a problem in which one is to find an unknown function $y(t)$ that satisfies both a given ODE and given boundary conditions, like $y(t_0) = 0$, $y(t_1) = 0$. Note that the conditions involve the function $y(t)$ (or its derivatives) evaluated at two different times.

The following theorem gives the form of solutions to the ODE (I.1).

Theorem I.2.

Assume that the coefficients $a_0(t)$, $a_1(t)$, $\cdots$, $a_{n-1}(t)$, $a_n(t)$ and $F(t)$ are continuous functions and that $a_0(t)$ is not zero.

(a) The general solution to the ODE (I.1) is of the form

$$y(t) = y_p(t) + C_1 y_1(t) + C_2 y_2(t) + \cdots + C_n y_n(t) \tag{I.2}$$

where

- n is the order of (I.1)
- $y_p(t)$ is any solution to (I.1)
- $C_1, C_2, \cdots, C_n$ are arbitrary constants
- $y_1, y_2, \cdots, y_n$ are n independent solutions to the homogenous equation

$$a_0(t)y^{(n)}(t) + a_1(t)y^{(n-1)}(t) + \cdots + a_{n-1}(t)y'(t) + a_n(t)y(t) = 0$$

associated to (I.1). "Independent" just means that no y_i can be written as a linear combination of the other y_j's. For example, $y_1(t)$ cannot be expressed in the form $b_2 y_2(t) + \cdots + b_n y_n(t)$.

In (I.2), y_p is called the "particular solution" and $C_1 y_1(t) + C_2 y_2(t) + \cdots + C_n y_n(t)$ is called the "complementary solution".

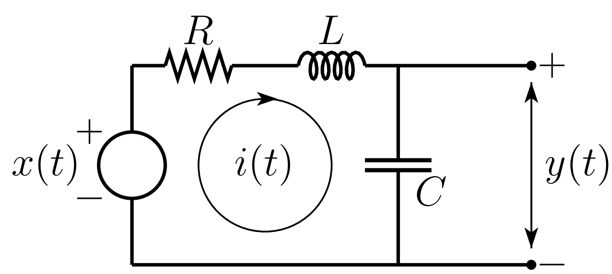
Theorem I.2 (continued).

(b) Given any constants $b_0, \cdots, b_{n-1}$ there is exactly one function $y(t)$ that obeys the ODE (I.1) and the initial conditions

$$y(0) = b_0 \qquad y'(0) = b_1 \qquad \cdots \qquad y^{(n-1)}(0) = b_{n-1}$$

Example I.3 (RLC circuit)

As an example of the most commonly used techniques for solving linear, constant coefficient ODE's, we consider the RLC circuit, which is the electrical circuit consisting of a resistor of resistance R, a coil (or solenoid) of inductance L, a capacitor of capacitance C and a voltage source arranged in series, as shown below. Here R, L and C are all nonnegative constants.



We're going to think of the voltage $x(t)$ as an input signal, and the voltage $y(t)$ as an output signal. The goal is to determine the output signal produced by a given input signal. If $i(t)$ is the current flowing at time t in the loop as shown and $q(t)$ is the charge on the capacitor, then the voltages across R, L and C, respectively, at time t are $Ri(t)$, $L\frac{di}{dt}(t)$ and $y(t) = \frac{q(t)}{C}$. By the Kirchhoff's law[1] that says that the voltage between any two points has to be independent of the path used to travel between the two points, these three voltages must add up to $x(t)$ so that

$$Ri(t) + L\frac{di}{dt}(t) + \frac{q(t)}{C} = x(t) \tag{I.3}$$

Assuming that R, L, C and $x(t)$ are known, this is still one differential equation in two unknowns, $i(t)$ and $q(t)$. Fortunately, there is a relationship between the two. Namely

$$i(t) = \frac{dq}{dt}(t) = Cy'(t) \tag{I.4}$$

This just says that the capacitor cannot create or destroy charge on its own; all charging of the capacitor must come from the current. Substituting (I.4) into (I.3) gives

$$LCy''(t) + RCy'(t) + y(t) = x(t)$$

1 Gustav Robert Kirchhoff (1824–1887) was a German physicist.

As a concrete example, we'll take an ac voltage source and choose the origin of time so that $x(0) = 0$, $x(t) = E_0 \sin(\omega t)$. Then the differential equation becomes

$$LCy''(t) + RCy'(t) + y(t) = E_0 \sin(\omega t) \tag{I.5}$$

This is a second order, linear, constant coefficient ODE. So we know, from Theorem I.2, that the general solution is of the form $y_p(t) + C_1 y_1(t) + C_2 y_2(t)$, where

- $y_p(t)$, the particular solution, is any one solution to (I.5),
- C_1, C_2 are arbitrary constants and
- $y_1(t)$, $y_2(t)$ are any two independent solutions of the corresponding homogeneous equation

$$LCy''(t) + RCy'(t) + y(t) = 0 \tag{I.5$_\text{h}$}$$

So to find the general solution to (I.5), we need to find three functions: $y_1(t)$, $y_2(t)$ and $y_p(t)$.

- *Finding $y_1(t)$ and $y_2(t)$:* The best way to find y_1 and y_2 is to guess them. Any solution, $y_h(t)$, of (I.5$_\text{h}$) has to have the property that $y_h(t)$, $RCy_h'(t)$ and $LCy_h''(t)$ cancel each other out for all t. We choose our guess so that $y_h(t)$, $y_h'(t)$ and $y_h''(t)$ are all proportional to a single function of t. Then it will be easy to see if $y_h(t)$, $RCy_h'(t)$ and $LCy_h''(t)$ all cancel. All derivatives of the function e^{rt} are again proportional to e^{rt}. Hence we try $y_h(t) = e^{rt}$, with the constant r to the determined. This guess is a solution of (I.5$_\text{h}$) if and only if

$$\begin{aligned} LCr^2 e^{rt} + RCre^{rt} + e^{rt} = 0 &\iff LCr^2 + RCr + 1 = 0 \\ &\iff r = \frac{-RC \pm \sqrt{R^2C^2 - 4LC}}{2LC} \equiv r_{1,2} \end{aligned} \tag{I.6}$$

  How we proceed depends on the sign of $R^2C^2 - 4LC$. That is, whether $R > 2\sqrt{\frac{L}{C}}$ or $R < 2\sqrt{\frac{L}{C}}$ or $R = 2\sqrt{\frac{L}{C}}$.

  - *Finding $y_1(t)$ and $y_2(t)$, when $R > 2\sqrt{\frac{L}{C}}$:* Then $R^2C^2 - 4LC > 0$, and r_1 and r_2 are two different real numbers. We may take $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ so that the complimentary solution is $C_1 y_1(t) + C_2 y_2(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$.
  - *Finding $y_1(t)$ and $y_2(t)$, when $R < 2\sqrt{\frac{L}{C}}$:* Then $R^2C^2 - 4LC < 0$ and r_1 and r_2 are the two different complex numbers $-\rho \pm i\nu$, where

$$\rho = \frac{R}{2L} \qquad \text{and} \qquad \nu = \frac{\sqrt{4LC - R^2C^2}}{2LC}$$

    We may again take $C_1 e^{r_1 t} + C_2 e^{r_2 t}$ as the complimentray solution. However we can also rewrite $C_1 e^{r_1 t} + C_2 e^{r_2 t}$ in terms of real valued functions by using that $e^{\pm i\theta} = \cos\theta \pm i \sin\theta$:

$$\begin{aligned} C_1 e^{r_1 t} + C_2 e^{r_2 t} &= e^{-\rho t}\big[C_1 e^{i\nu t} + C_2 e^{-i\nu t}\big] \\ &= e^{-\rho t}\big[C_1\{\cos(\nu t) + i\sin(\nu t)\} + C_2\{\cos(\nu t) - i\sin(\nu t)\}\big] \\ &= e^{-\rho t}\big[D_1 \cos(\nu t) + D_2 \sin(\nu t)\big] \end{aligned}$$

where[2] $D_1 = C_1 + C_2$, $D_2 = i(C_1 - C_2)$. So we may also take $y_1(t) = e^{-\rho t}\cos(\nu t)$, $y_2(t) = e^{-\rho t}\sin(\nu t)$ in the complementary solution.

There is yet a third useful way to write the complementary solution. Think of (D_1, D_2) as a point in the xy-plane. Call the polar coordinates of that point A and θ so that $D_1 = A\cos\theta$ and $D_2 = A\sin\theta$. Then, using the trig identity $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, with $\alpha = \nu t$ and $\beta = -\theta$,

$$\begin{aligned} e^{-\rho t}\big[D_1\cos(\nu t) + D_2\sin(\nu t)\big] &= e^{-\rho t}\big[A\cos(\nu t)\cos\theta + A\sin(\nu t)\sin\theta\big] \\ &= Ae^{-\rho t}\cos(\nu t - \theta) \end{aligned} \tag{I.7}$$

We have, in effect, replaced the two arbitrary constants D_1 and D_2, whose values would normally be determined by initial conditions, by two other arbitrary constants, R and θ, whose values would also normally be determined by initial conditions.

– *Finding $y_1(t)$ and $y_2(t)$, when $R = 2\sqrt{\frac{L}{C}}$:* Then $R^2C^2 - 4LC = 0$ so that $r_1 = r_2$. We may take $y_1 = e^{r_1 t}$, but $e^{r_2 t} = e^{r_1 t}$ is certainly not a second independent solution. So we still need to find y_2. Here is a trick (called reduction of order[3]) for finding the other solutions: look for solutions of the form $v(t)e^{-r_1 t}$. Here $e^{-r_1 t}$ is the solution we have already found and $v(t)$ is to be determined. To save writing, set $\rho = \frac{R}{2L}$ so that $r_1 = r_2 = \rho$. To save writing also divide (I.5$_\mathrm{h}$) by LC and substitute that $\frac{R}{L} = 2\rho$ and $\frac{1}{LC} = \frac{R^2}{4L^2} = \rho^2$. (Recall that we are assuming that $R^2 = \frac{4L}{C}$.) So (I.5$_\mathrm{h}$) is equivalent to

$$y_h''(t) + 2\rho\, y_h'(t) + \rho^2\, y_h(t) = 0$$

Substitute in

$$\begin{aligned} y_h(t) &= \quad v(t)e^{-\rho t} \\ y_h'(t) &= -\rho v(t)e^{-\rho t} + \quad v'(t)e^{-\rho t} \\ y_h''(t) &= \quad \rho^2 v(t)e^{-\rho t} - 2\rho v'(t)e^{-\rho t} + v''(t)e^{-\rho t} \end{aligned}$$

So when $y_h(t) = v(t)e^{-\rho t}$,

$$\begin{aligned} &y_h''(t) + 2\rho\, y_h'(t) + \rho^2\, y_h(t) \\ &\qquad = \big[\rho^2 - 2\rho^2 + \rho^2\big]v(t)e^{-\rho t} + \big[-2\rho + 2\rho\big]v'(t)e^{-\rho t} + v''(t)e^{-\rho t} \\ &\qquad = v''(t)e^{-\rho t} \end{aligned}$$

Thus $v(t)e^{-\rho t}$ is a solution of (I.5$_\mathrm{h}$) whenever the function $v''(t) = 0$ for all t. But, for any values of the constants C_1 and C_2, $v(t) = C_1 + C_2 t$ has vanishing second derivative so $\big(C_1 + C_2 t\big)e^{-\rho t} = \big(C_1 + C_2 t\big)e^{-r_1 t}$ solves (I.5$_\mathrm{h}$). This is of the form $C_1 y_1(t) + C_2 y_2(t)$ with $y_1(t) = e^{-r_1 t}$, the solution that we found first, and $y_2(t) = te^{-r_1 t}$, a second independent solution. So we may take $y_2(t) = te^{r_1 t}$.

2 Don't make the mistake of thinking that C_1 and C_2 have to be real numbers, forcing D_2 to be pure imaginary. In most applications, D_1 and D_2 will be pure real and C_1 and C_2 will be complex.

3 The modern method of reduction of order was created by the French mathematician, physicist and music theorist Jean le Rond d'Alembert (1717-1783). The interested reader can easily search out more about his life.

- *Finding $y_p(t)$:* The best way to find y_p is to guess it. We guess that the circuit responds to an oscillating input voltage with an output voltage that oscillates at the same frequency. So we try $y_p(t) = \mathcal{A}\sin(\omega t - \varphi)$ with the amplitude $\mathcal{A}$ and phase φ to be determined.

  For $y_p(t)$ to be a solution, we need

$$\begin{aligned} LCy_p''(t) + RCy_p'(t) + y_p(t) &= E_0 \sin(\omega t) \\ -LC\omega^2\mathcal{A}\sin(\omega t - \varphi) + RC\omega\mathcal{A}\cos(\omega t - \varphi) + \mathcal{A}\sin(\omega t - \varphi) &= E_0 \sin(\omega t) \\ &= E_0 \sin(\omega t - \varphi + \varphi) \end{aligned}$$

  and hence, applying $\sin(A + B) = \sin A \cos B + \cos A \sin B$ with $A = \omega t - \varphi$ and $B = \varphi$,

$$\begin{aligned} &\big(1 - LC\omega^2\big)\mathcal{A}\sin(\omega t - \varphi) + RC\omega\mathcal{A}\cos(\omega t - \varphi) \\ &\qquad = E_0\cos(\varphi)\sin(\omega t - \varphi) + E_0\sin(\varphi)\cos(\omega t - \varphi) \end{aligned}$$

  Matching coefficients of $\sin(\omega t - \varphi)$ and $\cos(\omega t - \varphi)$ on the left and right hand sides gives

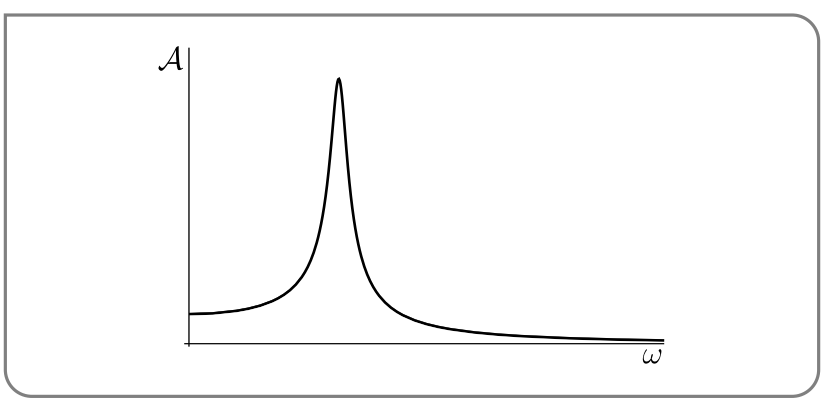
$$\big(1 - LC\omega^2\big)\mathcal{A} = E_0\cos(\varphi) \tag{I.8}$$

$$RC\omega\mathcal{A} = E_0\sin(\varphi) \tag{I.9}$$

  It is now easy to solve for $\mathcal{A}$ and φ

$$\begin{aligned} \frac{(\text{I.9})}{(\text{I.8})} &\implies \tan(\varphi) = \frac{RC\omega}{1 - LC\omega^2} \\ &\implies \varphi = \arctan\frac{RC\omega}{1 - LC\omega^2} \\ \sqrt{(\text{I.8})^2 + (\text{I.9})^2} &\implies \sqrt{\big(1 - LC\omega^2\big)^2 + R^2C^2\omega^2}\,\mathcal{A} = E_0 \\ &\implies \mathcal{A} = \frac{E_0}{\sqrt{(1 - LC\omega^2)^2 + R^2C^2\omega^2}} \end{aligned}$$

Naturally, different input frequencies ω give different output amplitudes $\mathcal{A}$. Here is a graph of $\mathcal{A}$ against ω, with all other parameters held fixed.



Note that there is a small range of frequencies that give a large amplitude response. This is the phenomenon of resonance. It is exploited in the design of radio and television tuning circuitry. It has also been dramatically illustrated in, for example, the collapse[4] of the Tacoma narrows bridge.

Example I.3

Example I.4 (Boundary Value Problems)

By part (b) of Theorem I.2, an initial value problem consisting of an n^{th} order linear ODE with reasonable[5] coefficients and n initial conditions always has exactly one solution. We shall now see that a boundary value problem may have no solutions at all. Or it may have exactly one solution. Or it may have infinitely many solutions. We shall start by finding all solutions to the ODE

$$y'' + y = 0 \tag{I.10}$$

We shall then impose various boundary conditions and see what happens.

The function $y(t) = e^{rt}$ is a solution to (I.10) if and only if

$$r^2 e^{rt} + e^{rt} = 0 \iff r^2 + 1 = 0 \iff r = \pm i$$

where i (which electrical engineers often denote[6] j) is a square root of -1. Thus the general solution to the second order linear ODE (I.10) is $y(t) = C_1' e^{it} + C_2' e^{-it}$, with C_1' and C_2' arbitrary constants. We may rewrite this general solution in terms of $\sin t$ and $\cos t$ by substituting in

$$e^{it} = \cos t + i \sin t \qquad e^{-it} = \cos t - i \sin t$$

This gives

$$y(t) = C_1'\big(\cos t + i \sin t\big) + C_2'(\cos t - i \sin t) = C_1 \cos t + C_2 \sin t$$

where $C_1 = C_1' + C_2'$, and $C_2 = i(C_1' - C_2')$. Note that there is nothing stopping C_1' and C_2' from being complex numbers. So there is nothing stopping $C_1 = C_1' + C_2'$, and $C_2 = i(C_1' - C_2')$ from being real numbers.

(a) Now consider the boundary value problem

$$y'' + y = 0 \qquad y(0) = 0 \qquad y(2\pi) = 1 \tag{I.11}$$

The function $y(t)$ satisfies the ODE if and only if it is of the form

$$y(t) = C_1 \cos t + C_2 \sin t$$

for some constants C_1 and C_2. A function of this form satisfies the boundary condition $y(0) = 0$ if and only if

$$0 = y(0) = C_1 \cos 0 + C_2 \sin 0 = C_1$$

4 There are videos of the collapse on the web.
5 For example, continuous.
6 This is to avoid confusion with the current, which is typically called i.

A function of this form satisfies the boundary condition $y(2\pi) = 1$ if and only if

$$1 = y(2\pi) = C_1 \cos 2\pi + C_2 \sin 2\pi = C_1$$

The two requirements $C_1 = 0$ and $C_1 = 1$ are incompatible. So the boundary value problem (I.11) has no solution at all.

(b) Next consider the boundary value problem

$$y'' + y = 0 \qquad y(0) = 0 \qquad y\left(\frac{\pi}{2}\right) = 0 \tag{I.12}$$

The function $y(t)$ satisfies the ODE if and only if it is of the form

$$y(t) = C_1 \cos t + C_2 \sin t$$

for some constants C_1 and C_2. A function of this form satisfies the boundary condition $y(0) = 0$ if and only if

$$0 = y(0) = C_1 \cos 0 + C_2 \sin 0 = C_1$$

A function of this form satisfies the boundary condition $y\left(\frac{\pi}{2}\right) = 0$ if and only if

$$0 = y\left(\frac{\pi}{2}\right) = C_1 \cos\left(\frac{\pi}{2}\right) + C_2 \sin\left(\frac{\pi}{2}\right) = C_2$$

So we have a solution if and only if $C_1 = C_2 = 0$ and the boundary value problem (I.12) has exactly one solution, namely $y(t) = 0$, which is a bit dull.

(c) Finally consider the boundary value problem

$$y'' + y = 0 \qquad y(0) = 0 \qquad y(2\pi) = 0 \tag{I.13}$$

The function $y(t)$ satisfies the ODE if and only if it is of the form

$$y(t) = C_1 \cos t + C_2 \sin t$$

for some constants C_1 and C_2. A function of this form satisfies the boundary condition $y(0) = 0$ if and only if

$$0 = y(0) = C_1 \cos 0 + C_2 \sin 0 = C_1$$

A function of this form satisfies the boundary condition $y(2\pi) = 0$ if and only if

$$0 = y(2\pi) = C_1 \cos(2\pi) + C_2 \sin(2\pi) = C_1$$

So we have a solution if and only if $C_1 = 0$ and the boundary value problem (I.13) has infinitely many solutions, namely $y(t) = C_2 \sin t$ with C_2 being an arbitrary constant.

Example I.4