Math 538, lecture 22, 3/4/2024

Last time: Applied "geometry of numbers" to obtain:
(fix # field K)

Thm 1: $Cl(K)$ finite.

Thm 2: $\mathcal{O}_K^\times \cong \left\{\begin{matrix}\text{roots of} \\ \text{unity} \\ \text{in } K\end{matrix}\right\} \times \mathbb{Z}^{r+s-1}$

where K has r real places, s complex ones

Chapter 5: Analytic theory

Fix #field K.

Question: How many ideals in $\mathcal{O}_K$ have norm at most X? How many primes up to X?

Remark: Can't count integers by norm due to the units

$(\mathcal{O}_K \setminus \{0\})/\mathcal{O}_K^\times \simeq$ Principal ideals

Plan: (1) Counting in general
(2) Apply to problem at hand

§1. Smooth cutoffs

Have a sequence $\{a_n\}_{n=1}^{\infty}$. Want to estimate sum $\sum_{n \le X} a_n$:

$$\sum_{n \le X} a_n = \sum_{n=1}^{\infty} a_n \cdot \mathbb{1}_{[0,X]}(n) = \sum_{n=1}^{\infty} a_n \cdot \mathbb{1}_{[0,1]}\left(\frac{n}{X}\right)$$

Idea: replace $\mathbb{1}_{[0,1]}$ with some $\varphi \in C_c^{\infty}(\mathbb{R})$.

Say take
$\varphi \equiv 1$ on $[0, 1-h]$
$\varphi \equiv 0$ on $[1+h, \infty)$
$0 \le \varphi \le 1$ on $[1-h, 1+h]$

Get $$\sum_{n \le X} a_n = \sum_{n \le 1}^{\infty} a_n \varphi\left(\frac{n}{X}\right) - \sum_{X < n < (1+h)X} a_n \varphi\left(\frac{n}{X}\right)$$

Better:

$$\left| \sum_{n \le X} a_n - \sum_{n=1}^{\infty} a_n \varphi\left(\frac{n}{X}\right) \right| \le \sum_{1-h < \frac{n}{X} < 1+h} |a_n|$$

if h is small, RHS is an error term

Remark: sometimes helpful to count dyadically:

count on $[1,2], [2,4], [4,8], [8,16]$ . .
(on $[X, 2X]$) use φ approximates $1_{[1,2]}$.

§2. The Mellin transform

Def: The Mellin transform of φ defined on $\mathbb{R}_{>0}^{\times}$ is

$$\tilde{\varphi}(s) = \int_0^\infty \varphi(x) x^s \frac{dx}{x}$$

note 1: $\frac{dx}{x}$ is the Haar measure of $\mathbb{R}_{>0}^{\times}$.

note 2: under the isomorphism $\exp: \mathbb{R}^{+} \to \mathbb{R}_{>0}^{\times}$ this is just the Fourier transform

Lemma: (1) If $\varphi \in C_c((0,\infty))$ then $\tilde{\varphi}$ is entire.
(2) If φ decays at ∞ at least polynomially then $\tilde{\varphi}$ is holomorphic in a strip half-plane.

Pf: (1) clear (integral converges locally unif. abs. for all s)

(2) If φ is bdd at 0, decays like $|x|^{-a}$,
if $\mathrm{Re}(s) > 0$, integral ok at 0
if $\mathrm{Re}(s) < a$, integral ok at ∞.

(note: $\left| \int_0^\infty \varphi(x) x^s \frac{dx}{x} \right| \leq \int_0^\infty |\varphi(x)| \cdot x^{\operatorname{Re} s} \cdot \frac{dx}{x}$)

<u>Thm</u>: (Morera) to show $F(s)$ is holomorphic in disc D enough to show (1) F is cts on D

(2) $\oint_\gamma F\, ds = 0$ for all closed contours $\gamma \subset D$.

<u>Fact</u>: Let φ be "reasonable". Then for σ large enough,

$$\varphi(x) = \frac{1}{2\pi i} \int_{(\sigma)} \tilde{\varphi}(s) x^{-s} \frac{ds}{s}$$

contour $(\sigma - i\infty, \sigma + i\infty)$

§3. <u>Putting things together</u>

Let φ be a smooth function of cpt support on $\mathbb{R}$. Then $\tilde{\varphi}$ is hol on $\operatorname{Re} s > 0$.

Get:

$$\sum_{n=1}^\infty a_n \varphi\left(\frac{n}{X}\right) = \frac{1}{2\pi i} \sum_{n=1}^\infty a_n \int_{(\sigma)} \left(\frac{n}{X}\right)^{-s} \tilde{\varphi}(s) \frac{ds}{s}$$

$$= \frac{1}{2\pi i} \int_{(\sigma)} D(s) \tilde{\varphi}(s) X^s \frac{ds}{s}$$

where:

$$\boxed{D(s) = \sum_{n=1}^{\infty} a_n n^{-s}}$$

Ex: $\left(\sum_{k=1}^{\infty} a_k k^{-s}\right)\left(\sum_{\ell=1}^{\infty} b_\ell \ell^{-s}\right) = \sum_{n=1}^{\infty}\left(\sum_{k\ell=n} a_k b_\ell\right) n^{-s}.$

Assume: $|a_n| \ll n^a$. Then $D(s)$ converges absolutely in $\mathrm{Re}(s) > a+1$, $\tilde{\varphi}(s)$ decays rapidly in vertical direction, can change order of $\int, \sum$ if $\sigma > a+1$.

Preliminary: $\left|\sum_{n=1}^{\infty} a_n \varphi\left(\frac{n}{X}\right)\right| \ll \int_{-\infty}^{+\infty} |D(\sigma+it)|\,|\tilde{\varphi}(\sigma+it)| \frac{dt}{|\sigma+it|} \cdot X^\sigma$

Suppose $D(s)$, $\tilde{\varphi}(s)$ continue meromorphically to the left. Then shift contour to the left.

Get:

$$\sum_{n=1}^{\infty} a_n \varphi\left(\frac{n}{X}\right) = \sum_{\sigma' < \mathrm{Re}\,\rho < \sigma} \mathrm{Res}_{s=\rho}\left(D(s)\tilde{\varphi}(s)\frac{X^s}{s}\right) \quad \text{of size } X^{\mathrm{Re}\,\rho}$$

$$+\frac{1}{2\pi i}\int_{(\sigma')} D(s)\tilde{\varphi}(s) X^s \frac{ds}{s} \quad \text{of size } X^{\sigma'}.$$

Simplest: $\varphi \in C_c^\infty((0,\infty))$. Then $\tilde{\varphi}$ is entire, decays in vertical direction. (no double poles)

$$\sum_{n=1}^\infty a_n \varphi\left(\frac{n}{X}\right) = \sum_{\sigma' < \operatorname{Re}\rho < \sigma} \tilde{\varphi}(\rho) X^\rho \operatorname{Res}_{s=\rho}\left(D(s)\frac{1}{s}\right) + O(X^{\sigma'})$$

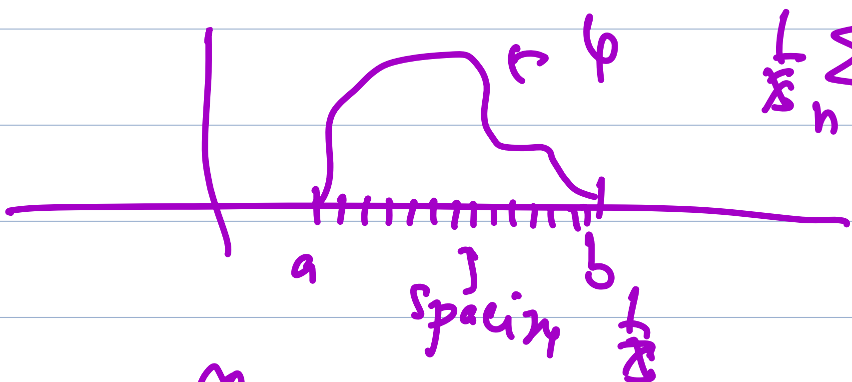
Expects $\sum_{n \le X} a_n \sim X^{\operatorname{Re}\rho}$, ρ first pole of $D(s)$

Example: $a_n = 1$ $\quad \sum_{n \le X} a_n = [X]$

$D(s) = \sum_{n=1}^\infty n^{-s} = \zeta(s)$ continues to a meromorphic fcn on $\mathbb{C}$, unique pole at $s=1$, residue $=1$.

Get: $$\sum_{n \ge 1}^\infty \varphi\left(\frac{n}{X}\right) = \tilde{\varphi}(1) X - \tilde{\varphi}(0)\zeta(0) + O(X^{\sigma'})$$

Say φ is supported on $[a,b]$



$\frac{1}{X}\sum_n \varphi\left(\frac{n}{X}\right) =$ Riemann sum for $\int_0^\infty \varphi(x)\,dx$

Note: $\tilde{\varphi}(1) = \int_0^\infty \varphi(x) \cdot x^1 \frac{dx}{x} = \int_0^\infty \varphi(x)\,dx$

Examples: counting integers mod 4.

Let $\chi_4(n) = \begin{cases} 1 & n \equiv 1\ (4) \\ -1 & n \equiv -1\ (4) \\ 0 & 2 \mid n \end{cases}$; $I_4(n) = \begin{cases} 1 & n \equiv 1,3\ (4) \\ 0 & n \equiv 2,0\ (4) \end{cases}$

Then $\dfrac{\chi_4 + I_4}{2} = \begin{cases} 1 & n \equiv 1\ (4) \\ 0 & n \not\equiv 1\ (4) \end{cases}$

$$\frac{I_4 - \chi_4}{2} = \begin{cases} 1 & n \equiv 3\ (4) \\ 0 & n \not\equiv 3\ (4) \end{cases}$$

$\overline{\sum_{n=1}^{\infty}} I_4(n) n^{-s} = \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} n^{-s}$; $\sum_{\substack{n=1 \\ \text{even}}}^{\infty} n^{-s} = \sum_{m=1}^{\infty} (2m)^{-s} = 2^{-s}\zeta(s)$

So $\sum_{n=1}^{\infty} I_4(n) n^{-s} = \zeta(s) - 2^{-s}\zeta(s) = (1-2^{-s})\zeta(s)$

<u>Define</u>: $L(s; \chi_4) := \sum_{n=1}^{\infty} \chi_4(n) n^{-s}$ (entire!)

$\Rightarrow$ Dirichlet series for counting mod 4 are

$$D(s) = \frac{1}{2}\left((1-2^{-s})\zeta(s) \pm L(s;\chi_4)\right)$$

$$\Rightarrow \sum_{n \equiv \pm 1\ (4)} \varphi\left(\frac{n}{X}\right) = \frac{1}{4}\tilde{\varphi}(1) X \pm \frac{1}{2}\tilde{\varphi}(0) L(0;\chi_4) + \text{small}$$

Next time: count primes.

Then $D(s) = -\frac{\zeta'(s)}{\zeta(s)}$ has ∞'ly many poles