Math 538, Lecture 15, 6/3/2024

Last time: $\{\text{Places of a field } F\} \leftrightarrow \{\text{primes}\} \sqcup \{\infty \text{ places}\}$

$\Rightarrow$ new proofs of results about primes (e.g. $Gal(K/F)$ acts transitively on primes of K over fixed prime)

+ Product formula for $x \in F^\times$

$$\prod_{v \in |F|} \|x\|_v = 1$$

$\|x\|_p = q_p^{-v_p(x)}$, $\|x\|_{\mathbb{R}}$ = usual absolute value
$\|x\|_{\mathbb{C}}$ = square of usual one

Aside: $\|x\|_v$ is the module of x: if m is a Haar measure on F_v, then
$$m(xE) = \|x\|_v \cdot m(E)$$

Today: Different & discriminant

Studying extension L/K of local fields or #fields, in either case we'll develop invariants different, discriminant, ideals in $\mathcal{O}_L$, $\mathcal{O}_K$ resp., measure ramification

Key property: $\mathfrak{P} \triangleleft \mathcal{O}_L$ divides $\partial_{L/K}$ iff $e(\mathfrak{P} : \mathfrak{P} \cap \mathcal{O}_K) > 1$

$\mathfrak{p} \triangleleft \mathcal{O}_K$ divides $D(L:K)$ iff $\exists \mathfrak{P} | \mathfrak{p}$ with $e(\mathfrak{P} : \mathfrak{p}) > 1$

Step 1: The trace form

Motivation: Start from identity ("Fourier expansion")

$$L^2(\mathbb{R}/\mathbb{Z}) \simeq \widehat{\bigoplus_{k \in \mathbb{Z}}} \mathbb{C} e_k$$

$$e(z) \overset{def}{=} \exp(2\pi i z), \qquad e_k(x) \overset{def}{=} e(kx)$$

Suppose K #field, then $K_\infty = K \otimes_{\mathbb{Q}} \mathbb{R} = \bigoplus_{v | \infty} K_v$ is an $\mathbb{R}$-algebra

$$\dim K_\infty = \sum_{v | \infty} f(K_v : \mathbb{R}) = n$$

$(n = [K:\mathbb{Q}])$

Observe the image of $\mathcal{O}_K$ in K_∞ is discrete : $N^K_{\mathbb{Q}}(\mathcal{O}_K \setminus \{0\}) \subset \mathbb{Z} \setminus \{0\}$

$N^{K_\infty}_{\mathbb{R}}$ is cts, so if $x \in \mathcal{O}_K^{\times}$ had image too close to 0, its norm would be less than 1.

Example: $\mathbb{Z}[\sqrt{2}] \hookrightarrow \mathbb{R} \times \mathbb{R}$ say $\alpha^2 = 2$

$$a + b\alpha \mapsto (a + b\sqrt{2}, a - b\sqrt{2})$$

$\{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\} \subset \mathbb{R}$ is dense!

But if $a + b\sqrt{2}$ and $a - b\sqrt{2}$ are close to 0 then $a^2 - 2b^2$ is close to zero.

Ors if $a + b\sqrt{2}$, $a - b\sqrt{2}$ small, so are

$$a = \frac{(a + b\sqrt{2}) + (a - b\sqrt{2})}{2}, \quad b = \frac{\sim}{2\sqrt{2}}.$$

$\Rightarrow \mathcal{O}_K \subset K_\infty$ is a discrete subgp

Ex: A discrete subgp Λ of $\mathbb{R}^n$ has the form

$$\Lambda = \bigoplus_{i=1}^{k} \mathbb{Z} \underline{v}_i \qquad \{\underline{v}_i\}_{i=1}^{k} \subset \mathbb{R}^n \text{ lin indep}$$

$$\Rightarrow \mathrm{rk}_{\mathbb{Z}} \Lambda = \dim_{\mathbb{Q}} \Lambda \otimes_{\mathbb{Z}} \mathbb{Q} = \dim_{\mathbb{Q}} \mathrm{span}_{\mathbb{Q}}(\Lambda)$$

$$= \dim_{\mathbb{R}} \mathrm{span}_{\mathbb{R}}(\Lambda) = \dim_{\mathbb{R}} \mathbb{R} \otimes_{\mathbb{Z}} \Lambda$$

($\Leftrightarrow$ in basis extending $\{\underline{v}_i\}$

$$\Lambda = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n : \begin{array}{l} \{x_i\}_{i=1}^{k} \subset \mathbb{Z} \\ \{x_i\}_{i=k+1}^{n} = \{0\} \end{array} \right\}$$

Know: $\mathrm{rk}_{\mathbb{Z}} \mathcal{O}_K = n$

$\Rightarrow \mathcal{O}_K \subset K_\infty$ has full rank

$\Rightarrow$ can identify K_∞ with $\mathbb{R}^n$ s.t. $\mathcal{O}_K$ is identified with $\mathbb{Z}^n$. $\Rightarrow K_\infty / \mathcal{O}_K$ is cpt

Ex: $k = (k_v)_{v|\infty}$, $x = (x_v)_{v|\infty} \in K_\infty$

define

$$e_k(x) = e\left(\mathrm{Tr}_{\mathbb{R}}^{K_\infty} kx\right)$$

$e_k(x+y) = e_k(x)\, e_k(y)$

$$\mathrm{Tr}_{\mathbb{R}}^{K_\infty} kx = \sum_{v|\infty} \mathrm{Tr}_{\mathbb{R}}^{K_v}(k_v x_v)$$

if k, x images of elements in K, have

$$\mathrm{Tr}_{\mathbb{R}}^{K_\infty} kx = \mathrm{Tr}_{\mathbb{Q}}^{K} kx$$

$\Rightarrow$ Fix k ask: is $e_k(x): K_\infty \to \mathbb{C}$ $\mathcal{O}_K$-periodic?

$e_k(x)$ $\mathcal{O}_K$-periodic iff $e_k(x) = 1$ when $x \in \mathcal{O}_K$, iff $\mathrm{Tr}_{\mathbb{R}}^{K_\infty} kx \in \mathbb{Z}$ for all $x \in \mathcal{O}_K$.

$\Rightarrow$ $e_k(x)$ is $\mathcal{O}_K$-periodic iff $k \in K$ and

$$\mathrm{Tr}_{\mathbb{Q}}^{K}(kx) \in \mathbb{Z} \quad \text{for all } x \in \mathcal{O}_K$$

Def: $\mathcal{O}_K^* = \left\{ k \in K \;\middle|\; \begin{array}{l} \forall x \in \mathcal{O}_K: \\ \mathrm{Tr}_{\mathbb{Q}}^{K}(kx) \in \mathbb{Z} \end{array} \right\}$

Thm: $L^2(K_\infty/\mathcal{O}_K) \simeq \widehat{\bigoplus_{\lambda\in\mathcal{O}_K^*}} \mathbb{C}\cdot e_\lambda$.

Let's analyze the trace form

The form $(x,y) = \mathrm{Tr}^L_K(xy)$ is a K-bilinear form on L, symmetric & it's non-degenerate: if $x\in L^\times$,

$$\mathrm{Tr}^L_K(x\cdot x^{-1}) = [L:K] \neq 0$$

$\Rightarrow$ $(\cdot,\cdot)$ identifies L with its dual vsp /K.

Def: Let $\Lambda\subset L$ be an $\mathcal{O}_K$-submodule. Define dual of Λ to be

$$\Lambda^* = \{x\in L \mid \mathrm{Tr}(x\Lambda)\subset\mathcal{O}_K\}$$

$$= \{x\in L \mid \forall y\in\Lambda: \mathrm{Tr}(xy)\in\mathcal{O}_K\}$$

Evidently an $\mathcal{O}_K$-submodule of L

Aside: If $\Lambda \simeq \mathbb{Z}^n$, $\Lambda^* = \mathrm{Hom}(\Lambda; \mathbb{Z})$.

Lemma: $\{w_i\}_{i=1}^n \subset L$ be a K-basis
$\{w_i^*\}_{i=1}^n \subset L$ the dual basis wrt $(\cdot,\cdot)$

Then $\left(\bigoplus_{i=1}^n \mathcal{O}_K w_i\right)^* = \bigoplus_{i=1}^n \mathcal{O}_K w_i^*$.

Pf: $\Lambda = \bigoplus_{i=1}^n \mathcal{O}_K w_i$. $Tr_K^L(w_i w_j^*) = \delta_{ij} \in \mathbb{Z}$

so all $w_j^* \in \Lambda^*$. Conversely say that

$\sum_j a_j w_j^* \in \Lambda^*$ then $(w_i, \sum_j a_j w_j^*) \in \mathcal{O}_K$
$= a_j$.

Cor: If $\Lambda \subset L$ is a fractional ideal, Λ^* is also a fractional ideal

Pf: (1) $\mathcal{O}_L^* \supset \mathcal{O}_L$ if $x \in \mathcal{O}_L$, $y \in \mathcal{O}_L$, $Tr(xy) \in \mathcal{O}_K$

(2) if $\alpha \in L^\times$, $(\alpha\Lambda)^* = \alpha^{-1}\Lambda^*$
$\Rightarrow$ wlog can replace Λ with $\alpha\Lambda$

Now let $\mathfrak{a} \subset L$ be a fractional ideal

If $x \in \mathfrak{a}^*$, $\alpha \in \mathcal{O}_L$ then

$$\mathrm{Tr}(\alpha x \mathfrak{a}) = \mathrm{Tr}(x\mathfrak{a}) \subset \mathcal{O}_K$$

so $\alpha x \in \mathfrak{a}^*$, so $\mathfrak{a}^*$ is an $\mathcal{O}_L$-module

If $\mathfrak{a} \subset \mathcal{O}_L$ then $\mathfrak{a}^* \supset \mathcal{O}_L^* \supset \mathcal{O}_L$
so $\mathfrak{a}^* \neq \{0\}$

Finally let $\alpha \in \mathfrak{a} \setminus \{0\}$, $\exists \{w_i\}_{i=1}^n \subset \mathcal{O}_L$ a K-basis

$$\mathfrak{a} \supset \alpha \mathcal{O}_L \supset \bigoplus_{i=1}^n \alpha \mathcal{O}_K w_i$$

$$\Rightarrow \mathfrak{a}^* \subset \alpha^{-1} \bigoplus_{i=1}^n \mathcal{O}_K w_i^*$$

let $m \in \mathbb{Z} \setminus \{0\}$ be such that $m w_i^* \in \mathcal{O}_L$

$$\Rightarrow m\mathfrak{a}^* \subset \alpha^{-1} \bigoplus_{i=1}^n \mathcal{O}_K m w_i^* \subset \alpha^{-1} \mathcal{O}_L. \blacksquare$$

The different

As before L/K finite extension of # fields or fields complete wrt discrete valuation.

Def: The complementary module (or inverse different) of L/K is the fractional ideal

$$\mathcal{C}_{L/K} \overset{\text{def}}{=} \mathcal{O}_L^\vee \quad \text{(duality wrt } Tr_K^L \text{ form)}$$

The relative different of L/K is the ideal

$$\mathcal{D}_{L/K} \overset{\text{def}}{=} \mathcal{C}_{L/K}^{-1}.$$

(since $\mathcal{C}_{L/K} \supset \mathcal{O}_L$, $\mathcal{C}_{L/K}^{-1} \subset \mathcal{O}_L$)

Lemma: Let $\mathfrak{a}$ be a fractional ideal. Then $\mathfrak{a}^* = \mathcal{C}_{L/K}\,\mathfrak{a}^{-1}$.

Pf: Certainly $Tr_K^L(\mathfrak{a}\,\mathcal{C}_{L/K}\,\mathfrak{a}^{-1}) = Tr_K^L(\mathcal{C}_{L/K}) \subseteq \mathcal{O}_K$, so $\mathfrak{C}_{L/K}\,\mathfrak{a}^{-1} \subset \mathfrak{a}^*$.

Conversely $Tr(\mathcal{O}_L \cdot \mathfrak{a}\mathfrak{a}^*) = Tr(\mathfrak{a}\mathfrak{a}^*) \subset \mathcal{O}_K$

so $\mathfrak{a}\mathfrak{a}^* \subset \mathfrak{C}_{L/K}$ so $\mathfrak{a}^* \subset \mathfrak{C}_{L/K}\,\mathfrak{a}^{-1}$. ∎

Lemma: Let $M/L/K$ be a tower of fields. Then

$$D_{M/K} = D_{M/L} \cdot D_{L/K}.$$

$$= \left\{ \sum_{i=1}^{r} x_i y_i \;\middle|\; \begin{matrix} x_i \in D_{M/L} \\ y_i \in D_{L/K} \end{matrix} \right\}$$

M
|
L
|
K

Pf: $Tr^M_K(\mathfrak{C}_{L/K}\,\mathfrak{C}_{M/L} \cdot \mathcal{O}_M)$

$= Tr^L_K\, Tr^M_L(\mathfrak{C}_{L/K}\,\mathfrak{C}_{M/L}\,\mathcal{O}_M)$

$= Tr^L_K(\mathfrak{C}_{L/K} \cdot Tr^M_L(\mathfrak{C}_{M/L}\,\mathcal{O}_M))$

$\subseteq Tr^L_K(\mathfrak{C}_{L/K}\,\mathcal{O}_L) \subseteq \mathcal{O}_K$.

$\Rightarrow \mathfrak{C}_{L/K} \cdot \mathfrak{C}_{M/L} \subset \mathfrak{C}_{M/K}$

Converse:

Tr^{M}_{L} is $\mathcal{O}_L$-linear

$$Tr^{L}_{K}\left(\mathcal{O}_L\, Tr^{M}_{L}(C_{M/K}\,\mathcal{O}_M)\right) = Tr^{L}_{K} Tr^{M}_{L}(C_{M/K}\,\mathcal{O}_M)$$

$$= Tr^{M}_{K}(C_{M/K}) \subseteq \mathcal{O}_K.$$

$$\Rightarrow Tr^{M}_{L}\, C_{M/K} \subset C_{L/K}$$

$$\Rightarrow Tr^{M}_{L}(C_{L/K}^{-1} C_{M/K}\,\mathcal{O}_M) = C_{L/K}^{-1} Tr^{M}_{L}(C_{M/K}\,\mathcal{O}_M) \subseteq \mathcal{O}_L$$

$$\Rightarrow C_{L/K}^{-1} C_{M/K} \subset C_{M/L}$$

$$\Rightarrow C_{M/K} \subseteq C_{M/L} \cdot C_{L/K}$$

$$\Rightarrow C_{M/K} = C_{M/L} \cdot C_{L/K}.$$

Take inverse (check: if $\mathfrak{a}$ fractional ideal of $\mathcal{O}_L$, $\mathfrak{a}^{-1}\cdot\mathcal{O}_M = (\mathfrak{a}\cdot\mathcal{O}_M)^{-1}$ as fractional ideals of $\mathcal{O}_M$)

∎

Aside: $\Gamma = PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$

acts on $\mathbb{H}^{(2)} \times \mathbb{H}^{(2)}$

Gives action of $\Gamma = PSL_2(\mathbb{Z}[\sqrt{2}])$

$\Rightarrow$ talk about Hilbert modular forms:

hol fcns $f: \mathbb{H}^{(2)} \times \mathbb{H}^{(2)} \to \mathbb{C}$ $\qquad k = (k_1, k_2)$

st $(f|_k \gamma)(z_1, z_2) = (cz_1 + d)^{k_1} (\bar{c} z_2 + \bar{d})^{k_2} f(\gamma z_1, \bar{\gamma} z_2)$.

$= f(z_1, z_2)$

In particular, $f(z_1, z_2)$ invariant under translation by $\mathcal{O}_K$ $\left\{ \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \right\}$, $b \in \mathcal{O}_K$

$\Rightarrow$ get Fourier expansion

$$f(z_1, z_2) = \sum_{k \in \mathcal{O}_{\mathbb{Q}(\sqrt{2})}} a_k \, e(\mathrm{Tr}(k(z_1, z_2))).$$

Book: Fourier analysis on number fields, Ramakrishnan - Valenza

GTM - 186

Silberman, MATH 539 notes

F totally real field

$\Gamma < PSL_2(\mathcal{O}_F)$ finite index

$\Gamma_{i\infty} = \{ \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} \in \Gamma \}$ finite index in $\mathcal{O}_F$

$\mathfrak{n} \triangleleft \mathcal{O}_F$

$\Gamma = \Gamma_0(\mathfrak{n}) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \in \mathfrak{n} \}$

$SL_2(\mathbb{C})$ acting on $SL_2(\mathbb{C})/SU(2) = \mathbb{H}^{(3)}$

$SL_2(\mathcal{O}_{\mathbb{Q}(i)})$ acts there

$\left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} : x \in \mathfrak{d}[i] \right\}$ acts by translation

$$\mathbb{H}^{(3)} : \left\{ iy + \underline{x} : \begin{array}{l} y > 0 \\ \underline{x} \in \mathbb{R}^2 \end{array} \right\}$$

$$SL_2(\mathcal{O}_F) \backslash SL_2(F_\infty)$$

$$\underline{y} = \prod_{v \mid \infty} \mathbb{R}^\times_{>0}$$

$$\underline{x} = \prod_{v \mid \infty} F_v$$

$$f(x + iy) = \sum_{k \in \mathcal{O}_{4/k}} a_k(y) e(kx)$$