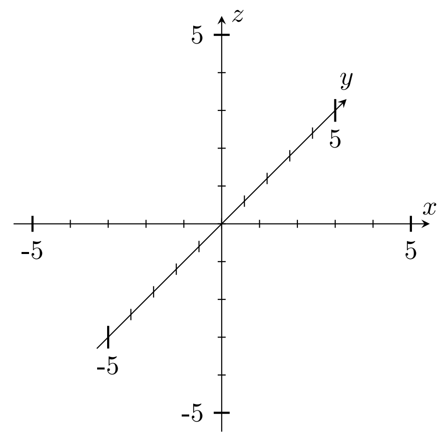
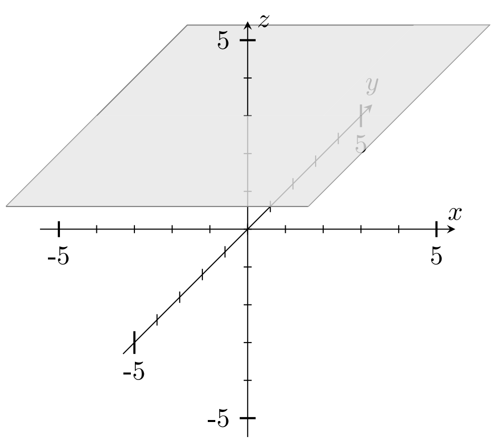
Math 100:V02 – SOLUTIONS TO WORKSHEET 17
MULTIVARIABLE DIFFERENTIATION

1. Plotting in three dimensions



(1) $\star$ Plot the points $(2, 1, 3)$, $(-2, 2, 2)$ on the axes provided.
(2) Let $f(x, y) = e^{x^2+y^2}$.
   (a) $\star$ What are $f(0, -1)$? $f(1, 2)$? Plot the point $(0, 1, f(0, 1))$ on the axes provided.
   (b) $\star$ What is the *domain* of f (that is: for what (x, y) values does f make sense?
   Solution: f makes sense for all (x, y) – equivalenly that is on the plane $\mathbb{R}^2$.
   (c) $\star$ What is the *range* of f (that is: what values does it take)?
   Solution: $x^2 + y^2$ takes all possible nonegative values, so $e^{x^2+y^2}$takes all values in $[1, \infty)$.
(3) $\star\star$ What would the graph of $z = \sqrt{1 - x^2 - y^2}$ look like?
   Solution: This is the same as $x^2 + y^2 + z^2 = 1$ with $z \geq 0$, so the graph would be the upper half of the sphere of radius 1.
(4) $\star$ Which plane is this?
   Solution: (C) $z = 3$.



(A) $x = 3$
(B) $y = 3$
(C) $z = 3$
(D) none
(E) not sure

2. Partial derivatives

(5) (a) $\star$ Let $f(x) = 2x^2 - a^2 - 2$. What is $\frac{df}{dx}$?
   Solution: $\frac{df}{dx} = 4x$.
   (b) $\star$ Let $f(x) = 2x^2 - y^2 - 2$ where y is a constant. What is $\frac{df}{dx}$?
   Solution: $\frac{df}{dx} = 4x$.
   (c) $\star$ Let $f(x, y) = 2x^2 - y^2 - 2$. What is the rate of change of f as a function of x *if we keep y constant*?
   Solution: $\frac{\partial f}{\partial x} = 4x$.

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(d) $\star$ What is $\frac{\partial f}{\partial y}$?
Solution: $\frac{\partial f}{\partial y} = -2y$.

(6) Find the partial derivatives with respect to both x, y of

(a) $\star$ $g(x,y) = 3y^2 \sin(x+3)$
Solution: $\frac{\partial g}{\partial x} = 3y^2 \cos(x+3)$ (note that $3y^2$ is *constant* if y is) while $\frac{\partial g}{\partial y} = 6y \sin(x+3)$ (note that $\cos(x+3)$ is constant when x is constant).

(b) $\star$ $h(x,y) = ye^{Axy} + B$
Solution: We have $\frac{\partial h}{\partial x} \overset{\text{linear}}{=} y \left(\frac{\partial}{\partial x} e^{Axy}\right) + \frac{\partial}{\partial x} B \overset{\text{chain}}{=} y \cdot Ay \cdot e^{Axy} = Ay^2 e^{Axy}$ and $\frac{\partial h}{\partial y} \overset{\text{pdt}}{=} \left(\frac{\partial}{\partial y} y\right) \cdot e^{Axy} + y \left(\frac{\partial}{\partial y} e^{Axy}\right) = e^{Axy} + Axye^{Axy} = e^{Axy}(1 + Axy)$.

(7) The the gravitational *potential* due to a point mass M (equivalently the electrical potential due to a point charge M) is given by the formula $U(x,y,z) = -\frac{GM}{r}$ where $r = \sqrt{x^2+y^2+z^2}$. Here G is the universal gravitational constant (equivalently G is the Coulomb constant).

(a) $\star$ The x-component of the field is given by the formula $F_x(x,y,z) = -\frac{\partial U}{\partial x}$. Find F_x
Solution: We have

$$\begin{aligned}
F_x &= GM \frac{\partial}{\partial x} \left(x^2+y^2+z^2\right)^{-1/2} \\
&= -\frac{GM}{2} \left(x^2+y^2+z^2\right)^{-3/2} 2x \\
&= -GM \left(x^2+y^2+z^2\right)^{-3/2} \cdot x \\
&= -\frac{GM}{r^3} x \,.
\end{aligned}$$

(b) $\star$ The magnitude of the field is given by $\left|\vec{F}\right| = \sqrt{F_x^2 + F_y^2 + F_z^2}$. How does it decay as a function of r?
Solution: Let $r = \left(x^2+y^2+z^2\right)^{1/2}$. Then $F_x = -\frac{GMx}{r^3}$ so $F_y = -\frac{GMy}{r^3}$ and $F_z = -\frac{GMz}{r^3}$ and we get:

$$\begin{aligned}
\left|\vec{F}\right|^2 &= \frac{(GM)^2 x^2}{r^6} + \frac{(GM)^2 y^2}{r^6} + \frac{(GM)^2 z^2}{r^6} \\
&= (GM)^2 \frac{x^2+y^2+z^2}{r^6} \\
&= \frac{r^2}{r^6} = (GM)^2 r^{-4} \,.
\end{aligned}$$

Thus

$$\left|\vec{F}\right| = \frac{GM}{r^2} \,.$$

This is the inverse square law.

(8) The *entropy* of an ideal gas of N molecules at temperature T and volume V is

$$S(N,V,T) = Nk \log \left[\frac{VT^{1/(\gamma-1)}}{N\Phi}\right] \,.$$

where k is *Boltzmann's constant* and γ, Φ are constants that depend on the gas.

(a) $\star$ Find the *heat capacity at constant volume* $C_V = T\frac{\partial S}{\partial T}$.
Solution: We have $S = Nk \log V + \frac{Nk}{\gamma-1} \log T - Nk \log N - Nk \log \Phi$

$$\begin{aligned}
T\frac{\partial S}{\partial T} &= T\frac{Nk}{(\gamma-1)T} \\
&= \frac{Nk}{\gamma-1} \,.
\end{aligned}$$

(b) $\star\star\star$ Using the relation ("ideal gas law") $PV = NkT$ write S as a function of N, P, T instead. Differentiating with respect to T **while keeping** P **constant** determine the *heat capacity at constant pressure* $C_P = T\frac{\partial S}{\partial T}$.

Solution: Substituting $V = \frac{NkT}{P}$ we get $S = Nk \log \left[\frac{kT^{\gamma/(\gamma-1)}}{\Phi P}\right] = -Nk \log P + \frac{\gamma Nk}{\gamma-1} \log T - Nk \log \frac{k}{\Phi}$ so now

$$T\frac{\partial S}{\partial T} = T\frac{\gamma Nk}{(\gamma-1)T}$$
$$= \frac{\gamma}{\gamma-1} Nk\,.$$

(9) We can also compute second derivatives. For example $f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2}{\partial y \partial x} f$. Evaluate:

(a) $\star$ $h_{xx} = \frac{\partial^2 h}{\partial x^2} =$

Solution: We have $\frac{\partial^2 h}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}\right) = \frac{\partial}{\partial x}\left(Ay^2 e^{Axy}\right) = A^2 y^3 e^{Axy}$.

(b) $\star$ $h_{xy} = \frac{\partial^2 h}{\partial y \partial x} =$

Solution: We have $\frac{\partial^2 h}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial h}{\partial x}\right) = \frac{\partial}{\partial y}\left(Ay^2 e^{Axy}\right) = 2Aye^{Axy} + A^2 xy^2 e^{Axy} = \left(2Ay + A^2 xy^2\right) e^{Axy}$.

(c) $\star$ $h_{yx} = \frac{\partial^2 h}{\partial x \partial y} =$

Solution: We have $\frac{\partial^2 h}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial y}\right) = \frac{\partial}{\partial x}\left(e^{Axy}\left(1 + Axy\right)\right) = Aye^{Axy}\left(1 + Axy\right) + e^{Axy} \cdot Ay = e^{Axy}\left(A^2 xy^2 + 2Ay\right)$.

(d) $\star$ $h_{yy} = \frac{\partial^2 h}{\partial y^2} =$

Solution: We have $\frac{\partial^2 h}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial h}{\partial y}\right) = \frac{\partial}{\partial y}\left(e^{Axy}\left(1 + Axy\right)\right) = Axe^{xy}\left(1 + Axy\right) + e^{Axy}\left(Ax\right) = Ax\left(2 + Axy\right) e^{Ax}$.

(10) $\star$ Repeat this exercise for the function g from problem 2(a).

Solution: We have

$$\frac{\partial^2 g}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial x}\right) = \frac{\partial}{\partial x}\left(3y^2 \cos(x+3)\right) = -3y^2 \sin(x+3)$$
$$\frac{\partial^2 g}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\right) = \frac{\partial}{\partial y}\left(3y^2 \cos(x+3)\right) = 6y \cos(x+3)$$
$$\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial y}\right) = \frac{\partial}{\partial x}\left(6y \sin(x+3)\right) = 6y \cos(x+3)$$
$$\frac{\partial^2 g}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial y}\right) = \frac{\partial}{\partial y}\left(6y \sin(x+3)\right) = 6 \sin(x+3)\,.$$

(11) You stand in the middle of a north-south street (say Health Sciences Mall). Let the x axis run along the street
(say oriented toward the south), and let the y axis run across the street. Let $z = z(x, y)$ denote the height of
the street surface above sea level.

(a) $\star$ What does $\frac{\partial z}{\partial y} = 0$ say about the street?

Solution: The street surface is level.

(b) $\star$ What does $\frac{\partial z}{\partial x} = 0.15$ say about the street?

Solution: The street has a 15% grade sloping up toward the south: for each $1m$ we walk south we gain $0.15m$ in altitude.

(c) $\star$ You want to follow the street downhill. Which way should you go?

Solution: Since altitude increases with increasing x (i.e. as you go south), you should go north.

(d) The intersection of Health Sciences Mall and Agronomy Road is a local maximum.
What does that say about $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ there?

Solution: Both derivatives must be zero, else the street would be sloped in some direction.

3. Bonus (<u>**nonexaminable!**</u>): multivariable linear and higher approximation

Definition 1. A function $f(x,y)$ is *differentiable* at x_0, y_0 if we have a linear approximation $f(x,y) = f(x_0,y_0)+A(x-x_0)+B(y-y_0)+\text{small}$ as $(x,y) \to (x_0,y_0)$. We then have $A = \frac{\partial f}{\partial x}(x_0,y_0)$ and $B = \frac{\partial f}{\partial y}(x_0,y_0)$. The definition for functions of more than two variables is analogous.

(12) Let $f(x) = \sqrt{2+x^2+y^2}$.

(a) Write the linear approximation to f about $(1,1)$ and use that to estimate $f(1.1,1.2)$.
Solution: We have $\frac{\partial f}{\partial x} = \frac{x}{\sqrt{2+x^2+y^2}}$ and $\frac{\partial f}{\partial y} = \frac{y}{\sqrt{2+x^2+y^2}}$. So at $(1,1)$ we have $f(1,1) = 2$, $f_x(1,1) = f_y(1,1) = \frac{1}{2}$ and the linear approximation is

$$f(x,y) \approx 2 + \frac{1}{2}(x-1) + \frac{1}{2}(y-1)\,.$$

In particular

$$f(1.1,1.2) \approx 2 + \frac{1}{2}\cdot\frac{1}{10} + \frac{1}{2}\cdot\frac{2}{10} = 2\frac{3}{20} = 2.15\,.$$

(b) Write the linear approximation to f about $(3,5)$ and use that to estimate $f(2.8,4.9)$.
Solution: At $(3,5)$ we have $f(3,5) = 6$, $f_x(3,5) = \frac{3}{6} = \frac{1}{2}$ and $f_y(3,5) = \frac{5}{6}$. Thus the linear approximation is

$$f(x,y) \approx 6 + \frac{1}{2}(x-3) + \frac{5}{6}(y-5)\,.$$

In particular

$$f(2.8,4.9) \approx 6 + \frac{1}{2}\left(-\frac{2}{10}\right) + \frac{5}{6}\left(-\frac{1}{10}\right) = 6 - \frac{1}{10} - \frac{1}{12} = 5\frac{49}{60}\,.$$