Math 100:V02 – SOLUTIONS TO WORKSHEET 15
OPTIMIZATION

1. Optimization of functions

(1) Let $f(x) = x^4 - 4x^2 + 4$.

(a) Find the absolute minimum and maximum of f on the interval $[-5, 5]$.
Solution: $f'(x) = 4x^3 - 8x$ is defined everywhere and vanishes at $0, \pm\sqrt{2}$. We have $f(\pm 5) = 625 - 100 + 4 = 529$, $f(\pm\sqrt{2}) = 4 - 8 + 4 = 0$ and $f(0) = 4$. The global maximum is therefore 529, attained at ± 5 and the global minimum is 0, attained at $\pm\sqrt{2}$.
Takeaway: straightforward application of the calculus: the global maximum/minimum must either be at the end of the interval, or at the interior, and if in the interior must be at a critical or singular point.

(b) Find the absolute minimum and maximum of f on the interval $[-1, 1]$.
Solution: Now the only critical point is 0 (note that $\sqrt{2} > 1$). We have $f(\pm 1) = 1 - 4 + 4 = 1$ and $f(0) = 4$. The global maximum is now 4, attained at 0, while the global minimum is 1, attained at ± 1.
Takeaway: The interval matters. "Critical points", "singular points" etc mean *points in the interval*.

(c) Find the absolute minimum and maximum of f (if they exist) on the interval $(-1, 1)$.
Solution: The function still has a *local* maximum at 0 where $f(0) = 4$ while $f'(x) < 0$ on $(0, 1)$ and $f'(x) > 0$ on $(-1, 0)$ so this is also the global maximum. There is no global minimum since $f(x) > 1 = f(\pm 1)$ for any x in the open interval (x can get *close* to ± 1 but it can't actually *equal* them).
Takeaway: If the interval is open (does not include the endpoints) then there need not be a global maximum/minimum and we need to analyze the function more carefully – usually by finding the intervals where it's increasing/decreasing.

(d) Find the absolute minimum and maximum of f (if they exist) on the real line.
Solution: Since as $x \to \pm\infty$ we have $f(x) \sim x^4 \to \infty$ there is no global maximum. It also follows that outside some closed interval $f(x)$ only takes large values, so to find the minimum it's enough to consider a big closed interval $[-L, L]$. The minimum won't be at the endpoints (the function is large there) but rather at a critical point. By part (a) we see that the global minimum is 0 attained at $\pm\sqrt{2}$.
Takeaway: On an open interval a contintinuous function can tend to infinity. We can use that to not have to analyze it far enough away.
Solution: (Alternative) Since as $x \to \pm\infty$ we have $f(x) \sim x^4 \to \infty$ there is no global maximum. Next, For $x \geq 5$ we have $f'(x) = x(4x^2 - 8) \geq x(20 - 8) \geq 12x > 0$ and for $x \leq -5$ similarly $4x^2 - 8 \geq 12$ so $f(x) = x(4x^2 - 8) \leq 12x < 0$ (if $x \leq -5$ it's negative). Thus f is increasing if $x \geq 5$ and decreasing if $x \leq -5$. Thus it only take values smaller than $f(\pm 5) = 629$ inside the interval $[-5, 5]$ and the global minimum must be there.
Takeaway: We can make an explicit estimate on when f starts to be large (we use $|x| \geq 5$ here because in part (a) we analyzed what happens in $[-5, 5]$).

(2) Let $f(x) = |x|$. Find the absolute minimum and maximum of f on the interval $[-1, 3]$.

Solution: We have $f'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$ so there are no critical points but there is a singular point at $x = 0$. We have $f(-1) = 1$, $f(0) = 0$, $f(3) = 3$ so the maximum of f on the interval is 3,

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attained at $x = 3$, and the minimum is 0, attained at $x = 0$.

Takeaway: Singular points matter too.

(3) Find the global extrema (if any) of $f(x) = \frac{1}{x}$ on the intervals $(0, 5)$ and $[1, 4]$.

Solution: Since f is defined and strictly decreasing on these intervals, there are no extrema in the first case (and in fact $\lim_{x\to 0^+} \frac{1}{x} = \infty$) but in the second case for any $1 \leq x \leq 4$ we have $1 = f(1) \geq f(x) \geq f(4) = \frac{1}{4}$ so 1 is the global maximum and $\frac{1}{4}$ is the global minimum.

Takeaway: can sometimes solve problems without calculus at all.

2. Optimization problems

(4) A standard model for the interaction between two neutral molecules is the *Lennard-Jones Potential* $V(r) = \epsilon\left[\left(\frac{r}{R}\right)^{-12} - 2\left(\frac{r}{R}\right)^{-6}\right]$. Here r is the distance between the molecules and $R, \epsilon > 0$ are parameters.

(a) What is the range of r values that makes sense?
Solution: Distances are non-negative numbers. Since the potential blows up at $r = 0$ the we have $r \in (0, \infty)$.

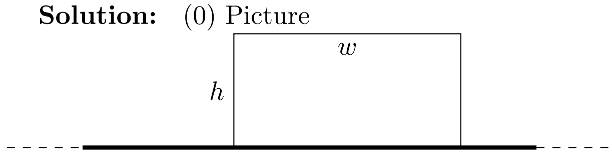
(b) Physical systems tend to settle into a state of least energy. Find the minimum of this potential.
Solution: $V'(r) = \epsilon\left[-12R^{12}r^{-13} + 12R^6r^{-7}\right] = 12\epsilon\frac{R^{12}}{r^{13}}\left[\left(\frac{r}{R}\right)^6 - 1\right]$. We then see the potential decreasing for $\frac{r}{R} \in (0, 1)$ and increasing for $\frac{r}{R} \in (1, \infty)$, so the unique minimum is at $r = R$ where $V(R) = -\epsilon$.

(c) Expand the potential to second order about the minimum.
Solution: We have $V''(r) = \epsilon\left[156R^{12}r^{-14} - 84R^6r^{-8}\right]$ so $V''(R) = 72\epsilon/R^2$. For r close to R we therefore have

$$V(r) \approx -\epsilon + \frac{36\epsilon}{R^2}(r - R)^2 = \epsilon\left[-1 + 36\left(\frac{r}{R} - 1\right)^2\right].$$

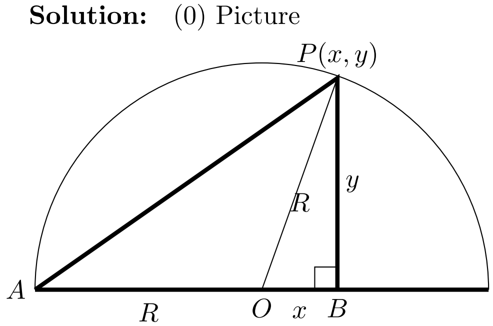
Remark: this is the same potential as for a harmonic osciilator (e.g. spring) where 72ϵ plays the role of the spring constant. One can thus compute the frequency of small oscillations about the equilibrium position (see physics textbooks).

(5) Suppose we have 100m of fencing to enlose a rectangular area against a long, straight wall. What is the largest area we can enclose?
Solution: (0) Picture



(1) Let the width of the rectangle be w, its height h, measured in metres.
(2) The total fencing used is then $2h + w$ so we must have $2h + w = 100$.
(3) The area of the rectangle is then $A = wh = h(100 - 2h)$.
(4) We must have $h \geq 0$ and since we have at most 100m of fencing we must have $h \leq 50$, so we need to optimize $A(h) = h(100 - 2h) = 100h - 2h^2$ on $[0, 50]$. We have $A'(h) = 100 - 4h$ which vanishes at $h = 25$. Since $A(0) = A(50) = 0$ (these are *degenerate rectangles*) and $A(25) = 25 \cdot 50 > 0$ $h = 25$m gives the maximum area.
(5) The maximum area we can enclose is 1250m^2.

(6) (Final 2012) The right-angled triangle ΔABP has the vertex $A = (-1, 0)$, a vertex P on the semicircle $y = \sqrt{1 - x^2}$, and another vertex B on the x-axis with the right angle at B. What is the largest possible area of such a triangle?
Solution: (0) Picture



(1) Put the coordinate system where the centre of the circle is at $(0, 0)$ and the diameter is on the x-axis. Let B be at $(x, 0)$, P at (x, y).
(2) Since P is on the circle we have $y = \sqrt{1 - x^2}$. The area of the triangle is then $A = \frac{1}{2}(\text{base}) \times$

(height) $= \frac{1}{2}(1+x)\sqrt{1-x^2}$ since the base of the triangle has length $1+x$.
(4) The function $A(x)$ is continuous on $[-1,1]$ so we can find its minimum by differentiation. By the product rule and chain rule,

$$\begin{aligned} A'(x) &= \frac{1}{2}\sqrt{1-x^2} + \frac{1}{2}(1+x)\frac{-2x}{2\sqrt{1-x^2}} \\ &= \frac{\left(\sqrt{1-x^2}\right)^2}{2\sqrt{1-x^2}} - \frac{x(1+x)}{2\sqrt{1-x^2}} = \frac{1-x^2-x-x^2}{2\sqrt{1-x^2}} \\ &= \frac{1-x-2x^2}{2\sqrt{1-x^2}}\,. \end{aligned}$$

This is defined on $(-1,1)$ and the critical points satisfy $2x^2+x-1=0$ so they are $x = \frac{-1\pm\sqrt{1+8}}{4} = \frac{-1\pm 3}{4} = -1, \frac{1}{2}$. The only critical point in the interior is then $x = \frac{1}{2}$. The area vanishes at the endpoints (the triangle becomes degenerate) and

$$A\left(\frac{1}{2}\right) = \frac{1}{2}\cdot\frac{3}{2}\cdot\sqrt{1-\frac{1}{2^2}} = \frac{3\sqrt{3}}{8}\,.$$

It follows that the largest possible area is $\frac{3\sqrt{3}}{8}$.

(7) A ferry operator is trying to optimize profits. Before each ferry trip workers spend some time loading cars after which the trip takes 1 hour. The ferry can carry up to 100 cars, each paying \$50 for the trip. Worker salaries total \$500/hour and the fuel for the trip costs \$250. The workers can load $N(t) = 100\frac{t}{t+1}$ cars in t hours.

(a) How much time should be devoted to loading to maximize profits *per trip*.
Solution: If we load cars for t hours, we have revenues (in dollars) of $R(t) = 50N(t) = 5,000\frac{t}{t+1}$ and costs $C(t) = 250 + 500(1+t) = 750 + 500t$ (the workers are paid for both loading the cars and for the trip; note the combination of fixed and variable costs). The profits are then

$$P(t) = R(t) - C(t) = 5000\frac{t}{t+1} - 500t - 750\,.$$

We note that $P(0) = -750$ (we lose money if we load no cars) and as $t\to\infty$ we have $P(t) \sim -500t \to -\infty$ (revenue is capped at 5000 – the loading time shows *diminishing returns*). Since $P(1) = 2500 - 500 - 750 > 0$ the maximum must be positive so somewhere in between, thus at a critical or singular point. We have $P(t) = 5000\frac{t+1}{t+1} - \frac{5000}{t+1} - 500t - 750 = 4250 - 5000\frac{1}{t+1} - 500t$ so $P'(t) = 5000\frac{1}{(1+t)^2} - 500$; this vanishes when $(1+t)^2 = 10$ so when $t_0 = \sqrt{10} - 1 \approx 2.16$ hours. We can also check that $P'(t) > 0$ if $t < t_0$ and $P'(t) < 0$ if $t > t_0$ so this really is a maximum.
Takeaway: If we are asked for the *time* there is no need to compute the precise profit at that time.

(b) The ferry runs continuously. How much time should be devoted to loading to maximize profits *per hour*?
Solution: If we load cars for t hours, our profits per hour are

$$\begin{aligned} Q(t) = \frac{P(t)}{t+1} &= 5000\frac{t}{(t+1)^2} - 500\frac{t}{t+1} - \frac{750}{t+1}\,. \\ &= \frac{5000t}{(t+1)^2} - \frac{250}{t+1} - 500 \\ &= \frac{4750t - 250}{(t+1)^2} - 500 \\ &= 250\frac{19t-1}{(t+1)^2} - 500 \end{aligned}$$

We note that $Q(0) = -750$ (we still lose money if we load no cars) and as $t\to\infty$ we have $Q(t) \sim \frac{-500t^2}{t^2} \to -500$ (if we load forever we just lose \$500 per hour paying the workers and make nothing on the trips). The maximum must therefore be somewhere in between, so at a critical or singular point. We have

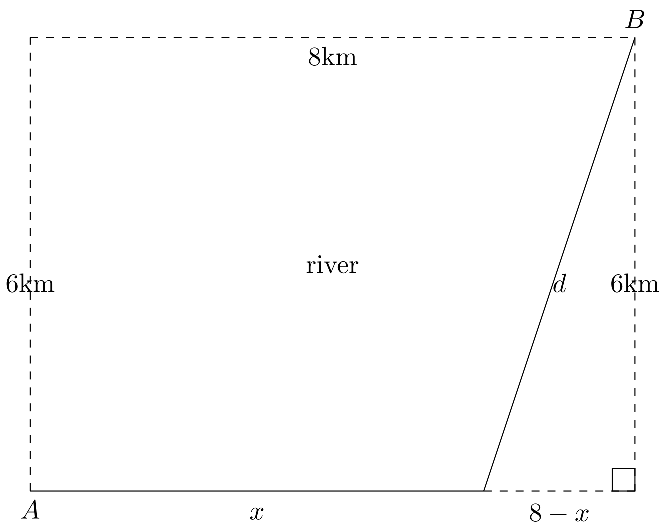
$$Q'(t) = 250\frac{19(t+1)^2 - (19t-1)2(t+1)}{(t+1)^4}$$
$$= 250\frac{19(t+1) - (38t-2)}{(t+1)^3} = 250\frac{21-19t}{(t+1)^3}\,.$$

This vanishes when $21 - 19t = 0$ so at $t_0 = \frac{21}{19} \approx 1.11$ hours; again we can verify that this is a maximum (e.g. by checking that $Q'(0) = 250 \cdot 9 > 0$ or by computing $Q(1) = 250 \cdot \frac{18}{4} - 500 = 250 \cdot 2\frac{1}{2} > 0$).

Takeaway: Changing the goal of the optimization can change the point of maximum: the time at which we maximize profits per hour is not the same as the time we maximize profits per trip.

(8) (Final 2010) A river running east-west is 6km wide. City A is located on the shore of the river; city B is located 8km to the east on the opposite bank. It costs \$40/km to build a bridge across the river, \$20/km to build a road along it. What is the cheapest way to construct a path between the cities?

Solution: (0) Picture



(1) Build a road of length x from A along the bank, then build a bridge of length d toward B.
(2) By Pythagoras, $d = \sqrt{6^2 + (8-x)^2}$.
(3) The total cost is

$$C(x) = 20x + 40\sqrt{6^2 + (8-x)^2} = 20x + 40\sqrt{6^2 + (x-8)^2}\,.$$

(4) The function $C(x)$ is defined everywhere ($6^2 + (8-x)^2 \geq 6^2 > 0$) and continuous there. We have

$$C'(x) = 20 + 40\frac{2(x-8)}{2\sqrt{6^2 + (x-8)^2}}\,.$$

This exists everywhere (the denominator is everywhere positive by the same calculation). It's enough to consider $0 \leq x \leq 8$ (no point in starting the bridge west of A or east of B). Looking for critical points we solve $C'(x) = 0$ that is:

$$\begin{aligned} 20 + 40\frac{x-8}{\sqrt{36+(x-8)^2}} &= 0 \\ 20 &= 40\frac{8-x}{\sqrt{36+(8-x)^2}} \\ \sqrt{36+(8-x)^2} &= 2(8-x) \\ 36+(8-x)^2 &= 4(8-x)^2 \\ 36 &= 3(8-x)^2 \\ (8-x) &= \sqrt{\frac{36}{3}} = \sqrt{12} = 2\sqrt{3} \end{aligned}$$

(only the positive root since $0 \le x \le 8$ forces $8 - x \ge 0$) so

$$x = 8 - 2\sqrt{3}\,.$$

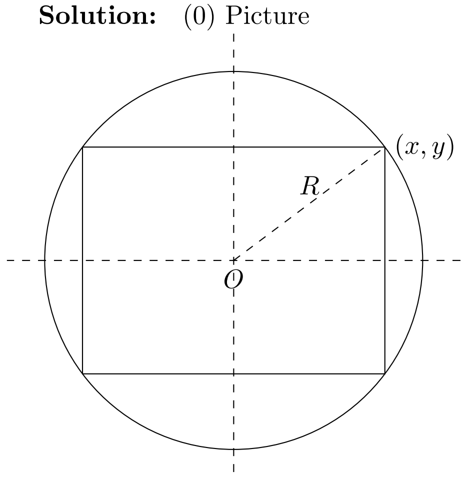
We then have $C(0) = 40\sqrt{6^2+8^2} = 40\sqrt{100} = 400$, $C(8) = 20 \cdot 8 + 40\sqrt{6^2} = 160 + 240 = 400$ and

$$\begin{aligned} C(8-2\sqrt{3}) &= 20\left(8-2\sqrt{3}\right) + 40\sqrt{6^2+(2\sqrt{3})^2} = 160 - 40\sqrt{3} + 40\sqrt{36+12} \\ &= 160 - 40\sqrt{3} + 40\sqrt{48} = 160 - 40\sqrt{3} + 40\sqrt{16\cdot 3} \\ &= 160 - 40\sqrt{3} + 40\cdot 4\sqrt{3} = 160 + 120\sqrt{3}\,. \end{aligned}$$

Now $\sqrt{3} < \sqrt{4} = 2$ so $C(8-2\sqrt{3}) = 160 + 120\sqrt{3} < 160 + 120 \cdot 2 = 400 = C(0) = C(8)$ and we conclude that $C(8-2\sqrt{3})$ is the minimum.
(5) The cheapest way to construct a bridge is construct a road of length $\left(8-2\sqrt{3}\right)$ km along the bank from A toward B, and then bridge from the end of the road to B.
(6) Sanity checks: $0 < 2\sqrt{3} < 2 \cdot 2 < 8$ so the indeed the bridge starts somewhere betwen the cities. Our answer is on the kilometer scale.

(9) (Final 2019) Among all rectangles inscribed in a given circle, which one has the largest perimeter? Prove your answer.

Solution: (0) Picture



(1) We rotate the rectangle so that it's aligned with the axes; suppose one corner is at (x, y). Call the radius of the circle R and the perimeter of the rectangle P.
(2) We have $x^2 + y^2 = R^2$ so $y = \sqrt{R^2 - x^2}$.
(3) The total perimeter is

$$P(x) = 2x + 2y + 2x + 2y = 4(x + y) = 4(x + \sqrt{R^2 - x^2})$$

where $0 \le x \le R$.
(4) The function P is defined and continuous on $[0, R]$. We have

$$P'(x) = 4\left(1 - \frac{2x}{2\sqrt{R^2 - x^2}}\right)$$

This exists everywhere except at the endpoint $x = R$ where the denominator vanishes. There are critical points where $C'(x) = 0$ that is where

$$\begin{aligned} 4\left(1 - \frac{x}{\sqrt{R^2 - x^2}}\right) &= 0 \\ 1 &= \frac{x}{\sqrt{R^2 - x^2}} \\ \sqrt{R^2 - x^2} &= x \\ R^2 - x^2 &= x^2 \\ 2x^2 &= R^2 \\ x &= \frac{1}{\sqrt{2}}R\,. \end{aligned}$$

We have $P(\frac{1}{\sqrt{2}}R) = 4\left(\frac{1}{\sqrt{2}}R + \sqrt{R^2 - \frac{1}{2}R^2}\right) = 4\left(\frac{2}{\sqrt{2}}R\right) = 4\sqrt{2}R$ while at the endpoints we have $P(0) = 4\left(0 + \sqrt{R^2}\right) = 4R$ and $P(R) = 4\left(R + \sqrt{0}\right) = 4R$. It follows that the largest perimeter occurs when $x = \frac{1}{\sqrt{2}}R$.
(5) This rectangle also has $y = \sqrt{R^2 - x^2} = \frac{1}{\sqrt{2}}R$ so the rectangle with the largest perimeter is the square.

(10) Owners of a car rental company have determined that if they charge customers d dollars per day to rent a car, the number of cars N they rent per day can be modelled by the function $N(d) = A - Bd$ where $A, B > 0$ are constants.

(a) What is the range of d for which this model makes sense?
Solution: The price should be positive, and the number of cars rented should be positive too, so we need $0 \le d \le \frac{A}{B}$.
Takeaway: Can sometimes determine the "sensible" range of the problem from the expressions.

(b) What price should they set to maximize their daily *revenue*?
Solution: The revenue for renting $N(d)$ cars at d dollars per day is $R(d) = N(d) \cdot d = (A - Bd)d = Ad - Bd^2$. This function is differentiable on $\left[0, \frac{A}{B}\right]$ were we have $R(0) = 0$ (if we don't charge rent we don't make money) and $R(\frac{A}{B}) = 0$ (if we rent no cars we don't make money). In between we have $R'(d) = A - 2Bd$ which vanishes at $d = \frac{A}{2B}$. Since f is positive in between the endpoints the maximum must be *somewhere* in the interval, and since there is only one critical point it must be the maximum, so the recommended number of cars is about $\frac{A}{2B}$. Alternative: evaluate $f\left(\frac{A}{2B}\right) = A \cdot \frac{A}{2B} - B \cdot \frac{A^2}{4B^2} = \frac{A^2}{4B} > 0$.
Takeaway: Can sometimes determine the "sensible" range of the problem from the expressions.
Solution: We have $R(d) = Ad - Bd^2 = \frac{A^2}{4B} - B\left(d - \frac{A}{2B}\right)^2$ so $R(d) \le \frac{A^2}{4B}$ for all d and we achieve equality when $d = \frac{A}{2B}$ exactly.
Takeaway: Can sometimes use algebra without any calculus.

(11) A car factory can produce up to 120 units per week. Find the (whole number) quantity q of units which maximizes *profit* if the total revenue in dollars is $R(q) = (750 - 3q)q$, the total cost in dollars is $C(q) = 10{,}000 + 148q$ (observe the combination of *fixed* and *variable* costs).
Solution: The profits are the revenues minus the costs, so we need to maximize

$$\begin{aligned} P(q) &= R(q) - C(q) \\ &= 750q - 3q^2 - 148q - 10{,}000 \\ &= 602q - 3q^2 - 10{,}000\,. \end{aligned}$$

This function is differentiable on the closed interval $[0, 120]$ where $P'(q) = 602 - 6q$ achieving its maximum on $q_0 = \frac{602}{6} = 100\frac{1}{3}$. This is not an integer, so guess we need to round q_0 either right or left – but we need further analysis to decide which way. We look at the shape of the graph: $P'(q) > 0$ if $q < q_0$ and $P'(q) < 0$ if $q > q_0$ so the function is increasing on $\left[0, 100\frac{1}{3}\right]$ and decreasing on $\left[100\frac{1}{3}, 120\right]$. In particular the largest value on $[0, 100]$ is at 100 and the largest value on $[101, 120]$ is at 101. Using a calculator we find $P(100) = 22,200$ and $P(101) = 22,199$ so the best choice is to make 100 cars per week.
Takeaway: Can use the shape of the graph to round solutions (but watch out).