Math 100:V02 – SOLUTIONS TO WORKSHEET 13
QUALITATIVE ASPECTS OF DIFFERENTIAL EQUATIONS

1. Fixed points

(1) (Review)

(a) For which value of ω is $y = A\sin(\omega t) + B\cos(\omega t)$ a solution of $\ddot{y} = -9y$?
Solution: $\frac{d}{dt}(\sin(\omega t)) = \omega\cos\omega t$ and $\frac{d}{dt}(\cos(\omega t)) = -\omega\sin\omega t$ so

$$\frac{d^2}{dt^2}\left(A\sin(\omega t) + B\cos(\omega t)\right) = -\omega^2\left(A\sin(\omega t) + B\cos(\omega t)\right)$$

and we get a solution iff $\omega^2 = 9$ that is if $\omega = \pm 3$. Since $\sin(-3t) = -\sin(3t)$and $\cos(-3t) = \cos(3t)$ might as well just take $\omega = 3$.

(b) Can you find the general solution of $\ddot{y} = 9y$?
Solution: Since $\frac{d}{dt}(e^{rt}) = re^{rt}$ we have $\frac{d^2}{dt^2}(e^{rt}) = r^2 e^{rt}$ so we get a solution if $r^2 = 9$ so if $r = \pm 3$ so $Ae^{3t} + Be^{-3t}$ is a solution.

(2) (Steady states = fixed points = equilibria)

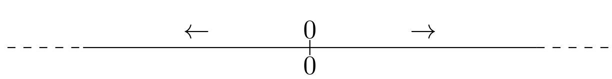
(a) Consider the Malthusian growth equation $\dot{y} = ry$, $r > 0$. Can you find a value a so that $y(t) \equiv a$ is a solution?
Solution: The constant function $y \equiv a$ has derivative zero, so we need $0 = ar$ so $a = 0$.

(b) What about the *logistic growth* model $\dot{y} = ry(1-y)$ *with* $r > 0$*?*
Solution: The constant function $y \equiv a$ has derivative zero, so we need $0 = ra(1-a)$ so either $a = 0$ or $a = 1$.

(c) What about $\dot{y} = y^3 - 5y^2 + 6y$?
Solution: We have $\dot{y} = y(y-2)(y-3)$ so the fixed points are at $0, 2, 3$.

(3) (Phase line)

(a) In the model $\dot{y} = ry$ with $r > 0$, what is the sign of $\dot{y}$ when $y < 0$? when $y > 0$? What would the solution look like if we started with y_0 in each range? Draw the phase line.
Solution: When $y > 0$ we have $\dot{y} > 0$ so y will keep increasing forever (tending to infinity since the solution is exponential)). When $y < 0$ we have $\dot{y} < 0$ and y will descrease forever (tending to negative infinity).



(b) What about the *logistic growth* model $\dot{y} = ry(1-y)$*?*
Solution: When $y < 0$ we have $\dot{y} < 0$ so y will decrease (to infinity since once $y < 1$ we have $\dot{y} < 2ry$. When $0 < y < 1$ we have $\dot{y} > 0$ so y will increase, tending toward $y = 1$. When

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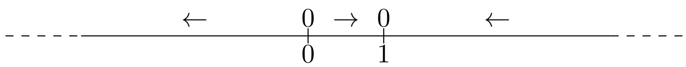
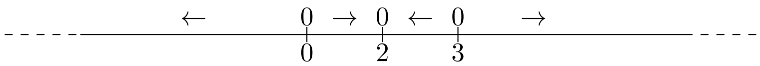


(c) What about $\dot{y} = y^3 - 5y^2 + 6y$?

Solution: The fixed points are $0, 2, 3$. For $y < 0$ we have $y' < 0$, for $0 < y < 2$ we have $\dot{y} > 0$, for $2 < y < 3$ we have $\dot{y} < 0$, for $y > 3$ we have $\dot{y} > 0$.



(4) Analyze $\frac{dy}{dt} = -2y^3 + 9y^2 - 12y$

2. Taylor expansion

(5) Consider the equation $\dot{y} = -\sin y$, $y(0) = \frac{\pi}{2}$.

(a) What is $\dot{y}(0)$?

Solution: We have $\dot{y}(0) = -\sin(y(0)) = -\sin\left(\frac{\pi}{2}\right) = -1$.

(b) What is $\ddot{y}(0)$?

Solution: We have $\frac{d}{dt}\dot{y} = -(\cos y)\,\dot{y} = \cos y \sin y$ so $\ddot{y}(0) = 0$.

(c) What is the third-order Taylor expansion of y about $t = 0$?

Solution: We have $y^{(3)}(t) = -\ddot{y}\cos y + (\dot{y})^2 \sin y$. At $t = 0$ we have $\ddot{y}(0) = 0$ and $\dot{y}(0) = -1$ so $y^{(3)}(0) = 1$. The expansion is therefore

$$y(t) \approx \frac{\pi}{2} - t + \frac{t^3}{6}\,.$$

(d) What are the fixed points of this equation? Are the stable or unstable?

Solution: $\sin y = 0$ whenever $y = \pi k$ for some $k \in \mathbb{Z}$. For $(2m-1)\pi < y < (2m)\pi$ we have $\sin y < 0$ and hence $\dot{y} > 0$ and for $(2m)\pi < y < (2m+1)\pi$ we have $\sin y > 0$ so $\dot{y} < 0$. It follows that the points $2m\pi$ are attractive and the point $(2m+1)\pi$ are repulsive.