Math 100:V02 – SOLUTIONS TO WORKSHEET 4
CALCULATING DERIVATIVES

1. Definition of the derivative

Definition. $f(a+h) \approx f(a) + f'(a)h$ (or $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$)

(1) Find $f'(a)$ if
  (a) $f(x) = x^2$, $a = 3$.
  Solution: $(3+h)^2 = 3 + 6h + h^2 \approx 3 + 6h$ to first order so $f'(3) = 6$.
  Solution: $\lim_{h\to 0} \frac{(3+h)^2-(3)^2}{h} = \lim_{h\to 0} \frac{9+6h+h^2-9}{h} = \lim_{h\to 0} \frac{6h+h^2}{h} = \lim_{h\to 0}(6+h) = 6$.
  (b) $f(x) = \frac{1}{x}$, any a.
  Solution: $\frac{1}{a+h} - \frac{1}{a} = \frac{a}{a(a+h)} - \frac{a+h}{a(a+h)} = -\frac{h}{a(a+h)} \sim -\frac{h}{a^2}$ so $f'(a) = -\frac{1}{a^2}$.
  Solution: $\lim_{h\to 0} \frac{\frac{1}{a+h}-\frac{1}{a}}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{a-(a+h)}{a(a+h)}\right) = \lim_{h\to 0} \frac{-h}{h\cdot a(a+h)} = -\lim_{h\to 0} \frac{1}{a(a+h)} = -\frac{1}{a^2}$
  (a) $f(x) = x^3 - 2x$, any a (you may use $(a+h)^3 = a^3 + 3a^2h + 3ah^2 + h^3$).
  Solution: We have

$$\begin{aligned}(a+h)^3 - 2(a+h) &= a^3 + 3a^2h + 3ah^2 + h^3 - 2a - 2h \\ &= (a^3 - 2a) + (3a^2 - 2)h + 3ah^2 + h^3 \\ &\approx (a^3 - 2a) + (3a^2 - 2)h\end{aligned}$$

  to first order in h so the derivative is $3a^2 - 2$.
  Solution: We have

$$\begin{aligned}\frac{(a+h)^3 - 2(a+h) - a^3 + 2a}{h} &= \frac{a^3 + 3a^2h + 3ah^2 + h^3 - 2a - 2h - a^3 + 2a}{h} \\ &= \frac{3a^2h + 3ah^2 + h^3 - 2h}{h} \\ &= 3a^2 - 2 + 3ah + h^2 \xrightarrow[h\to 0]{} 3a^2 - 2\,.\end{aligned}$$

(2) Express the limits as derivatives: $\lim_{h\to 0} \frac{\cos(5+h)-\cos 5}{h}$, $\lim_{x\to 0} \frac{\sin x}{x}$
  Solution: These are the derivative of $f(x) = \cos x$ at the point $a = 5$ and of $g(x) = \sin x$ at the point $a = 0$.

(3) (Final, 2015, variant – gluing derivatives) Is the function

$$f(x) = \begin{cases} x^2 & x \le 0 \\ x^2 \cos\frac{1}{x} & x > 0 \end{cases}$$

differentiable at $x = 0$?
  Solution: We have $f(0) = 0$, so we'd have $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x} = \lim_{x\to 0} \frac{f(x)}{x}$ provided the limit exists, and since we have different expresions for $f(x)$ on both sides of 0 we compute the limit as two one-sided limits. On the left we have

$$\lim_{x\to 0^-} \frac{f(x)}{x} = \lim_{x\to 0^-} \frac{x^2}{x} = \lim_{x\to 0^-} x = 0\,.$$

Alternatively, we could recognize the limit as giving the derivative of $f(x) = x^2$ at $x = 0$. Using differentiation rules (to be covered later in the course) we know that $\left[\frac{d}{dx}x^2\right]_{x=0} = [2x]_{x=0} = 0$ and

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it would again follow that $\lim_{x\to 0^-}\frac{f(x)}{x}=0$.
On the right we have

$$\lim_{x\to 0^+}\frac{f(x)}{x}=\lim_{x\to 0^+}\frac{x^2\cos\frac{1}{x}}{x}=\lim_{x\to 0^+}x\cos\left(\frac{1}{x}\right)=0$$

since $x\to 0$ while $\cos\left(\frac{1}{x}\right)$ is bounded. Thus the function is differentiable and its derivative is zero.

2. The tangent line

(4) (Final, 2015) Find the equation of the line tangent to the function $f(x)=\sqrt{x}$ at $(4,2)$.
Solution: $f'(x)=\frac{1}{2\sqrt{x}}$, so the slope of the line is $f'(4)=\frac{1}{4}$, and the equation for the line line itself is $y-2=\frac{1}{4}(x-4)$ or $y=\frac{1}{4}(x-4)+2$ or $y=\frac{1}{4}x+1$.

(5) (Final 2015) The line $y=4x+2$ is tangent at $x=1$ to which function: x^3+2x^2+3x, x^2+3x+2, $2\sqrt{x+3}+2$, x^3+x^2-x, x^3+x+2, none of the above?
Solution: The line has slope 4 and meets the curve at $(1,6)$. The last two functions don't evaluate to 6 at 1.We differentiate the first three.

$$\frac{d}{dx}|_{x=1}\left(x^3+2x^2+3x\right)=\left(3x^2+4x+3\right)|_{x=1}=10$$
$$\frac{d}{dx}|_{x=1}\left(x^2+3x+2\right)=(2x+3)|_{x=1}=5$$
$$\frac{d}{dx}|_{x=1}\left(2\sqrt{x+3}+2\right)=\left(\frac{2}{2\sqrt{x+3}}\right)|_{x=1}=\frac{1}{2}.$$

The answer is "none of the above".

(6) Find the lines of slope 3 tangent to the curve $y=x^3+4x^2-8x+3$.
Solution: $\frac{dy}{dx}=3x^2+8x-8$, so the line tangent at (x,y) has slope 3 iff $3x^2+8x-8=3$, that is iff $3(x^2-1)+8(x-1)=0$. We can factor this as $(x-1)(3x+11)=0$ so the x-coordinates of the points of tangency are $1,-\frac{11}{3}$ and the lines are:

$$y=3(x-1)$$
$$y=3(x+\frac{11}{3})+\left(\left(\frac{11}{3}\right)^3+4\left(\frac{11}{3}\right)^2-8\left(\frac{11}{3}\right)+3\right).$$

(7) The line $y=5x+B$ is tangent to the curve $y=x^3+2x$. What is B?
Solution: At the point (x,y) the curve has slope $\frac{dy}{dx}=3x^2+2$, so the curve has slope 5 at the points where $x=\pm 1$, that is the points $(-1,-3)$ and $(1,3)$. The line needs to meet the curve at the point, so there are two solutions:

$$y=5x+2 \qquad \text{(tangent at } (-1,-3))$$
$$y=5x-2 \qquad \text{(tangent at } (1,3))$$

3. Linear approximation

Definition. $f(a+h)\approx f(a)+f'(a)h$

(8) Estimate
(a) $\star$ $\sqrt{1.2}$
Solution: Let $f(x)=\sqrt{x}$ so that $f'(x)=\frac{1}{2\sqrt{x}}$. Then $f(1)=1$ and $f'(1)=\frac{1}{2}$ so $f(1.2)\approx f(1)+f'(1)\cdot 0.2=1+\frac{1}{2}\cdot 0.2=1.1$.
Better: $f(1.21)=1.1$ and $f'(1.21)=\frac{1}{2.2}$ so $f(1.2)=f(1.21-0.01)\approx 1.1-0.01\cdot\frac{1}{2.2}\approx 1.09545$.
(b) $\star$ (Final, 2015) $\sqrt{8}$
Solution: Using the same f we have $f(9-1)\approx f(9)+f'(9)\cdot(-1)=3-\frac{1}{6}=2\frac{5}{6}$.

(c) $\star$ (Final, 2016) $(26)^{1/3}$
Solution: Let $f(x) = x^{1/3}$ so that $f'(x) = \frac{1}{3}x^{-2/3}$. Then $f(27) = 3$ and $f'(27) = \frac{1}{3 \cdot 27^{2/3}} = \frac{1}{27}$ so

$$f(26) = f(27-1) \approx f(27) + (-1) \cdot f'(27) = 3 - \frac{1}{27} = 2\frac{26}{27}\,.$$

4. Arithmetic of derivatives

(2) Differentiate

(a) $\star$ $f(x) = 6x^{\pi} + 2x^{e} - x^{7/2}$
Solution: This is a linear combination of power laws so $f'(x) = 6\pi x^{\pi-1} + 2ex^{e-1} - \frac{7}{2}x^{5/2}$.

(b) $\star$ (Final, 2016) $g(x) = x^2 e^x$ (and then also $x^a e^x$)
Solution: Applying the product rule we get $\frac{dg}{dx} = \frac{d\left(x^2\right)}{dx} \cdot e^x + x^2 \cdot \frac{d(e^x)}{dx} = (2x + x^2)e^x = x(x+2)e^x$, and in general

$$\frac{d}{dx}(x^a e^x) = ax^{a-1}e^x + x^a e^x = x^{a-1}(x+a)e^x\,.$$

(c) $\star$ (Final, 2016) $h(x) = \frac{x^2+3}{2x-1}$
Solution: Applying the quotient rule the derivative is $\frac{2x \cdot (2x-1) - (x^2+3) \cdot 2}{(2x-1)^2} = \frac{4x^2-2x-2x^2-6}{(2x-1)^2} = 2\frac{x^2-x-3}{(2x-1)^2}$.

(d) $\star$ $\frac{x^2+A}{\sqrt{x}}$
Solution: We write the function as $x^{3/2} + Ax^{-1/2}$ so its derivative is $\frac{3}{2}x^{1/2} - \frac{A}{2}x^{-3/2}$.

(3) $\star$ Let $f(x) = \frac{x}{\sqrt{x}+A}$. Given that $f'(4) = \frac{3}{16}$, give a quadratic equation for A.
Solution: $f'(x) = \frac{1 \cdot \left(\sqrt{x}+A\right) - x\left(\frac{1}{2}x^{-1/2}\right)}{\left(\sqrt{x}+A\right)^2} = \frac{\sqrt{x}+A-\frac{1}{2}\sqrt{x}}{\left(\sqrt{x}+A\right)^2} = \frac{\frac{1}{2}\sqrt{x}+A}{\left(\sqrt{x}+A\right)^2}$. Plugging in $x = 4$ we have

$$\frac{3}{16} = f'(4) = \frac{1+A}{(2+A)^2}$$

so we have

$$3(2+A)^2 = 16(1+A)$$

that is

$$3A^2 + 12A + 12 = 16 + 16A$$

that is

$$3A^2 - 4A - 4 = 0\,.$$

In fact this gives $A = -\frac{2}{3}, 2$.

(4) Suppose that $f(1) = 1$, $g(1) = 2$, $f'(1) = 3$, $g'(1) = 4$.

(a) $\star$ What are the linear approximations to f and g at $x = 1$? Use them to find the linear approximation to fg at $x = 1$.
Solution: We have

$$\begin{aligned} f(x) &\approx f(1) + f'(1)(x-1) = 1 + 3(x-1) \\ g(x) &\approx g(1) + g'(1)(x-1) = 2 + 4(x-1) \end{aligned}$$

multiplying them we have

$$\begin{aligned} (fg)(x) &\approx (1 + 3(x-1))\,(2 + 4(x-1)) \\ &= 2 + 1 \cdot 4(x-1) + 2 \cdot 3(x-1) + 12(x-1)^2 \\ &\approx 2 + 10(x-1) \end{aligned}$$

to first order.

(b) $\star$ Find $(fg)'(1)$ and $\left(\frac{f}{g}\right)'(1)$.

Solution: $(fg)'(1) = f'(1)g(1) + f(1)g'(1) = 3 \cdot 2 + 1 \cdot 4 = 10$.

$$\left(\frac{f}{g}\right)'(1) = \frac{f'(1)g(1) - f(1)g'(1)}{(g(1))^2} = \frac{3 \cdot 2 - 1 \cdot 4}{2^2} = \frac{1}{2}\,.$$

(5) Evaluate

(a) $\star$ $(x \cdot x)'$ and $(x') \cdot (x')$. What did we learn?

Solution: $(x \cdot x)' = (x^2)' = 2x$ while $(x') \cdot (x') = 1 \cdot 1 = 1$ – the "rule" $(fg)' = f'g'$ is **wrong**.

(b) $\star$ $\left(\frac{x}{x}\right)'$ and $\frac{(x')}{(x')}$. What did we learn?

Solution: $\left(\frac{x}{x}\right)' = (1)' = 0$ while $\frac{(x')}{(x')} = \frac{1}{1} = 1$ – the "rule" $\left(\frac{f}{g}\right)' = \frac{f'}{g'}$ is **wrong**.