Math 100, lecture 24, 9/4/2024

Last time: (1) Graph of $z = f(x,y)$, $w = g(x,y,z)$, ...

(2) Partial derivatives $\frac{\partial f}{\partial x}(x,y)$, $\frac{\partial g}{\partial y}(x,y,z)$

(3) Directional change if $\frac{\partial f}{\partial x} < 0$, $\frac{\partial f}{\partial x} > 0$

$\Rightarrow$ Critical points: points where all partial derivatives vanish

Eg. for $f(x,y)$ need
$$\begin{cases} \frac{\partial f}{\partial x}(x,y) = 0 \\ \frac{\partial f}{\partial y}(x,y) = 0 \end{cases}$$

Math 100:V02 – WORKSHEET 18
MULTIVARIABLE OPTIMIZATION

1. Critical points; multivariable optimization

(1) ⋆How many critical points does $f(x,y) = x^2 - x^4 + y^2$ have?

$\frac{\partial f}{\partial x} = 2x - 4x^3 = 2x(1-2x^2)$; $\frac{\partial f}{\partial y} = 2y$

so critical pts satisfy $\begin{cases} 2x(1-2x^2) = 0 \\ 2y = 0 \end{cases}$

$\Rightarrow x \in \{0, \pm\frac{1}{\sqrt{2}}\}$, $y = 0$,

$\Rightarrow$ pts are $(0,0)$, $(\frac{1}{\sqrt{2}}, 0)$, $(-\frac{1}{\sqrt{2}}, 0)$

pts are $(0,0,0)$, $(\pm\frac{1}{2}, 0, \frac{1}{4})$

Date: 9/4/2023, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(2) $\star$Find the critical points of $f(x,y) = x^2 - x^4 + xy + y^2$.

$\frac{\partial f}{\partial x} = 2x - 4x^3 + y$; $\frac{\partial f}{\partial y} = x + 2y$

Need to solve $\begin{cases} 2x - 4x^3 + y = 0 \\ x + 2y = 0 \end{cases}$

Eliminate y (= solve one equation for y, substitute in other)

Here $y = -\frac{1}{2}x$ so set $2x - 4x^3 - \frac{1}{2}x = 0$

so $3x - 4x^3 = 0$ so $4x\left(\frac{3}{4} - x^2\right) = 0$

so $x \in \left\{0, \frac{\pm\sqrt{3}}{2}\right\}$ At critical pts $y = -\frac{1}{2}x$

so critical pts are $(0,0), \left(\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{4}\right), \left(-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{4}\right)$

(3) (MATH 105 Final, 2013) $\star$ Find the critical points of $f(x,y) = xye^{-2x-y}$.

$f_x = y\left(e^{-2x-y} + xe^{-2x-y}\cdot(-2)\right) = y(1-2x)e^{-2x-y}$

$f_y = x\left(e^{-2x-y} + ye^{-2x-y}\cdot(-1)\right) = x(1-y)e^{-2x-y}$

Critical pts are $\begin{cases} y(1-2x)e^{-2x-y} = 0 \\ x(1-y)e^{-2x-y} = 0 \end{cases}$

From 1st equation either $y=0$ or $x = \frac{1}{2}$.

If $y=0$, 2nd equation reads: $x\cdot(1-0) = 0$ so $x=0$.

If $x = \frac{1}{2}$, " " " $\frac{1}{2}(1-y) = 0$ so $y=1$

so critical pts are $(0,0), \left(\frac{1}{2}, 1\right)$.

Summary

To find critical pts : (1) find f_x, f_y

(2) solve system $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$

solutions are locations (x,y)

or points $(x, y, f(x,y))$ on graph

Optimization

Fact: If $f(x,y)$ is defined on closed, bounded domain R, then max/min occur at one of:

(1) critical pts
(2) singular pts
(3) boundary of R

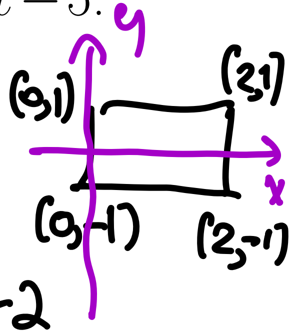
As usual, if f is cts min & max exist.
If domain isn't closed, or isn't bdd need to make special arguments

(5) $\star$ Find the critical points of $(7x + 3y + 2y^2)e^{-x-y}$.

2. Optimization

(6) $\star\star$Find the minimum of $f(x, y) = 2x^2 + 3y^2 - 4x - 5$:

(a) on the rectangle $0 \le x \le 2$, $-1 \le y \le 1$.



$\frac{\partial f}{\partial x} = 4x - 4$; $\frac{\partial f}{\partial y} = 6y$ so crit pt $(1, 0, -7)$

On bottom boundary segment have $f(x,-1) = 2x^2 - 4x - 2$

$\frac{d}{dx}(2x^2 - 4x - 2) = 4x - 4$ so crit over $x = 1$. $f(1,-1) = -4$, $f(0,-1) = -2$

On top, $f(x,1) = 2x^2 - 4x - 2 \geq -4$ $\quad f(2,-1) = -2$.

(b) on the rectangle $2 \le x \le 3$, $-1 \le y \le 1$.

on left $f(0,y) = 3y^2 - 5 \geq -5$

on right $f(2,y) = 3y^2 - 5 \geq -5$

so min is -7 attained at $(1,0)$

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on rectangle $[2,3] \times [-1,1]$ no critical pts