Math 100, Lecture 15, 5/3/2024

Last time: Ordinary Differential Equations

DE: (1) equation; (2) unknown function; (3) involves derivatives of the unknown
for

Solution: function that satisfies the equation

family of solutions: solution that depends on a parameter

particular solution: one member of family.

- Can find a particular solution satisfying a condition

(find $y(t) = A + 3 \cdot t^2$ s.t. $y(3) = 5$)

- Ansatz: (family) of guesses for solutions

Today: qualitative questions.

concentrate on autonomous equation: no explicit dependence on independent variable.

Math 100:V02 – WORKSHEET 13
QUALITATIVE ASPECTS OF DIFFERENTIAL EQUATIONS

1. Fixed points

(1) (Review)

(a) For which value of ω is $y = A\sin(\omega t) + B\cos(\omega t)$ a solution of $\ddot{y} = -9y$?

If $y = A\sin(\omega t) + B\cos(\omega t)$

then $\dot{y} = A\cos(\omega t)\cdot\omega - B\sin(\omega t)\cdot\omega$

and $\ddot{y} = -A\sin(\omega t)\cdot\omega^2 - B\cos(\omega t)\omega^2 = -\omega^2(A\sin(\omega t) + B\cos(\omega t))$

$= -\omega^2 y$

so y is a solution if $\omega^2 = 9$ so $\omega = 3$, $y = A\sin(3t) + B\cos(3t)$

(what about $\omega = -3$? $A\sin(-3t) + B\cos(-3t) = (-A)\sin(3t) + B\cos(3t)$)

(b) Can you find the general solution of $\ddot{y} = 9y$?

Date: 5/3/2024, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(2) (Steady states = fixed points = equilibria)

(a) Consider the Malthusian growth equation $\dot{y} = ry$, $r > 0$. Can you find a value a so that $y(t) \equiv a$ is a solution?

Need $0 = ra$ $\left(\frac{d(a)}{dt} \equiv 0\right)$ so $a = 0$

[Aside: we know general solution Ce^{rt}
steady state has $C = 0$]

(b) What about the *logistic growth* model $\dot{y} = ry(1 - y)$ *with* $r > 0$*?*

need $0 = r \cdot a(1-a)$ so either $a = 1$ or $a = 0$

$y^3 - 5y^2 + 6y$

(c) What about $\dot{y} = \cancel{y^3 - 5y + 6}$?

$y = a$ is a fixed point if $a^3 - 5a^2 + 6a = 0$

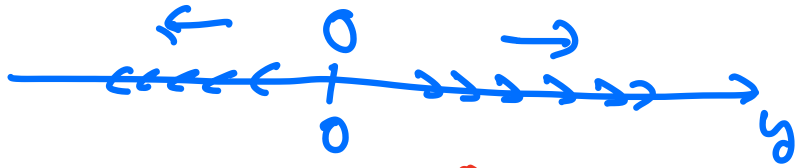
$a(a^2 - 5a + 6) = a(a-2)(a-3)$

so fixed points at $0, 2, 3$

(3) (Phase line)

(a) In the model $\dot{y} = ry$ with $r > 0$, what is the sign of $\dot{y}$ when $y < 0$? when $y > 0$? What would the solution look like if we started with y_0 in each range? Draw the phase line.

if $y<0$, $ry<0$ so $\dot{y}<0$, if $y>0$, $ry>0$, $\dot{y}>0$



note: here steady state is unstable

phase line.

Has (1) fixed pts
(2) behaviour between them

(b) What about the *logistic growth* model $\dot{y} = ry(1-y)$?

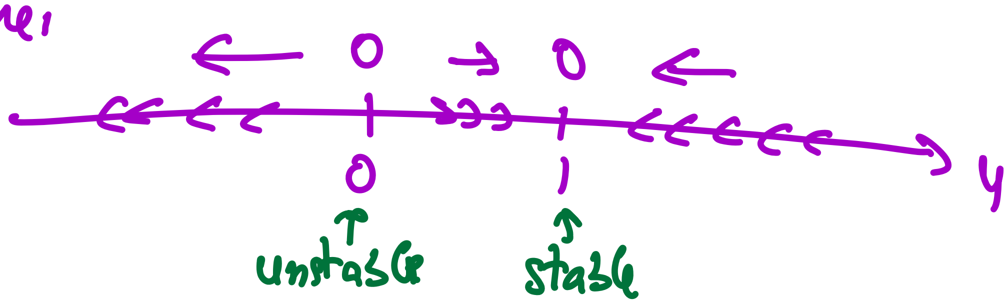
Fixed points at 0, 1.

if $y<0$, $1-y>1>0$, so $\dot{y} = ry(1-y) < 0$

if $0<y<1$, $1-y>0$, so $\dot{y} = ry(1-y) > 0$

if $y>1>0$, $1-y<0$, so $\dot{y} = ry(1-y) < 0$

phase line:



(c) What about $\dot{y} = y^3 - 5y^2 + 6y$

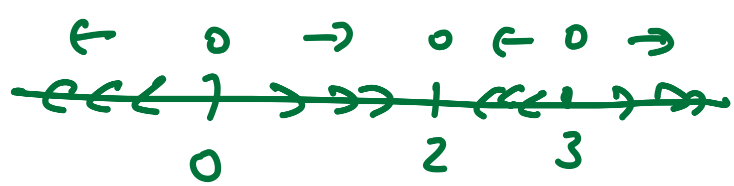
Have $\dot{y} = y(y-2)(y-3)$ fixed pts $y = 0, 2, 3$

if $y<0$ $\quad \dot{y} = (-)(-)(-) < 0$

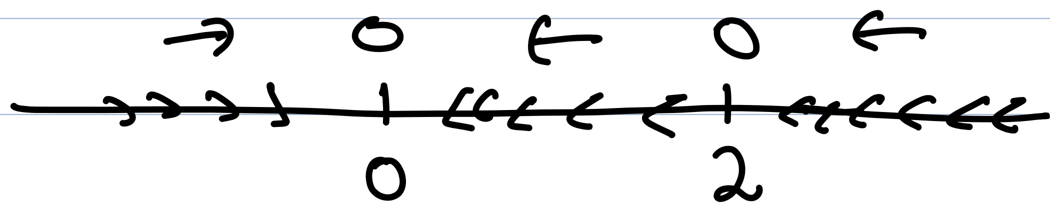
if $0<y<2$ $\quad \dot{y} = (+)(-)(-) > 0$

if $2<y<3$ $\quad \dot{y} = (+)(+)(-) < 0$

if $3<y$ $\quad \dot{y} = (+)(+)(+) > 0$



Compare with $\dot{y} = -y(y-2)^2$



plot $y(t)$