Math 100, lecture 10, 8/2/2024

Last time: curve sketching

Today: Higher-order approximation
= Taylor expansion

Linear approximation: $f(x) \approx f(a) + f'(a)(x-a)$
or $f(x+h) \approx f(x) + f'(x)h$

function $f(x)$, linear approx $f(a) + f'(a)(x-a)$
(0) have same value at a $(f(a))$
(1) have same derivative at a $(f'(a))$

Math 100:V02 – WORKSHEET 10
TAYLOR EXPANSION

1. Taylor expansion

(1) (Review) Use linear approximations to estimate:

(a) $\log\frac{4}{3}$ and $\log\frac{2}{3}$. Combine the two for an estimate of $\log 2$.

At $a=1$, $\log 1 = 0$, $(\log x)'|_{x=1} = \left[\frac{1}{x}\right]_{x=1} = 1$ so $\log x \approx 0 + 1\cdot(x-1)$
$= (x-1)$

so $\log\frac{4}{3} \approx \frac{1}{3}$, $\log\frac{2}{3} \approx -\frac{1}{3}$

so $\log 2 = \log\frac{4/3}{2/3} = \log\frac{4}{3} - \log\frac{2}{3} \approx \frac{1}{3} - \left(-\frac{1}{3}\right) \approx \frac{2}{3}$.

(b) $\sin 0.1$ and $\cos 0.1$.

$\cos 0 = 1$, $(\cos\theta)'|_{\theta=0} = (-\sin\theta)|_{\theta=0} = 0$

so $\cos\theta \approx 1 + 0\cdot\theta = 1$ to 1st order in θ

$\cos 0.1 \approx 1$ to 1st order

(linear term could be zero)

Date: 8/2/2024, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(2) Let $f(x) = e^x$

(a) Find $f(0), f'(0), f^{(2)}(0), \cdots$

(b) Find a polynomial $T_0(x)$ such that $T_0(0) = f(0)$.

(c) Find a polynomial $T_1(x)$ such that $T_1(0) = f(0)$ and $T_1'(0) = f'(0)$.

(d) Find a polynomial $T_2(x)$ such that $T_2(0) = f(0)$, $T_2'(0) = f'(0)$ and $T_2^{(2)}(0) = f^{(2)}(0)$.

(e) Find a polynomial $T_3(x)$ such that $T_3^{(k)}(0) = f^{(k)}(0)$ for $0 \le k \le 3$.

(a) $f(0)=1$, all k $f^{(k)}(x) = e^x$. so $f^{(k)}(0)=1$

(b) $T_0(x) = 1$ works

(c) $T_1(x) = 1 + 1 \cdot x = 1+x$ works (linear approx)

try $T_1(x) = a + bx$ want $T_1(0) = f(0) = 1$ so take $a=1$

(if $a=0$, $T_1(0) = b \cdot 0 = 0$, so not good choice)

want $T_1'(0) = f'(0) = 1$. So take $b=1$, get $1+x$.

(d) try $a + bx + cx^2$

want $T_2(0) = 1$, so need $a=1$

want $T_2'(0) = 1$, so $b + 2c \cdot 0 = 1$ so $b=1$

want $T_2''(0) = 1$, so $2c = 1$, so $c = \frac{1}{2}$

take $1 + x + \frac{1}{2}x^2$

$(x^2)'' = 2 \cdot 1 = 2$

(e) try $1 + x + \frac{1}{2}x^2 + dx^3$ 3rd derivative is $6d$

$(x^3)''' = 3 \cdot 2 \cdot 1 = 6$

so take $d = \frac{1}{6}$, $T_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$

Conclusion: to match kth derivative we added a correction of order k: a term $c_k x^k$.

choose c_k s.t. kth derivative of $(c_k x^k)$ is same as $f^{(k)}(0)$, so $c_k = \frac{1}{1\cdot 2\cdot 3\cdots (k)} f^{(k)}(0)$

In general if we are expanding about $x=a$

Use $\boxed{c_k = \frac{1}{k!} f^{(k)}(a)}$ $\quad k! = 1\cdot 2\cdot 3 \cdots \cdot k$

"k factorial".

$0! = 1$

Degree n Taylor polynomial / Taylor expansion of f about a is the polynomial

$$T_n(x) = f(a) + \frac{f'(a)}{1}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

(also say: $f(x) \simeq T_n(x)$ "to nth order at a")

Example: $f(x) = e^x$, $a=0$, $f^{(k)}(0) = 1$ all k,

$$e^x \simeq 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots + \frac{1}{n!}x^n.$$

Let $c_k = \frac{f^{(k)}(a)}{k!}$. The nth order Taylor expansion of $f(x)$ about $x = a$ is the polynomial
$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n$$

(4) Find the 4th order MacLaurin expansion of $\frac{1}{1-x}$ (=Taylor expansion about $x = 0$)

chain rule

Let $f(x) = \frac{1}{1-x} = (1-x)^{-1}$ so $f'(x) = -(1-x)^{-2} \cdot (-1) = (1-x)^{-2}$

$f''(x) = 2 \cdot 1 (1-x)^{-3}$, $f'''(x) = 3 \cdot 2 \cdot 1 \, (1-x)^{-4}$, $f^{(4)}(x) = 4 \cdot 3 \cdot 2 \cdot 1 \, (1-x)^{-5}$

So $f(0) = 1$, $f^{(1)}(0) = 1$, $f^{(2)}(0) = 2 \cdot 1$, $f^{(3)}(0) = 3 \cdot 2 \cdot 1$,

$f^{(4)}(0) = 4 \cdot 3 \cdot 2 \cdot 1$

So $T_4(x) = 1 + 1x + \frac{2 \cdot 1}{2 \cdot 1} x^2 + \frac{3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} x^3 + \frac{4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} x^4$

$2!$ $\quad 3!$ $\quad 4!$

$= 1 + x + x^2 + x^3 + x^4$

Know: To nth order in u

$$\boxed{\begin{aligned} e^u &\approx 1 + u + \tfrac{1}{2} u^2 + \tfrac{1}{3!} u^3 + \cdots + \tfrac{1}{n!} u^n \\ \frac{1}{1-u} &\approx 1 + u + u^2 + u^3 + \cdots + u^n \end{aligned}}$$

Warning: common error

Evaluate derivatives at $\underline{\underline{a}}$ (centre of the expansion) not x.

$\underline{\text{Eg}}$: Linear approx to e^x is $1+x$

not $\cancel{e^x + e^x \cdot x}$

Quadratic approx to $\frac{1}{1-x}$ is $1+x+x^2$

not $1 + \frac{1}{(1-x)^2} \cdot x + \frac{2}{2\cdot(1-x)^3} x^2 + \dots$

(5) $\star\star$ Find the nth order expansion of $\cos x$, and approximate $\cos 0.1$ using a 3rd order expansion

set

$g(x) = \cos x$; $g^{(1)}(x) = -\sin x$; $g^{(2)}(x) = -\cos x$; $g^{(3)}(x) = \sin x$

$g^{(4)}(x) = \cos x$; $g^{(5)}(x) = -\sin x$; -- (repeat)

$g(0) = 1$, $g^{(1)}(0) = 0$, $g^{(2)}(0) = -1$, $g^{(3)}(0) = 0$

$g^{(4)}(0) = 1$, $g^{(5)}(0) = 0$, $g^{(6)}(0) = -1$, $g^{(7)}(0) = 0$ (repeat)

so $\boxed{\cos x \approx 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 \cdots}$

$\boxed{\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 + \cdots}$

(6) (Final, 2015) $\star$ Let $T_3(x) = 24+6(x-3)+12(x-3)^2+4(x-3)^3$ be the third-degree Taylor polynomial of some function f, expanded about $a = 3$. What is $f''(3)$?

Solution 1: $C_2 = \frac{f^{(2)}(3)}{2!}$ so $12 = \frac{f^{(2)}(3)}{2}$ so $f''(3) = 24$

Solution 2: $T_3''(x) = 12\cdot 2 + 4\cdot 3\cdot 2(x-3)$ so $T_3''(3) = 24 = f''(3)$

(7) In special relativity we have the formula $E = \frac{mc^2}{\sqrt{1-v^2/c^2}}$ for the kinetic energy of a moving particle. Here m is the "rest mass" of the particle and c is the speed of light. Examine the behaviour of this formula for small velocities by expanding it to second order in the *small parameter* $x = v^2/c^2$. What is the 4th order expansion of the energy? Do you recognize any of the terms?

2. New expansions from old

Near $u = 0$: $\frac{1}{1-u} = 1 + u + u^2 + u^3 + u^4 \cdots$
$\exp u = 1 + \frac{1}{1!}u + \frac{1}{2!}u^2 + \frac{1}{3!}u^3 + \frac{1}{4!}u^4 + \cdots$

(8) $\star$ (Final, 2016) Use a 3rd order Taylor approximation to estimate $\sin 0.01$. Then find the 3rd order Taylor expansion of $(x+1)\sin x$ about $x = 0$.

(9) Find the 3rd order Taylor expansion of $\sqrt{x} - \frac{1}{4}x$ about $x = 4$.

(10) Find the 8th order expansion of $f(x) = e^{x^2} - \frac{1}{1+x^3}$. What is $f^{(6)}(0)$?

(11) Find the quartic expansion of $\frac{1}{\cos 3x}$ about $x = 0$.

(12) (Change of variable/rebasing polynomials)

(a) Find the Taylor expansion of the polynomial $x^3 - x$ about $a = 1$ using the identity $x = 1 + (x - 1)$.

(b) Expand $e^{x^3 - x}$ to third order about $a = 1$.

(13) Expand $\exp(\cos 2x)$ to sixth order about $x = 0$.

(14) Show that $\log \frac{1+x}{1-x} \approx 2(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots)$. Use this to -get a good approximation to $\log 3$ via a careful choice of x.

(15) (2023 Piazza @389) Find the asymptotics as $x \to \infty$

(a) $\sqrt{x^4 + 3x^3} - x^2$

(b) $\sqrt[3]{x^6 - x^4} - \sqrt{x^4 - \frac{2}{3}x^2}$

(16) Evaluate $\lim_{x\to 0} \frac{e^{-x^2/2}-\cos x}{x^4}$.