Math 100, lecture 10, 8/2/2024

Last time: curve sketching

Linear approximation - $f(x) \leq f(a) + f'(a)(x-a)$ or $f(x+h) \approx f(x) + f'(x)h$

function f(x), linear approx f(a) + f'(a) (x-a) (o) have same value at a (f(a)) (i) have some derivative at a (f'(a))

Math 100:V02 – WORKSHEET 10 TAYLOR EXPANSION

1. TAYLOR EXPANSION

(1) (Review) Use linear approximations to estimate: (a) $\log \frac{4}{3}$ and $\log \frac{2}{3}$. Combine the two for an estimate of $\log 2$. At $\theta = 1$, $\log | = 0$, $(\log x)'|_{X=1} = \begin{bmatrix} 1 \\ x \end{bmatrix}_{Y=1} = 1$ so $\log x = 0 + 1 \cdot (x-1)$ =(X-1)80 log = x = log = x - = (b) $\sin 0.1$ and $\cos 0.1$. $\cos 0 = 1$, $(\cos 0)'|_{0=0} = (-\sin 0)|_{0=0} = 0$ So co 0 3 1 + 0.0=1 to 1st order in 0 cosal×1 to 1st orden (linear term could be zero)

Date: 8/2/2024, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

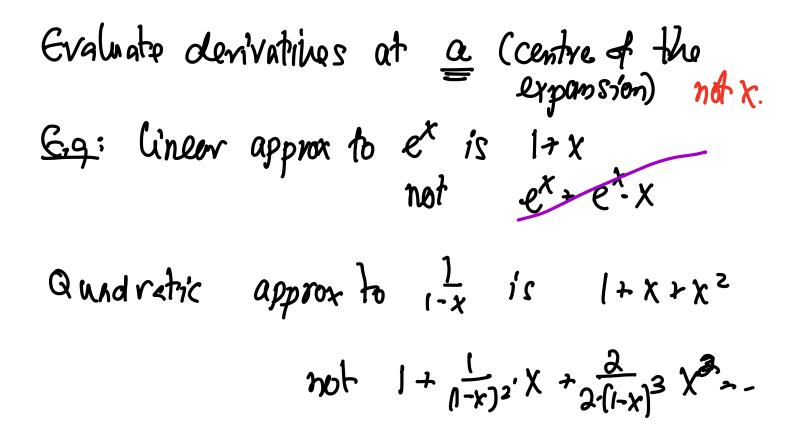
(2) Let $f(x) = e^x$ (a) Find $f(0), f'(0), f^{(2)}(0), \cdots$ (b) Find a polynomial $T_0(x)$ such that $T_0(0) = f(0)$. (c) Find a polynomial $T_1(x)$ such that $T_1(0) = f(0)$ and $T'_1(0) = f'(0)$. (d) Find a polynomial $T_2(x)$ such that $T_2(0) = f(0)$, $T'_{2}(0) = f'(0)$ and $T^{(2)}_{2}(0) = f^{(2)}(0)$. (e) Find a polynomial $T_3(x)$ such that $T_3^{(k)}(0) = f^{(k)}(0)$ for 0 < k < 3. (a) f(o) =1, QU k f^(k) (x)=e^x. So f^(b)(o)=1 (1) T. (y) =1 works (c) Tr (x) = 1+ 1-x = (+x works (linear approx) try T, (x)= a+2x want T, (0)= f(0)=1 so take a=1 (if a=0, T, (0)= 1, 0=0, so not good choice) Want T'(0)=f'(0)=1. So take b=1, get 1+x. a) try g+2x+cx2 want $T_2(0) = 1$, so held q = 1 $(x^2)^{\frac{n}{2}} = 2 \cdot | = 2$ Want T2'(6)=1, so b+2c.0=1 so b=1 Want Tr" (0)=1, So 2 C=1, SA C=5 tolo 1+X+1X2 $(x^3)^{\frac{1}{2}} = 3 \cdot 2 \cdot 1 = 6$ (e) $1 + x + \frac{1}{2}x^2 + dx^3 = 3^{rd}$ derivative is 6d So take $d = \frac{1}{6}$, $T_3(x) = 1 + x^2 + \frac{1}{2}x^2 + \frac{1}{6}x^3$

Conclusion: to match tith derivative we added
a correction of order t: a term
$$C_{k} \times k^{k}$$
.
choose C_{k} st. tith derivative \mathcal{A} $(C_{k} \times k^{k})$ is
same as $f^{(m)}(o)$, so $C_{k} = \frac{1}{1 \cdot 2 \cdot 3 \cdots (k)} f^{(m)}(o)$
In seneral if we are expanding about $X = a$.
Use
 $C_{k} = \frac{1}{k!} \frac{f^{(m)}(a)}{r^{k}}$
 $K! = 1 \cdot 2 \cdot 3 \cdots K$
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 $C_{k} = \frac{1}{k!} \frac{f^{(m)}(a)}{r^{k}}$
 $O_{k} = 1$
Degree n Taylor polynomial (Taylor expansion
of f about a is the polynomial
 $T_{n}(x) = f(a) + \frac{f'(a)}{1} (x \cdot a) + \frac{f^{(m)}(a)}{2!} (x \cdot a)^{n}$
(also say: $f(x) = T_{n}(x)$ to nith order at a')
 $\frac{Example:}{e^{x} \in 1 + x + \frac{1}{2}x^{2} + \frac{1}{6}x^{3} + \cdots + \frac{1}{n!}x^{n}$.

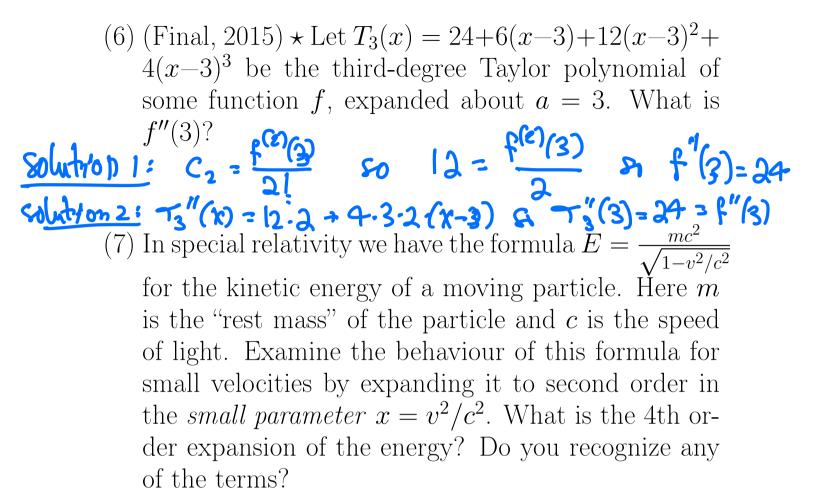
Let
$$c_k = \frac{f^{(k)}(a)}{k!}$$
. The *n*th order Taylor expansion of $f(x)$
about $x = a$ is the polynomial
 $T_n(x) = c_0 + c_1(x - a) + \dots + c_n(x - a)^n$
(4) Find the 4th order MacLaurin expansion of $\frac{1}{1-x}$ (=Tay-
lor expansion about $x = 0$)
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$$\frac{1}{1-u} = 1 + u + \frac{1}{2}u^2 + \frac{1}{3!}u^3 + \dots + \frac{1}{n!}u^n$$

Warming: Common error



(5) ** Find the *n*th order expansion of
$$\cos x$$
, and approximate $\cos 0.1$ using a 3rd order expansion
9(x) = Cosx; 9^(*)(x) = -Sinx; 9^(k)(x) = -Cosx; 9^(*)(x) = Sinx
9^(*)(x) = Cosx; 9^(*)(x) = -Sinx; -- (repeat)
9^(*)(x) = Cosx; 9^(*)(x) = -Sinx; -- (repeat)
9^(*)(x) = Cosx; 9^(*)(x) = -Sinx; -- (repeat)
9^(*)(x) = Cosx; 9^(*)(x) = -Sinx; -- (repat)
9^(*)(x) = -- (repat)
9



2. New expansions from old

Near u = 0: $\frac{1}{1-u} = 1 + u + u^2 + u^3 + u^4 \cdots$ exp $u = 1 + \frac{1}{1!}u + \frac{1}{2!}u^2 + \frac{1}{3!}u^3 + \frac{1}{4!}u^4 + \cdots$

(8) \star (Final, 2016) Use a 3rd order Taylor approximation to estimate sin 0.01. Then find the 3rd order Taylor expansion of $(x + 1) \sin x$ about x = 0. (9) Find the 3rd order Taylor expansion of $\sqrt{x} - \frac{1}{4}x$ about x = 4.

(10) Find the 8th order expansion of $f(x) = e^{x^2} - \frac{1}{1+x^3}$. What is $f^{(6)}(0)$? (11) Find the quartic expansion of $\frac{1}{\cos 3x}$ about x = 0.

(12) (Change of variable/rebasing polynomials) (a) Find the Taylor expansion of the polynomial x^3-x about a = 1 using the identity x = 1 + (x - 1).

(b) Expand e^{x^3-x} to third order about a = 1.

(13) Expand $\exp(\cos 2x)$ to sixth order about x = 0.

(14) Show that $\log \frac{1+x}{1-x} \approx 2(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots)$. Use this to -get a good approximation to $\log 3$ via a careful choice of x.

(15) (2023 Piazza @389) Find the asymptotics as $x \to \infty$ (a) $\sqrt{x^4 + 3x^3} - x^2$

(b)
$$\sqrt[3]{x^6 - x^4} - \sqrt{x^4 - \frac{2}{3}x^2}$$

(16) Evaluate $\lim_{x\to 0} \frac{e^{-x^2/2} - \cos x}{x^4}$.