Math 100, Lecture 3, 6/2/2024

Last time: Applying chain rule. Can differentiate identities with respect to any variable

(1) If $F(x,y)=0$ defines a curve can find $\frac{dy}{dx}$ by differentiating along the curve (using chain rule)

(2) If $F(x,y)=0$ enforces a relation between x,y can also diff wrt t (take $\frac{d}{dt}$) use chain rule to get relation between $x, y, \frac{dx}{dt}, \frac{dy}{dt}$.

(3) If we have a function $f(x,y)$ of two (or more) variables we can diff wrt x holding y constant get partial derivative $\frac{\partial f}{\partial x}$.

Notation for higher derivatives:

① $y', y'', y''', y'''', \ldots$
$y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}, \ldots$

② $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots$

③ $\frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^3 f}{\partial x^2 \partial y}$

$f_x \qquad f_{xy} \qquad f_{yxx}$

Math 100:V02 – WORKSHEET 9
CURVE SKETCHING

1. Partial derivatives

(1) Let $f(x,y) = x^3 + 3y^3 + 5xy^2$. Evaluate:

(a) $\frac{\partial f}{\partial x} = 3x^2 + 5y^2$ $\qquad$ $\frac{\partial f}{\partial y} = 9y^2 + 10xy$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(9y^2 + 10xy\right)$$

(b) $\frac{\partial^2 f}{\partial x^2} = 6x$ $\qquad$ $\frac{\partial^2 f}{\partial x \partial y} = 10y$ $\qquad$ $\frac{\partial^2 f}{\partial y^2} = 18y + 10x$

Date: 6/2/2024, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

<u>Today</u>: Curve sketching
(how f', f'' affect shape of graph $y = f(x)$)

2. Convexity and Concavity

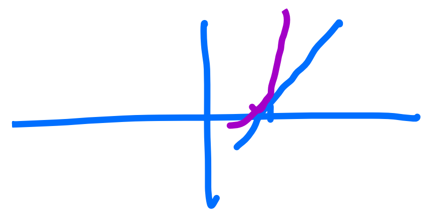
(2) Consider the curve $y = x^3 - x$.

(a) Find the line tangent to the curve at $x = 1$.

$\frac{dy}{dx} = 3x^2 - 1$ so $y(1) = 1 - 1 = 0$, $y'(1) = 3 - 1 = 2$ so line is $y = 2(x-1)$

(b) Near $x = 1$, is the line above or below the curve? Hint: how does the slope of the curve behave to the right and left of the point?

$f'' > 0 \iff$ tangent line below graph $\iff$ "concave up" U



opposite if $f'' < 0$ ∩

<u>inflection point</u>: change of concavity

(3) For each curve find its domain; where is it concave up or down? Where are the inflection points.

(a) $y = x \log x - \frac{1}{2}x^2$.

y is defined if $x > 0$ so $\log x$ is defined

$y' = \log x + x \cdot \frac{1}{x} - x = \log x + 1 - x$; $\quad y'' = \frac{1}{x} - 1$

so $y'' > 0$ where $\frac{1}{x} - 1 > 0 \iff \frac{1}{x} > 1 \iff 0 < x < 1$

$y'' < 0$ " $\frac{1}{x} - 1 < 0 \iff \frac{1}{x} < 1 \iff x > 1$

concave up on $(0,1)$, concave down on $(1,\infty)$

inflection pt over $x = 1$, at $(1, -\frac{1}{2})$.

(b) $y = \sqrt[3]{x}$.

3. Curve sketching

(4) Let $f(x) = \frac{x^2}{x^2+1}$ for which $f'(x) = \frac{2x}{(x^2+1)^2}$ and $f''(x) = \frac{2(1-3x^2)}{(x^2+1)^3}$.

(a) What are the domain and intercepts of f? What are the asymptotics at $\pm\infty$? Are there any vertical asymptotes? What are the asymptotices there?

f defined every where / on $\mathbb{R}$ / on $(-\infty, \infty)$

As $x \to \pm\infty$, $\frac{x^2}{x^2+1} \sim \frac{x^2}{x^2} \sim 1$.

f defined by formula $\Rightarrow$ cts $\Rightarrow$ no vertical asymptotes

(b) What are the intervals of increase/decrease? The local and global extrema?

$f'(x) = \frac{2x}{(x^2+1)^2}$ has same sign as x.

so f is increasing on $(0, \infty)$

decreasing on $(-\infty, 0)$

global minimum at $(0, 0)$.

(c) What are the intervals of concavity? Any inflection points?

$f''(x) = \frac{2(1-3x^2)}{(x^2+1)^3}$ has same sign as $1-3x^2$

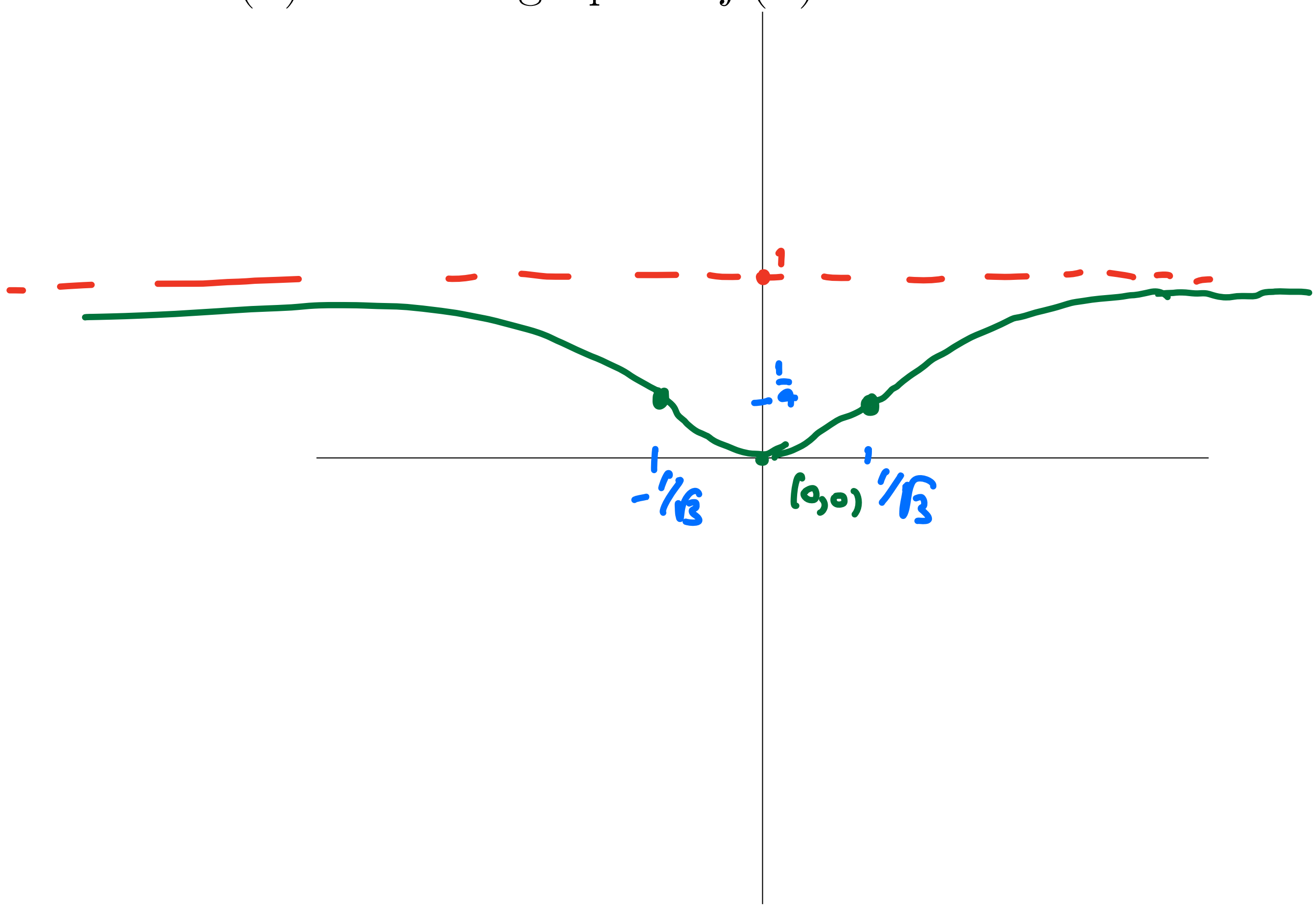
Now $1-3x^2>0$ if $1>3x^2 \iff \frac{1}{3}>x^2 \iff -\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}$

$1-3x^2<0$ if $1<3x^2 \iff \frac{1}{3}<x^2 \iff x>\frac{1}{\sqrt{3}}$ or $x<-\frac{1}{\sqrt{3}}$

$\Rightarrow f$ concave up on $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, down on $(-\infty, -\frac{1}{\sqrt{3}})$, $(\frac{1}{\sqrt{3}}, \infty)$

inflection points over $x=-\frac{1}{\sqrt{3}}$, $x=\frac{1}{\sqrt{3}}$: $(\pm\frac{1}{\sqrt{3}}, \frac{1}{4})$

(d) Sketch a graph of $f(x)$.



(5) $\star\star$ Let $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$.

(a) What are the domain and intercepts of f? What are the asymptotics at $\pm\infty$? Are there any vertical asymptotes? What are the asymptotices there?

(b) What are the intervals of increase/decrease? The local and global extrema?

(c) What are the intervals of concavity? Any inflection points?

(d) Sketch a graph of $f(x)$.

(6) (Final, December 2007) $\star\star$ Let $f(x) = x\sqrt{3-x}$.

(a) Find its domain, intercepts, and asymptotics at the endpoints.

(b) What are the intervals of increase/decrease? The local and global extrema?

(c) Given $f''(x) = \frac{3x-12}{4}(3-x)^{-3/2}$, what are the intervals of concavity? Any inflection points?

(d) Sketch a graph of $f(x)$.