Math 100, lecture 8, 1/2/2024

Last time: Chain Rule: $(f(g(x)))' = f'(g(x)) \cdot g'(x)$

$$\Rightarrow \frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} \quad \text{if } y = y(x)$$

Today: ① Apply chain rule
(i) implicit diff
(ii) "related rates"

② Partial derivatives

1. Review

(1) Differentiate

(a) $e^{\sqrt{\cos x}}$

$$\left(e^{\sqrt{\cos x}}\right)' = e^{\sqrt{\cos x}} \cdot \frac{1}{2\sqrt{\cos x}} \cdot (-\sin x)$$

(or: if $y = e^{\sqrt{\cos x}}$, $\log y = \sqrt{\cos x}$, diff both sides, ...)

(2) (Final, 2014) Let $y = x^{\log x}$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x only.

$$\Rightarrow \quad \log y = \log\left(x^{\log x}\right) = (\log x)(\log x) = (\log x)^2$$

diff both sides wrt x

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = 2(\log x)\cdot\frac{1}{x} \quad \text{so} \quad \frac{dy}{dx} = 2(\log x)\cdot\frac{1}{x}\cdot y$$

$$= 2(\log x)\cdot x\cdot x^{\log x}$$

$$\frac{d(\log y)}{dx} = \frac{d(\log y)}{dy}\cdot\frac{dy}{dx}$$

2. Implicit Differentiation

(3) Find the line tangent to the curve $y^2 = 4x^3 + 2x$ at the point $(2, 6)$.

Diff along the curve get $\frac{d}{dx}(y^2) = \frac{d}{dx}(4x^3+2x)$

$\Rightarrow 2y \cdot \frac{dy}{dx} = 12x^2 + 2$

At $(2,6)$ get $12 \cdot y' = 48 + 2 = 50$

so $y' = \frac{50}{12} = \frac{25}{6}$ so line is $\boxed{y = \frac{25}{6}(x-2)+6}$

(4) (Final, 2015) Let $xy^2 + x^2y = 2$. Find $\frac{dy}{dx}$ at the point $(1, 1)$.

Diff both sides along the curve: $\frac{d}{dx}(xy^2 + x^2y) = \frac{d}{dx}(2)$

$\frac{d(2)}{dx} = 0$; $\frac{d(xy^2+x^2y)}{dx} \underset{\text{sum rule}}{=} \frac{d(xy^2)}{dx} + \frac{d(x^2y)}{dx}$

$\underset{\text{pdt rule}}{=} \left[\frac{dx}{dx}y^2 + x\frac{d(y^2)}{dx}\right] + \left[\frac{d(x^2)}{dx}y + x^2\frac{dy}{dx}\right] \overset{\text{chain rule}}{=} y^2 + x\frac{d(y^2)}{dy}\cdot\frac{dy}{dx} + 2xy + x^2\frac{dy}{dx} =$

$= y^2 + 2xy\frac{dy}{dx} + 2xy + x^2\frac{dy}{dx}$

so at $(1,1)$ $1 + 2\frac{dy}{dx} + 2 + \frac{dy}{dx} = 0 \Rightarrow 3\frac{dy}{dx} = -3$

so $dy/dx = -1$, line is $\boxed{y = -(x-1)+1} = 2 - x$

In general set $(x^2+2xy)\frac{dy}{dx} + (y^2+2xy) = 0$

so $$\frac{dy}{dx} = -\frac{y^2+2xy}{x^2+2xy}$$

curve is $x^2y + xy^2 = 2$ so $x \neq 0$ on curve

if $x \neq 0$ can still have $x^2+2xy=0$ iff

$x+2y=0$ iff $x=-2y$

if $x=-2y$ $4y^3-2y^3=2$ so $2y^3=2$ so $y^3=1$

so $y=1$, $x=-2$

(5) (Final 2012) Find the slope of the line tangent to the curve $y + x\cos y = \cos x$ at the point $(0, 1)$.

(6) Find y'' (in terms of x, y) along the curve $x^5 + y^5 = 10$ (ignore points where $y = 0$).

Diff along curve set: $5x^4 + 5y^4 \cdot y' = 0$

Diff again, set $20x^3 + 5(4y^3 y')\cdot y' + 5y^4 \cdot y'' = 0$

so $y^4 \cdot y'' = -4x^3 - 4y^3 (y')^2$

so $y'' = -\frac{4x^3}{y^4} - \frac{4}{y}(y')^2$

also $y' = -\frac{5x^4}{5y^4} = -\frac{x^4}{y^4}$

$= -4\frac{x^3}{y^4} - \frac{4}{y}\left(-\frac{x^4}{y^4}\right)^2 = -\frac{4x^3}{y^4} - \frac{4x^8}{y^9}$.

Discussion:
Calculus step: diff twice

Algebra step: solve for y', y''

Summary: If we have relation $F(x,y)=0$
can diff both sides, wrt x
use chain rule to get linear equation for $\frac{dy}{dx}$.

What if x,y are both functions of t?
Can diff wrt t, get linear equation connecting
dx/dt, dy/dt.

3. Related Rates

(5) A particle is moving along the curve $y^2 = x^3 + 2x$. When it passes the point $(1, \sqrt{3})$ we have $\frac{\mathrm{d}y}{\mathrm{d}t} = 1$. Find $\frac{\mathrm{d}x}{\mathrm{d}t}$.

Solution 1: diff wrt x get $2y \cdot \frac{dy}{dx} = 3x^2 + 2$

so at given point $\frac{dy}{dx} = \frac{5}{2\sqrt{3}}$, $\frac{dy}{dt} = 1$

so $\frac{dx}{dt} = \frac{dy}{dt} \Big/ \frac{dy}{dx} = \frac{2\sqrt{3}}{5}$

Solution 2: $\frac{d}{dy}$: $2y = 3x^2 \cdot \frac{dx}{dy} + 2\frac{dx}{dy}$

$\Rightarrow$ at $(1, \sqrt{3})$, $\frac{dx}{dy} = \frac{2\sqrt{3}}{5}$

so $\frac{dx}{dt} = \frac{dx}{dy} \cdot \frac{dy}{dt} = \frac{2\sqrt{3}}{5} \cdot 1 = \frac{2\sqrt{3}}{5}$

Solution 3: hit with $\frac{d}{dt}$: $2y\frac{dy}{dt} = 3x^2 \cdot \frac{dx}{dt} + 2\frac{dx}{dt}$

at $(1, \sqrt{3})$, with $\frac{dy}{dt} = 1$ get $2\sqrt{3} \cdot 1 = 3 \cdot \frac{dx}{dt} + 2 \cdot \frac{dx}{dt}$

so $\boxed{\frac{dx}{dt} = \frac{2\sqrt{3}}{5}}$

(6) The state of a quantity of gas in a piston must satisfy the *ideal gas law*

$$PV = nRT\,,$$

where P is the pressure, V is the volume, n is the number of moles of gas, T is the (absolute) temperature and R is the ideal gas constant. Suppose $P = 1\text{atm}$ and $V = 22.4\text{L}$. How fast is the pressure of the gas changing when $\frac{dV}{dt} = 2.5\frac{\text{L}}{\text{min}}$, if the expansion is *isothermal*, that is with T held constant?

4. Partial derivatives

(7) Returning to the equation $PV = nRT$ now treat the temperature as a *function* of both pressure and volume.

(a) Suppose the volume is constant. What is the rate of change of temperature with respect to pressure?

(b) Suppose the pressure is constant. What is the rate of change of temperature with respect to pressure?

(c) What is the rate of change of the temperaure with respet to the number of moles of gas, pressure and volume being constant?