1A3

13. Multivariable optimization (29/11/2023)

Goals.
(1) Critial points in 2d
(2) Multivariable optimization
(3) Constrained optimization

Last Time. Multivariable functions

(1) points, lines, planes in 3d
sketching 3d objects in 2d
(2) Graphs $z = f(x,y)$
(3) Partial derivatives $\frac{\partial f}{\partial x}$
(4) critical points: all partial derivatives vanish

finding critical points of $f(x_1, .., x_n)$

$\Leftrightarrow$ solving system of n equations $\begin{cases} \frac{\partial f}{\partial x_1}(x_1, .., x_n) = 0 \\ \vdots \\ \frac{\partial f}{\partial x_n}(x_1, .., x_n) = 0 \end{cases}$

Math 100A – **WORKSHEET 13**
MULTIVARIABLE OPTIMIZATION

1. Critical points; multivariable optimization

(1) ⋆How many critical points does $f(x, y) = x^2 - x^4 + y^2$ have?

$\frac{\partial f}{\partial x} = 2x - 4x^3$; $\frac{\partial f}{\partial y} = 2y$

crit. pts at $\begin{cases} 2x(1-2x^2) = 0 & \Rightarrow x \in \{0, \pm\frac{1}{\sqrt{2}}\} \\ 2y = 0 & \Rightarrow y = 0 \end{cases}$

are $(0,0)$, $(\frac{1}{\sqrt{2}}, 0)$, $(-\frac{1}{\sqrt{2}}, 0)$

points: $(0,0,0)$, $(\pm\frac{1}{\sqrt{2}}, 0, \frac{1}{4})$

Date: 29/11/2023, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(2) ⋆Find the critical points of $f(x,y) = x^2 - x^4 + xy + y^2$.

$\frac{\partial f}{\partial x} = 2x - 4x^3 + y$; $\frac{\partial f}{\partial y} = x + 2y$

need to solve $\begin{cases} 2x - 4x^3 + y = 0 \\ x + 2y = 0 \end{cases}$ from 2nd equation $x = -2y$

so $-4y + 32y^3 + y = 0 \Leftrightarrow y(32y^2 - 3) = 0$

solution have $y \in \{0, \pm\sqrt{3/32}\}$

critical points over $(0,0), \left(-2\sqrt{\frac{3}{32}}, \sqrt{\frac{3}{32}}\right), \left(2\sqrt{\frac{3}{32}}, -\sqrt{\frac{3}{32}}\right)$

(3) (MATH 105 Final, 2013) ⋆ Find the critical points of $f(x,y) = xye^{-2x-y}$.

$\frac{\partial f}{\partial x} = ye^{-2x-y} + xy\frac{\partial}{\partial x}\left(e^{-2x-y}\right) = y(1-2x)e^{-2x-y}$

$\frac{\partial f}{\partial y} = xe^{-2x-y} - xye^{-2x-y} = x(1-y)e^{-2x-y}$

need to solve $\begin{cases} y(1-2x) = 0 \\ x(1-y) = 0 \end{cases}$ from 1st equation $y=0$ or $x = \frac{1}{2}$

because $e^{-2x-y} > 0$

if $y = 0$, $x = 0$ from 2nd equation

if $x = \frac{1}{2}$, $\frac{1}{2}(1-y) = 0$ in 2nd equation

so $y = 1$.

critical points over $(0,0), \left(\frac{1}{2}, 1\right)$

" " are $(0,0,0), \left(\frac{1}{2}, 1, \frac{1}{2}e^{-2}\right)$

(4)

(a) ⋆⋆ Let $f(x, y) = 4x^2 + 8y^2 + 7$. Find the critical point(s) of $f(x, y)$, and determine (if possible) whether each critical point corresponds to a local maximum, local minimum, or neither ("saddle point").

$\frac{\partial f}{\partial x} = 8x$; $\frac{\partial f}{\partial y} = 16y$ only critical point over $(0,0)$, it is a local min because it is the global minimum.
$4x^2 + 8y^2 + 7 \geq 7$ for all x, y.

(b) (MATH 105 Final, 2017) ⋆⋆ Let $f(x, y) = -4x^2 + 8y^2 - 3$. Find the critical point(s) of $f(x, y)$, and determine (if possible) whether each critical point corresponds to a local maximum, local minimum, or neither ("saddle point").

$\frac{\partial f}{\partial x} = -8x$, $\frac{\partial f}{\partial y} = 16y$, critical point $(0, 0, -3)$
it is a saddle point: if we move in x direction values decrease, if we move in y direction values increase

"Closed interval method"

Say f is defined on a domain Ω

① If f has a global max (or min) it will occur at one of:

(i) critical pts

(ii) singular pts

(iii) boundary of the domain

② If domain is closed and bounded, f cts, then always have global max & min

Examples: 1d: boundary of $[a,b]$ is $\{a,b\}$

2d: boundary of disk $\{x^2+y^2 \leq 7\}$ is circle $\{x^2+y^2=7$

For $f(x,y)$ the boundary is a curve, so to find max/min on boundary is 1d optimization

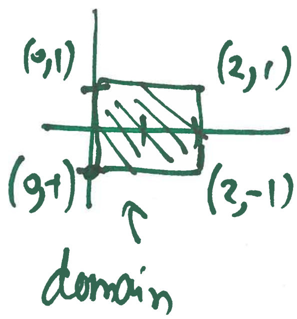
(5) $\star$ Find the critical points of $(7x + 3y + 2y^2)e^{-x-y}$.

2. Optimization

(6) $\star\star$Find the minimum of $f(x, y) = 2x^2 + 3y^2 - 4x - 5$:

(a) on the rectangle $0 \le x \le 2$, $-1 \le y \le 1$.

Crit pts over $\begin{cases} 4x - 4 = 0 \\ 6y = 0 \end{cases}$ at $(1, 0, -7)$



• boundary: trace over $x=0$ is $f(0,y) = 3y^2 - 5$ on $[-1,1]$ has min -5 at $(0,0)$

trace over $x=2$ is $f(2,y) = 3y^2 - 5$, on $-1 \le y \le 1$ has minimum -5; Trace over $y = \pm 1$ is $f(x, \pm 1) = 2x^2 - 4x$. min at $x=1$ $f(1, \pm 1) = -4$.

(b) on the rectangle $2 \le x \le 3$, $-1 \le y \le 1$.

So min on rectangle is -7, attained at $(1, 0)$

(b) no critical points! $(1,0)$ outside rectangle

trace over $y = \pm 1$ still $f(x, \pm 1) = 2x^2 - 4x - 2$

$= 2(x^2 - 2x + 1) - 4 = 2(x-1)^2 - 4$

increasing on $[2,3]$ so min is $f(2, \pm 1) = -2$

trace over $x=2$ is $f(y) = 3y^2 - 5$, min is -5 at $(2, 0)$ ← global min

trace over $x=3$ is $f(3,y) = 3y^2 + 1$, min is 1 at $(3,0)$

(7) Find the maximum of $(7x+3y+2y^2)e^{-x-y}$ for $x \geq 0$, $y \geq 0$,

$$\frac{\partial f}{\partial x} = (7 - (7x + 3y + 2y^2))e^{-x-y}$$

$$\frac{\partial f}{\partial y} = (3 + 4y - (7x + 3y + 2y^2))e^{-x-y}$$

crit pts over $\begin{cases} 7 = 7x + 3y + 2y^2 \\ 3 + 4y = 7x + 3y + 2y^2 \end{cases}$ so $7 = 3 + 4y$ so $y = 1$

so $7 = 7x + 3 + 2$ so $x = \frac{2}{7}$, crit pt at $(\frac{2}{7}, 1, 7e^{-9/7})$

as x or $y \to \infty$, e^{-x}, e^{-y} decay exponentially dominate $7x + 3y + 2y^2$, so $\lim\limits_{\substack{x \to \infty \\ \text{or} \\ y \to \infty}} f(x,y) = 0$

but $f(x,y) > 0$ if $x, y > 0$

so max occurs at finite pts, either at $(\frac{2}{7}, 1)$ or on $x = 0$ or on $y = 0$

If $x = 0$, trace is $f(0,y) = (3y + 2y^2)e^{-y}$ want max for $0 \leq y < \infty$

$$\frac{df(0,y)}{dy} = \frac{\partial f}{\partial y}(0,y) = (3 + 4y - 3y - 2y^2)e^{-y} = (3 + y - 2y^2)e^{-y}$$

solutions to $2y^2 - y - 3 = 0$ are $\frac{1 \pm \sqrt{1+24}}{4} = \frac{3}{2}, -1$. only $\frac{3}{2}$ rele[vant]

$f(0,0) = 0$, $f(0, \frac{3}{2}) = 9e^{-3/2}$, $\lim\limits_{y \to \infty} f(0,y) = 0$.

max on vertical boundary

Trace on $y=0$ is $f(x,0) = 7xe^{-x}$, we want max for $0 \le x < \infty$.

$$\frac{d}{dx} f(x,0) = \frac{\partial f}{\partial x}(x,0) = (7-7x)e^{-x} = 7(1-x)e^{-x}$$

Crit pt over $x=1$.

$f(0,0) = 0$, $f(1,0) = 7e^{-1}$, $\lim_{x \to \infty} f(x,0) = 0$

max on horizontal boundary.

$\Rightarrow$ max is largest of $7e^{-9/7}$, $7 \cdot e^{-1}$, $9e^{-3/2}$.

clearly $7e^{-9/7} < 7e^{-1}$ need to compare $\frac{7}{e}$, $\frac{9}{e^{3/2}}$

$\Leftrightarrow$ comparing $\frac{49}{e^2}$, $\frac{81}{e^3}$, same as comparing e, $\frac{81}{49}$

but $e > 2 > \frac{81}{49}$, so $\frac{49}{e^2} > \frac{81}{e^{3}}$, so $\frac{7}{e} > \frac{9}{e^{3/2}}$.

So global max is $\frac{7}{e}$, attained at $(1,0)$

(8) A company can make widgets of varying quality. The cost of making q widgets of quality t is $C = 3t^2 + \sqrt{t} \cdot q$. At price p the company can sell $q = \frac{t-p}{3}$ widgets.

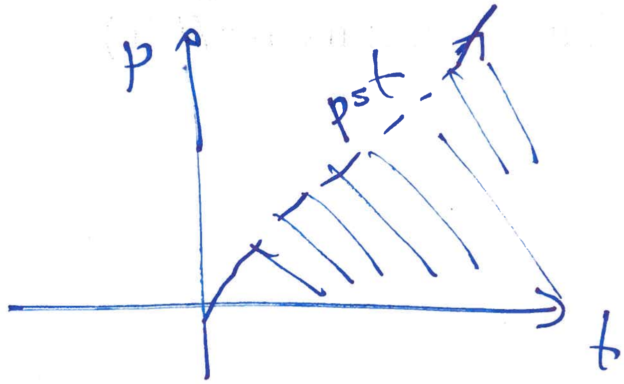
(a) Write an expression for the profit function $f(\overset{p}{\cancel{x}}, t)$.

$$f(p,t) = \underbrace{\tfrac{1}{3}(t-p)}_{\text{quantiy}} \cdot \underset{\uparrow\,\text{price}}{p} - \underbrace{3t^2 - \sqrt{t} \cdot \tfrac{1}{3}(t-p)}_{\text{costs}}$$

(b) How many widgets of what quality should the company make to maximize profits?

domain is $t \geq 0$, $0 \leq p \leq t$

$\nwarrow$ $q = \frac{t-p}{3}$ should be ≥ 0



(9) Find the maximum and minimum values of $f(x, y) = -x^2 + 8y$ in the disc $R = \{x^2 + y^2 \leq 25\}$.

in interior look for critical pts

on boundary circle optimize $f(x,y) = -x^2 + 8y$

subject to $x^2 + y^2 = 25$

(eg. along boundary $2x + 2y \frac{dy}{dx} = 0$

$$\frac{df}{dx} = -2x + 8\frac{dy}{dx} \quad)$$

$$= -2x - 8\frac{x}{y}$$

So crit pts on bdry are $\begin{cases} -2x - 8\frac{x}{y} = 0 \\ x^2 + y^2 = 25 \end{cases}$