1A3

12. Multivariable calculus (24/11/2023)

Goals.

(1) 3d space: coordinates and graphs
(2) Partial derivatives

Last Time. **Numerical methods**

(1) Euler scheme: Method for solving ODE approximated

Input: (1) ODE : $y' = f(y;t)$
(2) initial condition (t_0, y_0)
(3) Endpoint b

Goal: solve ODE on $[t_0, b]$

Algorithm: choose number n, divide $[t_0, b]$ into n steps of length $h = \frac{b-t_0}{n}$, set points t_0, $t_1 = t_0 + h$, $t_2 = t_0 + 2h$, ... $t_k = t_0 + kh$, .., $t_n = b$.

set $y_1 = y_0 + f(y_0;t_0)h$; $y_2 = y_1 + f(y_1;t_1)h$

...., $\boxed{y_{k+1} = y_k + f(y_k; t_k)h}$

use previous point to estimate slope

② Newton's Method

points where $f(z) = 0$

Method for finding zeroes of functions

<u>Input:</u> function f, initial guess x_0 (hopefully close to a zero of f)

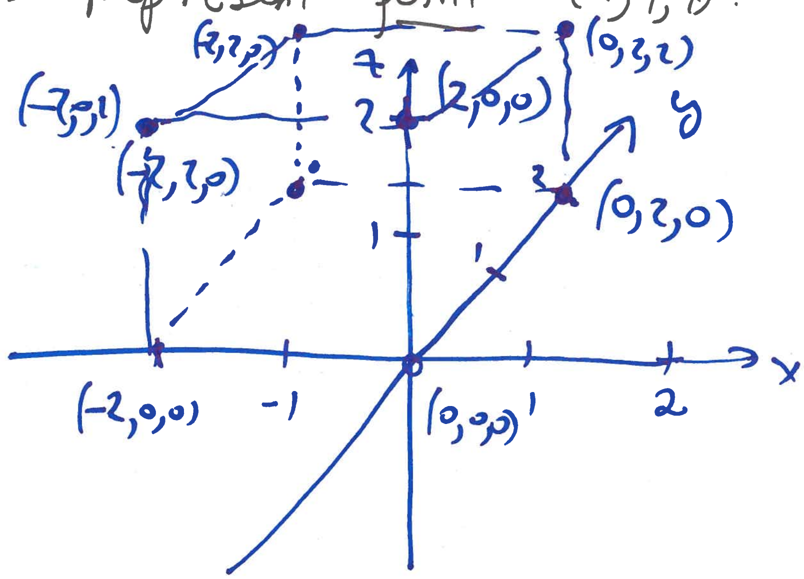
<u>Algorithm:</u> Given guess x_k ~~[illegible]~~ find the linear approximation to f about x_k, set x_{k+1} to be the point where linear approx crosses axis:

$$\boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}}$$

<u>Observation:</u> graph of $z = f(x,y)$ is a <u>surface</u> in 3d space (graph of $y = f(x)$ is a <u>curve</u> in 2d space)

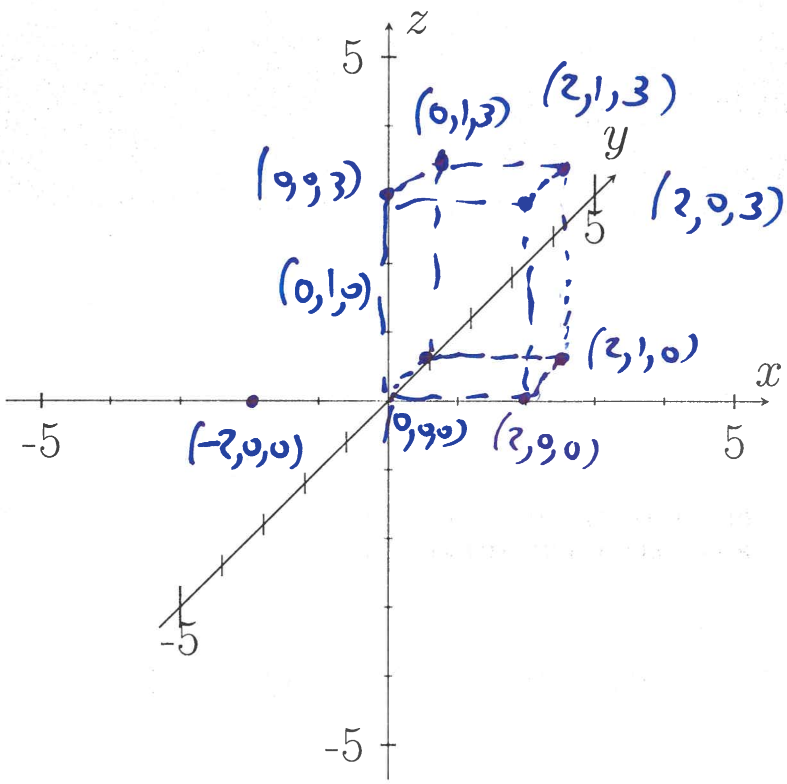
<u>Problem:</u> how to represent this?

<u>First task:</u> represent <u>point</u> (x,y,z).



Math 100C – WORKSHEET 10
MULTIVARIABLE CALCULUS

1. Plotting in three dimensions

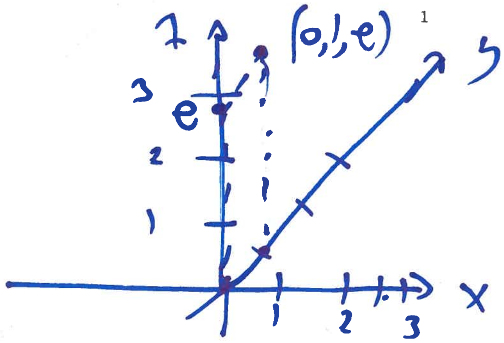


(1) $\star$ Plot the points $(2,1,3)$, $(-2,2,2)$ on the axes provided.

(2) Let $f(x,y) = e^{x^2+y^2}$. $f(0,-1) = e^{0^2+(-1)^2} = e^1 = e$, $f(1,2) = e^5$

(a) $\star$ What are $f(0,-1)$? $f(1,2)$? Plot the point $(0,1,f(0,1))$ on the axes provided.

Date: 24/11/2022, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.



(b) ⋆ What is the *domain* of f (that is: for what (x, y) values does f make sense?

All of the x-y plane

(c) ⋆ What is the *range* of f (that is: what values does it take)?

need z values for which have x, y s.t. $x^2+y^2=\log z$

$x^2+y^2\geq 0$ means $e^{x^2+y^2}\geq e^0=1$

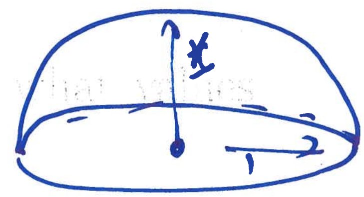
range is $[1,\infty)$

(3) ⋆⋆ What would the graph of $z = \sqrt{1-x^2-y^2}$ look like?

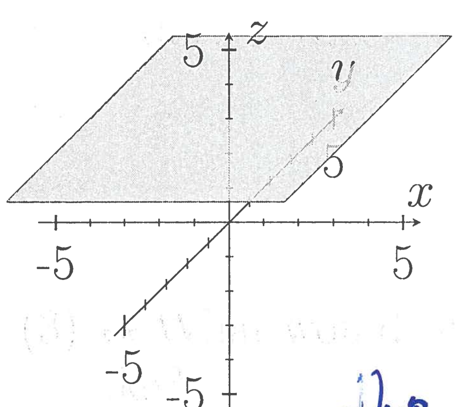
domain: $1-x^2-y^2\geq 0 \iff x^2+y^2\leq 1$

function is: $\begin{cases} z^2+x^2+y^2=1 \\ z\geq 0 \end{cases}$

graph: hemisphere



(4) ⋆ Which plane is this?



(A) $x = 3$
(B) $y = 3$
(C) $z = 3$
(D) none
(E) not sure

this plane is parallel to xy plane, meets the z-axis at (0,0,3)

so graph of $z=3$

∂ = \partial

2. Partial derivatives

(5)(a) $\star$ Let $f(x) = 2x^2 - a^2 - 2$. What is $\frac{df}{dx}$?

$$\frac{df}{dx} = 4x$$

(b) $\star$ Let $f(x) = 2x^2 - y^2 - 2$ where y is a constant. What is $\frac{df}{dx}$?

$$\frac{df}{dx} = 4x$$

(c) $\star$ Let $f(x, y) = 2x^2 - y^2 - 2$. What is the rate of change of f as a function of x *if we keep y constant*?

still have $\frac{\partial f}{\partial x} = 4x$

Call this the **partial derivative** of $f(x,y)$ wrt x holding y constant.

(d) $\star$ What is $\frac{\partial f}{\partial y}$?

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(2x^2) - \frac{\partial}{\partial y}(y^2) - \frac{\partial}{\partial y}(2) = -2y$$

sides try $u(x,y,z) = g \cdot z$

(7) The the gravitational *potential* due to a point mass M (equivalently the electrical potential due to a point charge M) is given by the formula $U(x,y,z) = -\frac{GM}{r}$ where $r = \sqrt{x^2+y^2+z^2}$. Here G is the universal gravitational constant (equivalently G is the Coulomb constant).

(a) ⋆ The x-component of the field is given by the formula $F_x(x,y,z) = -\frac{\partial U}{\partial x}$. Find F_x

$$u(x,y,z) = -(x^2+y^2+z^2)^{-\frac{1}{2}} GM, \quad -\frac{\partial u}{\partial x} = (-1)(-1)GM\left(-\frac{1}{2}\right)(x^2+y^2+z^2)^{-3/2} 2x$$

$$= -GM \frac{x}{(x^2+y^2+z^2)^{3/2}} = -\frac{GMx}{r^3}$$

(b) ⋆ The magnitude of the field is given by $\left|\vec{F}\right| = \sqrt{F_x^2+F_y^2+F_z^2}$. How does it decay as a function of r?

$$|\vec{F}| = \left[\left(\frac{GMx}{r^3}\right)^2 + \left(\frac{GMy}{r^3}\right)^2 + \left(\frac{GMz}{r^3}\right)^2\right]^{\frac{1}{2}} = \left[\frac{G^2M^2}{r^6}(x^2+y^2+z^2)\right]^{\frac{1}{2}}$$

$$= \frac{GM}{r^3}(x^2+y^2+z^2)^{\frac{1}{2}} = \frac{GM}{r^2}$$

(8) The *entropy* of an ideal gas of N molecules at temperature T and volume V is

$$S(N,V,T) = Nk\log\left[\frac{VT^{1/(\gamma-1)}}{N\Phi}\right].$$

where k is *Boltzmann's constant* and γ, Φ are constants that depend on the gas.

(a) $\star$ Find the *heat capacity at constant volume* $C_V = T\frac{\partial S}{\partial T}$.

$$S = Nk\log\frac{V}{N\Phi} + Nk\log T^{1/(\gamma-1)} = Nk\log\frac{V}{N\Phi} + \frac{Nk}{\gamma-1}\log T$$

so $\quad \dfrac{\partial S}{\partial T} = \dfrac{Nk}{\gamma-1}\cdot\dfrac{1}{T} \quad$ so $\quad C_V = T\dfrac{\partial S}{\partial T} = \dfrac{Nk}{\gamma-1}$

(b) $\star\star\star$ Using the relation ("ideal gas law") $PV = NkT$ write S as a function of N, P, T instead. Differentiating with respect to T **while keeping P constant** determine the *heat capacity at constant pressure* $C_P = T\frac{\partial S}{\partial T}$.

$$S = Nk\log\left[\frac{kT^{1+\frac{1}{\gamma-1}}}{P\Phi}\right] = Nk\log\left[\frac{k}{P\Phi}\right] + \frac{Nk\gamma}{\gamma-1}\log T$$

$$C_P = T\frac{\partial S}{\partial T} = \frac{Nk\gamma}{\gamma-1}.$$

constant P

(6) Find the partial derivatives with respect to both x, y of

(a) $\star$ $g(x,y) = 3y^2 \sin(x+3)$

$$\frac{\partial g}{\partial x} = 3y^2 \frac{\partial}{\partial x} \sin(x+3) = 3y^2 \cos(x+3)$$

linearity

$$\frac{\partial g}{\partial y} = 3\sin(x+3) \cdot \frac{\partial}{\partial y}(y^2) = 6y\sin(x+3)$$

(b) $\star$ $h(x,y) = ye^{Axy} + B$

$$\frac{\partial h}{\partial x} = y \cdot e^{Axy} \cdot Ay = Ay^2 e^{Axy}$$

$$\frac{\partial h}{\partial y} = e^{Axy} + y \cdot Ax \cdot e^{Axy} = (1+Axy)e^{Axy}$$

(9) We can also compute second derivatives. For example $f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2}{\partial y \partial x} f$. Evaluate:

(a) $\star$ $h_{xx} = \frac{\partial^2 h}{\partial x^2} = \frac{\partial}{\partial x}\left(Ay^2 e^{Axy}\right) = A^2 y^3 e^{Axy}$

(b) $\star$ $h_{xy} = \frac{\partial^2 h}{\partial y \partial x} = \frac{\partial}{\partial y}\left(Ay^2 e^{Axy}\right) = \left(2Ay + A^2 x y^2\right) e^{Axy}$

$\parallel$

(c) $\star$ $h_{yx} = \frac{\partial^2 h}{\partial x \partial y} = \frac{\partial}{\partial x}\left((1+Axy) e^{Axy}\right) = \left(Ay + Ay + A^2 x y^2\right) e^{A}$

(d) $\star$ $h_{yy} = \frac{\partial^2 h}{\partial y^2} = \frac{\partial}{\partial y}\left((1+Axy) e^{Axy}\right) = \left(Ax + Ax + A^2 x^2 y\right) e^{Ax}$

Fact: $f_{xy} = f_{yx}$ outside of exotic situations

(10) You stand in the middle of a north-south street (say Health Sciences Mall). Let the x axis run along the street
(say oriented toward the south), and let the y axis run across the street. Let $z = z(x, y)$ denote the height of
the street surface above sea level.

(a) $\star$ What does $\frac{\partial z}{\partial y} = 0$ say about the street?

the street is level

(b) $\star$ What does $\frac{\partial z}{\partial x} = 0.15$ say about the street?

the street has a grade of 15%: for every m we go south, altitude changes by 15cm

(c) $\star$ You want to follow the street downhill. Which way should you go?

North

(d) The intersection of Health Sciences Mall and Agronomy Road is a local maximum.
What does that say about $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ there?

$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$