1A3

8. Taylor expansion (25/10/2023)

Goals.

(1) Review: Linear approximation
(2) Higher order approximation
(3) Manipulating expansions

Last Time. **Curve sketching**

0th derivative info: asymptotics, $f>0$, $f<0$, discontinuity, blowup
1st derivative info: increase/decrease, local max/min critical/singular points
2nd derivative info: concavity, inflection pts

create picture representing all info

Today: use higher derivatives to study local behavior of functions

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WS 1

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linear approx: $f(x) \approx f(a) + f'(a)(x-a)$

$f(a+h) \approx f(a) + f'(a) \cdot h$

Math 100A – WORKSHEET 8
TAYLOR EXPANSION

1. Taylor expansion

(1) (Review) Use linear approximations to estimate:

(a) $\star$ $\log\frac{4}{3}$ and $\log\frac{2}{3}$. Combine the two for an estimate of $\log 2$.

Here $\log 1 = 0$ $\quad (\log x)'\big|_{x=1} = \frac{1}{x}\big|_{x=1} = 1$

so $\log\frac{4}{3} \approx 0 + 1\cdot\left(\frac{4}{3}-1\right) = \frac{1}{3}$; $\log\left(2/3\right) \approx -\frac{1}{3}$

So $\log 2 \approx \log\left(\frac{4/3}{2/3}\right) = \log\frac{4}{3} - \log 2/3 \approx 2/3$

(b) $\star$ $\sin 0.1$ and $\cos 0.1$.

$\sin 0 = 0$, $(\sin\theta)'\big|_{\theta=0} = \cos 0 = 1$; $\cos 0 = 1$, $(\cos\theta)'\big|_{\theta=0} = 0$

so $\sin(0.1) \approx 0 + 1\cdot 0.1 = 0.1$, $\quad \cos(0.1) \approx 1 + 0\cdot 0.1 = 1$
(to 1st order)

Date: 25/10/2023, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(2) Let $f(x) = e^x$

(a) Find $f(0), f'(0), f^{(2)}(0), \cdots$

(b) Find a polynomial $T_0(x)$ such that $T_0(0) = f(0)$.

(c) Find a polynomial $T_1(x)$ such that $T_1(0) = f(0)$ and $T_1'(0) = f'(0)$.

(d) Find a polynomial $T_2(x)$ such that $T_2(0) = f(0)$, $T_2'(0) = f'(0)$ and $T_2^{(2)}(0) = f^{(2)}(0)$.

(e) Find a polynomial $T_3(x)$ such that $T_3^{(k)}(0) = f^{(k)}(0)$ for $0 \le k \le 3$.

1a) $f(0) = 1$, $f'(0) = e^0 = 1$, $f^{(k)}(0) = [e^x]_{x=0} = 1$

(b) $T_0(x) = 1$ works

(c) $T_1(x) = 1 + x$ has right value & right slope

(d) [need coeff of x^2] [call it c]

Try $T_2(x) = 1 + x + cx^2$

$T_2(0) \overset{\checkmark}{=} 1$; $T_2'(0) = [1 + 2cx]_{x=0} \overset{\checkmark}{=} 1$; $T_2''(0) = 2c$

so to set $T_2''(0) = 1$ need $2c = 1$, so $c = \frac{1}{2}$

and $\boxed{T_2(x) = 1 + x + \frac{1}{2}x^2.}$

(e) Try $T_3(x) = 1 + x + \frac{1}{2}x^2 + dx^3$

$T_3^{(3)}(x) = 6d$ so to set $T_3^{3}(0) = 1$ choose $d = \frac{1}{6}$

set $\boxed{T_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3}$

Conclusion:

Given function f, point a, can approximate $f(x)$ for x near a by:

$$f(x) \simeq c_0 + c_1(x-a) + c_2(x-a)^2 + \dots + c_n(x-a)^n$$

with $c_0 = f(a)$, $c_1 = f'(a)$, $c_2 = \frac{1}{1\cdot 2} f^{(2)}(a)$, $c_3 = \frac{1}{1\cdot 2\cdot 3} f^{(3)}(a)$

$$\dots \boxed{c_k = \frac{1}{k!} f^{(k)}(a)}$$

notation for k'th derivative

"k factorial" $= 1\cdot 2\cdot 3\cdot \dots \cdot k$

Call $T_n(x) = c_0 + c_1(x-a) + \dots + c_n(x-a)^n$

$$= f(a) + \frac{1}{1!} f^{(1)}(a)(x-a) + \dots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

the nth order **Taylor polynomial** or **Expansion** $\Big\}$ of f about a

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Typically, $R_n(x) = f(x) - T_n(x)$ decays like $(x-a)^{n+1}$ as $x \to a$

(always decays faster than $(x-a)^n$)

Common error:

Confusing $f^{(k)}(a)$ with $f^{(k)}(x)$

E.g. ~~we~~ write $\log x \approx \frac{1}{x} \cdot (x-1)$ $\boxed{\text{wrong}}$

$\uparrow (\log x)'$

want $(\log x)'|_{x=1}$

or $\exp x \approx 1 + e^{x} \cdot x + \frac{e^x}{2} x^2 + \frac{e^x}{6} x^3 + \ldots$ $\boxed{\text{wrong}}$

(want e^0 not e^x)

① not a polynomial

② cannot be used to understand e^x.

real: $e^x \approx 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \ldots$

(3) Do the same with $f(x) = \log x$ about $x = 1$.

$f'(x) = \frac{1}{x}$, $f^{(2)}(x) = -\frac{1}{x^2}$, $f^{(3)}(x) = \frac{1\cdot 2}{x^3}$, $f^{(4)}(x) = -\frac{1\cdot 2\cdot 3}{x^4}$, ...

$f(1) = 0$, $f'(1) = 1$, $f^{(2)}(1) = -1$, $f^{(3)}(1) = 1\cdot 2$, $f^{(4)}(1) = -1\cdot 2\cdot 3$.

$$\Rightarrow \log x \simeq (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \dots$$

(Expanding around 1, so $x-1$ is the small parameter.)

$$\Leftrightarrow \log(1+h) \simeq h - \frac{1}{2}h^2 + \frac{1}{3}h^3 - \frac{1}{4}h^4 + \dots$$

Let $c_k = \frac{f^{(k)}(a)}{k!}$. The nth order Taylor expansion of $f(x)$ about $x = a$ is the polynomial

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n$$

(4) ⋆ Find the 4th order MacLaurin expansion of $\frac{1}{1-x}$ (=Taylor expansion about $x = 0$)

let $g(x) = \frac{1}{1-x} = (1-x)^{-1}$; $g^{(1)}(x) = 1\cdot(1-x)^{-2}$; $g^{(2)}(x) = 1\cdot 2(1-x)^{-3}$;

$g^{(3)}(x) = 1\cdot 2\cdot 3(1-x)^{-4}$; $g^{(4)}(x) = 1\cdot 2\cdot 3\cdot 4(1-x)^{-5}$

so $g^{(k)}(0) = k!$

so $T_4(x) = 1 + \underbrace{x}_{\frac{1}{1}} + \underbrace{x^2}_{\frac{1\cdot 2}{1\cdot 2}} + \underbrace{x^3}_{\frac{1\cdot 2\cdot 3}{1\cdot 2\cdot 3}} + x^4$ . .

(if $|x|<1$, then $\boxed{1 + x + x^2 + \cdots = \frac{1}{1-x}}$

(6) (Final, 2015) ⋆ Let $T_3(x) = 24+6(x-3)+12(x-3)^2+4(x-3)^3$ be the third-degree Taylor polynomial of some function f, expanded about $a = 3$. What is $f''(3)$?

$12 = C_2 = \frac{1}{2!} f^{(2)}(3) \Rightarrow f^{(2)}(3) = 24$ | or $f^{(2)}(3) = T_3^{(2)}(3)$

(7) ⋆⋆ In special relativity we have the formula $E = \frac{mc^2}{\sqrt{1-v^2/c^2}}$ for the kinetic energy of a moving particle. Here m is the "rest mass" of the particle and c is the speed of light. Examine the behaviour of this formula for small velocities by expanding it to second order in the *small parameter* $x = v^2/c^2$. What is the 4th order expansion of the energy? Do you recognize any of the terms?

$E(x) = mc^2(1-x)^{-\frac{1}{2}}$ $\qquad E(0) = mc^2$

$E'(x) = \frac{1}{2}mc^2(1-x)^{-3/2}$ $\qquad E'(0) = \frac{1}{2}mc^2$

$E''(x) = \frac{3}{4}mc^2(1-x)^{-5/2}$ $\qquad E''(0) = \frac{3}{4}mc^2$ ✓

so $E \approx mc^2 + \frac{1}{2}mc^2\, x + \frac{1}{2}\frac{3}{4}mc^2\, x^2$

$\frac{1}{2!}$

$= \underbrace{mc^2}_{\text{rest energy}} + \underbrace{\frac{1}{2}mv^2}_{\text{Newtonian formula}} + \underbrace{\frac{3}{8}mv^2\frac{v^2}{c^2}}_{\text{first relativistic correction}}$

2. New expansions from old

Near $u = 0$:	$\frac{1}{1-u} = 1 + u + u^2 + u^3 + u^4 \cdots$
$\exp u = 1 + \frac{1}{1!}u + \frac{1}{2!}u^2 + \frac{1}{3!}u^3 + \frac{1}{4!}u^4 + \cdots$	

(8) $\star$ (Final, 2016) Use a 3rd order Taylor approximation to estimate $\sin 0.01$. Then find the 3rd order Taylor expansion of $(x+1)\sin x$ about $x = 0$.

Checks $\sin\theta \approx \theta - \frac{1}{6}\theta^3$ to 3^{rd} order

$1+x \approx 1+x$ " " "

so $(1+x)\cdot\sin x \approx (1+x)\left(x - \frac{1}{6}x^3\right)$

$= x + x^2 - \frac{1}{6}x^3 - \frac{1}{6}x^4$

$\approx x + x^2 - \frac{1}{6}x^3$ to 3^{rd} order.

Facts Can add, subtract, multiply expansions, result correct to same order

(divide using $\frac{1}{1-u} = 1+u+\ldots$)

(9) ⋆⋆ Find the 3rd order Taylor expansion of $\sqrt{x} - \frac{1}{4}x$ about $x = 4$.

(10) ⋆⋆ Find the 8th order expansion of $f(x) = e^{x^2} - \frac{1}{1+x^3}$. What is $f^{(6)}(0)$?

$e^u \approx 1 + u + \frac{1}{2}u^2 + \cdots$ so $e^{x^2} \approx 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{24}x^8$

$u = x^2$

$\frac{1}{1-u} = 1 + u + u^2 + \cdots$ so $\frac{1}{1+x^3} \approx 1 + (-x^3) + (-x^3)^2$

$u = -x^3$

don't need $(x^3)^3$

so $\boxed{e^{x^2} - \frac{1}{1+x^3} = x^2 + x^3 + \frac{1}{2}x^4 - 5/6\, x^6 + \frac{1}{24}x^8}$

to 8th order.

(15) (2023 Piazza @389) Find the asymptotics as $x \to \infty$

(a) ⋆⋆ $\sqrt{x^4+3x^3} - x^2$

since $\sqrt{x^4+3x^3} \sim x^2$ ~~need~~ need precise cancellation

Extract x^2:

$$\sqrt{x^4+3x^3} - x^2 = x^2\left(\sqrt{1+3/x} - 1\right)$$

Let $h = 3/x$, $\sqrt{1+h} \simeq 1 + \frac{1}{2}h - \frac{1}{8}h^2$

so $$x^2\left(\sqrt{1+3/x} - 1\right) \simeq x^2\left(\frac{1}{2}\cdot\frac{3}{x} - \frac{1}{8}\frac{3^2}{x^2}\right)$$

$$= \frac{3}{2}x - \frac{9}{8} + \ldots \sim \frac{3}{2}x$$

(b) ⋆⋆⋆ $\sqrt[3]{x^6 - x^4} - \sqrt{x^4 - \frac{2}{3}x^2}$

(worked to 2nd order, 1st order in h would have been enough)