1A3

7. Curve sketching; ~~Taylor expansion~~ (18/10/2023)

Goals.

(1) Convexity
(2) Curve sketching

Last Time. **Implicit diff.** if $f(x,y) = g(x,y)$ along some curve. Can diff relation ↗ wrt x, get relation between $x, y, \frac{dy}{dx}$.

Advice: use Liebnitz notation E.g. $\frac{d(\log y)}{dy} = \frac{1}{y}$

$$\frac{d(\log y)}{dx} = \frac{d(\log y)}{dy} \cdot \frac{dy}{dx} = \frac{1}{y}\frac{dy}{dx}$$

Related rates

In same situation can have x, y depend on s.
Then diff wrt s, get relation between $x, y, \frac{dx}{ds}, \frac{dy}{ds}$

Inverse trig: (1) defined arcsin, arccos, arctan (needed to restrict domain)

(2) differentiation: memorize

Math 100A – WORKSHEET 7
CURVE SKETCHING

1. Convexity and Concavity

(1) Consider the curve $y = x^3 - x$.

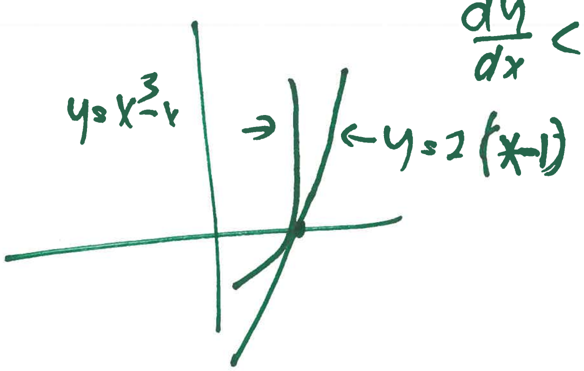
(a) ⋆ Find the line tangent to the curve at $x = 1$.

$\frac{dy}{dx} = 3x^2 - 1$ so $\left.\frac{dy}{dx}\right|_{x=1} = 2$, $y(1) = 0$ so line is

$y = 2(x-1)$

(b) ⋆⋆ Near $x = 1$, is the line above or below the curve? Hint: how does the slope of the curve behave to the right and left of the point?

near $x=1$, ~~slope~~ $\frac{dy}{dx} > 2$ if $x > 1$ so curve curves
$\frac{dy}{dx} < 2$ if $x < 1$ away from line:



Or: $x^3 - x = (x-1)(x^2+x) = (x-1)\left((x^2-2x+1) + 3(x-1) + 2\right)$
$= 2(x-1) + 3(x-1)^2 + (x-1)^3$

Date: 18/10/2023, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

$\approx 2(x-1) + 3(x-1)^2 \geq 2(x-1)$
↑
to 2nd order.

Conclusion: $f''(x) > 0 \iff$ tangent line lying beneath curve

Say f is **concave up**

$f''(x) < 0 \iff$ tangent line lying above curve

Say f is **concave down**

Def: If f is defined, cts at $x=a$ & if concavity changes at $x=a$ call a (or point $(a, f(a))$) an inflection point.

Either $f''(a)=0$ or $f''(a)$ is undefined

(2) For each curve find its domain; where is it concave up or down? Where are the inflection points.

(a) $y = x \log x - \frac{1}{2}x^2$.

$\frac{dy}{dx} = \log x + \frac{x}{x} - x$; $\frac{d^2y}{dx^2} = \frac{1}{x} - 1$

Domain: $x > 0$ ($\log x$ defined there)

Concave up: $\frac{1}{x} - 1 > 0 \iff x < 1$, ie. on $(0,1)$ — not $(-\infty,1)$ (outside domain)

Concave down: $\frac{1}{x} - 1 < 0 \iff x > 1$, ie. on $(1,\infty)$

inflection point at $(1, -\frac{1}{2})$ — but not $(-\infty,0)$

(a) Consider the curve $y = \sqrt[3]{x}$. Where is it continuous? Find where it is concave up and down.

domain: $(-\infty, \infty)$ ($x=0$ singular: $y'(0)$ undef)

$y'' = -\frac{2}{9}x^{-5/3} = -\frac{2}{9}\frac{1}{x^{5/3}}$

$y'' > 0$ if $x < 0$ concave up on $(-\infty, 0)$ $\cup$

$y'' < 0$ if $x > 0$ concave down on $(0, \infty)$ $\cap$

inflection point at $(0,0)$

2. Curve sketching

(3) ⋆⋆ Let $f(x) = \frac{x^2}{x^2+1}$ for which $f'(x) = \frac{2x}{(x^2+1)^2}$ and $f''(x) = \frac{2(1-3x^2)}{(x^2+1)^3}$.

(a) What are the domain and intercepts of f? What are the asymptotics at $\pm\infty$? Are there any vertical asymptotes? What are the asymptotices there?

Since $x^2+1>0$ for all x, f is defined on $(-\infty, \infty)$

As $x \to \pm\infty$, $\frac{x^2}{x^2+1} \sim \frac{x^2}{x^2} = 1$ so have horizontal asymptote $y=1$ on both sides

(no vertical asymptote).

(b) What are the intervals of increase/decrease? The local and global extrema?

Since $\frac{2}{(x^2+1)^2} > 0$, $f'(x)$ has same sign as x.

so f is increasing on $(0,\infty)$

decreasing on $(-\infty, 0)$

so have global minimum at $x=0$, where $f(0)=0$

(critical point)

(c) What are the intervals of concavity? Any inflection points?

since $\frac{2}{(x^2+1)^3} > 0$ for all x, $f''(x)$ has same sign as $1-3x^2$:

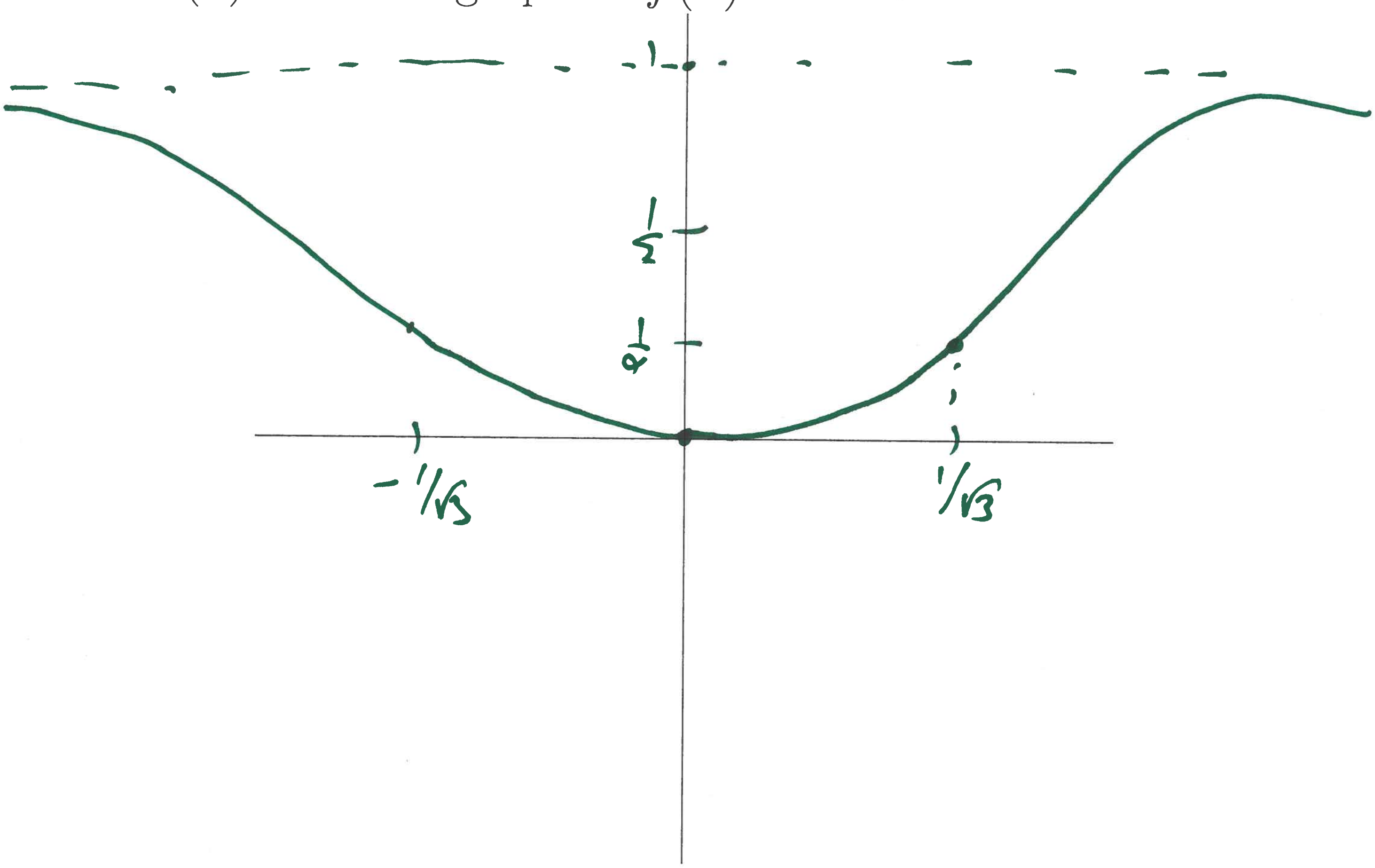
So $f''(x) > 0$ if $1-3x^2 > 0$ if $3x^2 < 1$ if $|x| < \frac{1}{\sqrt{3}}$

< 0 " < 0 " $3x^2 > 1$ if $|x| > \frac{1}{\sqrt{3}}$

So concave up on $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, concave down on $(-\infty, -\frac{1}{\sqrt{3}})$, $(\frac{1}{\sqrt{3}}, \infty)$

inflection pts at $x = \pm\frac{1}{\sqrt{3}}$, $y = 1/4$

(d) Sketch a graph of $f(x)$.



(4) ★★ Let $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$.

(a) What are the domain and intercepts of f? What are the asymptotics at $\pm\infty$? Are there any vertical asymptotes? What are the asymptotices there?

Domain is $(-\infty, \infty)$. As $x \to \pm\infty$, $-\frac{(x-\mu)^2}{2\sigma^2} \to -\frac{x^2}{2\sigma^2} \to -\infty$

so $\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$ will decay rapidly & has the horizontal asymptotes $y=0$.

(b) What are the intervals of increase/decrease? The local and global extrema?

$f'(x) = -\frac{1}{\sqrt{2\pi\sigma^6}}(x-\mu)e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ has same sign as $-(x-\mu)$

$\left(\frac{1}{\sqrt{2\pi\sigma^6}} \exp(*) > 0\right)$

so f increasing if $x<\mu$

decreasing if $x>\mu$

has its maximum at $\left(\mu, \frac{1}{\sqrt{2\pi\sigma^2}}\right)$. (critical pt)

(c) What are the intervals of concavity? Any inflection points?

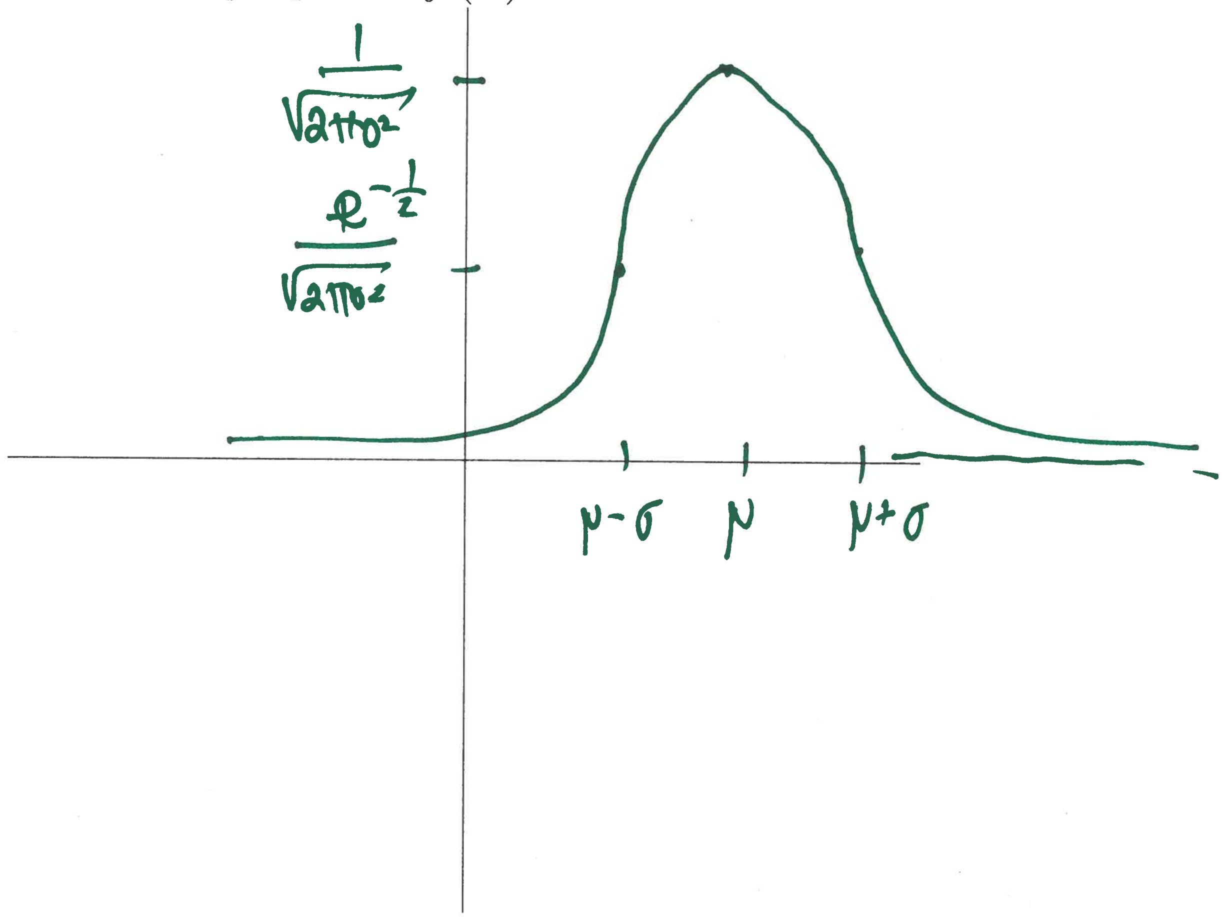
$f''(x) = \frac{1}{\sqrt{2\pi\sigma^{10}}} \exp(*) \left(\frac{(x-\mu)^2}{\sigma^2} - 1 \right)$ which has same sign as $\left((x-\mu)^2 - \sigma^2 \right)$

So f is concave up if $|x-\mu| > \sigma$ – on $(-\infty, \mu-\sigma)$, $(\mu+\sigma,$

down if $|x-\mu| < \sigma$ – on $(\mu-\sigma, \mu+\sigma)$

$\Rightarrow$ inflection pts at $\left(\mu \pm \sigma, \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}} \right)$

(d) Sketch a graph of $f(x)$.



(5) (Final, December 2007) ⋆⋆ Let $f(x) = x\sqrt{3-x}$.

(a) Find its domain, intercepts, and asymptotics at the endpoints.

(b) What are the intervals of increase/decrease? The local and global extrema?

$f(x) = x\sqrt{3-x}$

common denom

$$f'(x) = \sqrt{3-x} - \frac{x}{2\sqrt{3-x}} = \frac{2(3-x) - x}{2\sqrt{3-x}} = \frac{6-3x}{2\sqrt{3-x}}$$

$$= \frac{3}{2} \cdot \frac{2-x}{\sqrt{3-x}}$$