1A3

4. Computing derivatives (27/9/2023)

Goals.

(1) Combining linear approximations
(2) Linearity of the derivative
(3) The product and quotient rules

Last Time. Defined the **derivative**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(if limit exists, else f is not **differentiable** at x)

$\iff$ **linear approximation**

$$f(x+h) \approx f(x) + f'(x)h$$

Message: all (diff.) functions are approximately linear

WS 1 (a), (b), (c)

Math 100A – WORKSHEET 4
COMPUTING DERIVATIVES

1. Review of the derivative

(1) Expand $f(x+h)$ to linear order in h for the following functions and read the derivative off:

(a) $\star$ $f(x) = bx$

$f(x+h) = b(x+h) = bx + bh$

so $f'(x) = b$

(b) $\star$ $g(x) = ax^2$

to 1st order in h

$g(x+h) = a(x+h)^2 = ax^2 + 2axh + ah^2 \approx ax^2 + (2ax)h$

So $g'(x) = 2ax$

(c) $\star$ $h(x) = ax^2 + bx$.

① Can write $h(x+h) = a(x+h)^2 + \cancel{bh}$ $b(x+h)$ $= ax^2 + 2axh + ah^2 + bx + bh$

$\approx (ax^2 + bx) + (2ax+b)h$

neglecting ah^2.

Get $h'(x) = 2ax + b$

② Or: $f(x+h) \approx bx + bh$, $g(x+h) \approx ax^2 + (2ax)h$ so

Date: 27/9/2023, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

$(f+g)(x+h) = f(x+h) + g(x+h) \approx (ax^2 + bx) + (2ax+b)h$.

(can add the approximations)

In general, if ~~f(x)~~ $f(x+h) \approx f(x) + f'(x)h$

$g(x+h) \approx g(x) + g'(x)h$

Then $\underbrace{f(x+h) + g(x+h)}_{= (f+g)(x+h)} \approx (\underbrace{f(x) + g(x)}_{(f+g)(x)}) + (f'(x) + g'(x))h$

must be

Conclusion: $\boxed{\begin{array}{c}(f+g)'(x) = f'(x) + g'(x) \\ \text{or} \quad (f+g)' = f' + g'\end{array}}$ ("sum rule")

In general: $\boxed{(af + bg)' = af' + bg'}$ (a, b constants)

("linearity of the derivative")

1 (e)

(d) ⋆⋆ $i(x) = \frac{1}{b+x}$

(e) ⋆⋆⋆ $j(x) = 4x^4 + 5x$ (hint: use the known linear approximation to $2x^2$)

$j(x) = (2x^2)^2 + 5x$ so $j(x+h) = (2(x+h)^2)^2 + 5(x+h)$

$\approx (2x^2 + 4x \cdot h)^2 + 5x + 5h$

neglected $16x^2h^2 \ll h$ $\approx 4x^4 + 16x^3 \cdot h + 5x + 5h$

$\approx (4x^4 + 5x) + (16x^3 + 5)h$

<u>lesson</u>: can replace $2(x+h)^2$ with its linear approx

so $j'(x) = 16x^3 + 5$

Similarly, suppose $f(x+h) \approx f(x) + f'(x)h$

$g'(x+h) \approx g(x) + g'(x)h$

Then $(fg)(x+h) = f(x+h)g(x+h)$

$$\approx (f(x)+f'(x)h)(g(x)+g'(x)h)$$

$$\approx f(x)g(x) + f'(x)h\,g(x) + f(x)g'(x)h + f'(x)g'(x$$

$$\approx (fg)(x) + [f'(x)g(x)+f(x)g'(x)]h$$

to 1^{st} order in h

Conclusion: ("product rule").

$$\boxed{\begin{aligned} (fg)'(x) &= f'(x)g(x) + f(x)g'(x) \\ (fg)' &= f'g + fg' \end{aligned}}$$

Similarly ("quotient rule") if $g(x) \neq 0$,

$$\boxed{\begin{aligned} \left(\frac{f}{g}\right)'(x) &= \frac{f'(x)}{g(x)} - \frac{f(x)}{g(x)^2}g'(x) = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2} \\ \left(\frac{f}{g}\right)' &= \frac{f'g-fg'}{g^2}, \quad \frac{1}{g} = -\frac{g'}{g^2} \end{aligned}}$$

(2) Differentiate

(a) $\star$ $f(x) = 6x^{\pi} + 2x^{e} - x^{7/2}$

By linearity, and the power law rule

$$f'(x) = 6\pi x^{\pi-1} + 2e \cdot x^{e-1} - \frac{7}{2} x^{5/2}$$

(b) $\star$ (Final, 2016) $g(x) = x^2 e^x$ (and then also $x^a e^x$)

$$g'(x) = (x^2 \cdot e^x)' = (x^2)' \cdot e^x + x^2 \cdot (e^x)' = 2xe^x + x^2 e^x$$

(c) $\star$ (Final, 2016) $h(x) = \frac{x^2+3}{2x-1}$

$$h'(x) = \frac{(2x)(2x-1)-(x^2+3)\cdot 2}{(2x-1)^2} = 2\frac{x^2-x-3}{(2x-1)^2}$$

(d) $\star$ $\frac{x^2+A}{\sqrt{x}}$

$$\left(\frac{x^2+A}{\sqrt{x}}\right)' = \frac{2x\cdot\sqrt{x}-(x^2+A)\frac{1}{2\sqrt{x}}}{(\sqrt{x})^2} = 2\sqrt{x} - \frac{1}{2}\sqrt{x} - \frac{A}{2}\frac{1}{\sqrt{x}} = \frac{3}{2}x^{\frac{1}{2}} - \frac{A}{2}$$

or:

$$\frac{x^2+A}{\sqrt{x}} = x^{3/2} + Ax^{-\frac{1}{2}} \quad \text{so} \quad \frac{d}{dx}\left(\frac{x^2+A}{\sqrt{x}}\right) = \frac{3}{2}x^{\frac{1}{2}} - \frac{A}{2}x^{-\frac{3}{2}}$$

(3) $\star$ Let $f(x) = \frac{x}{\sqrt{x}+A}$. Given that $f'(4) = \frac{3}{16}$, give a quadratic equation for A.

interpreting this fact

$$\frac{df}{dx}(x) = \frac{\sqrt{x}+A-x\cdot\frac{1}{2\sqrt{x}}}{(\sqrt{x}+A)^2} = \frac{\frac{1}{2}\sqrt{x}+A}{(\sqrt{x}+A)^2}$$

$$\text{so} \quad f'(4) = \frac{1+A}{(2+A)^2} = \frac{A+1}{(A+2)^2} \quad \text{so} \quad \frac{A+1}{(A+2)^2} = \frac{3}{16}$$

$$\Leftrightarrow 3(A+2)^2 = 16(A+1) \Leftrightarrow 3A^2 - 4A - 4 = 0$$

(4) Suppose that $f(1) = 1$, $g(1) = 2$, $f'(1) = 3$, $g'(1) = 4$.

(a) $\star$ What are the linear approximations to f and g at $x = 1$? Use them to find the linear approximation to fg at $x = 1$.

$$\begin{cases} f(1+h) \approx 1 + 3h \\ g(1+h) \approx 2 + 4h \end{cases}$$

so

$$(fg)(1+h) \approx (1+3h)(2+4h) = 2 + 10h + 12h^2 \approx 2 + 10h$$

(or: $(fg)(1) = 2$, $(fg)'(1) = 10$ so $(fg)(1+h) \approx 2 + 10h$)

(b) $\star$ Find $(fg)'(1)$ and $\left(\frac{f}{g}\right)'(1)$.

$$(fg)'(1) = f'(1)g(1) + f(1)g'(1) = 3 \cdot 2 + 1 \cdot 4 = 10$$

$$\left(\frac{f}{g}\right)'(1) = \frac{f'(1)g(1) - f(1)g'(1)}{(g(1))^2} = \frac{3 \cdot 2 - 1 \cdot 4}{2^2} = \frac{1}{2}.$$

(Lesson: can compute $(fg)'(1)$ without formula for f, g)

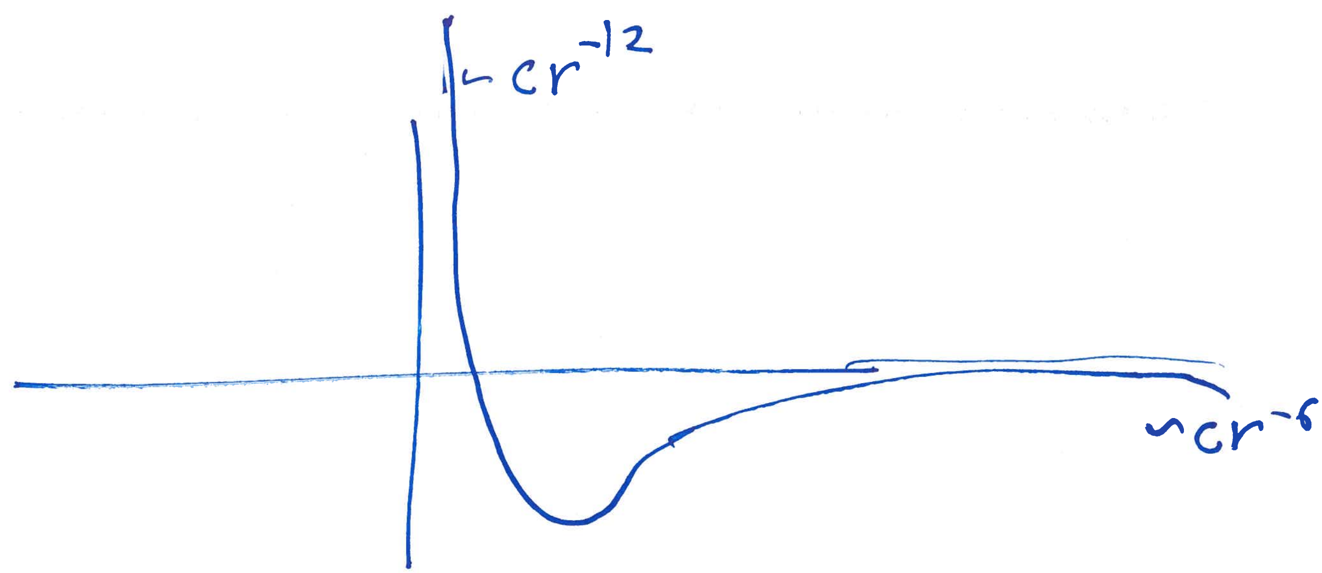
(6) The *Lennart–Jones potential* $V(r) = \epsilon\left(\left(\frac{R}{r}\right)^{12} - 2\left(\frac{R}{r}\right)^{6}\right)$ models the electrostatic potential energy of a diatomic molecule. Here $r > 0$ is the distance between the atoms and $\epsilon, R > 0$ are constants.

(a) $\star$ What are the asymptotics of $V(r)$ as $r \to 0$ and as $r \to \infty$?

As $r \to 0$, $V(r) \sim \varepsilon\left(\frac{R}{r}\right)^{12}$

As $r \to \infty$, $V(r) \sim -2\varepsilon\left(\frac{R}{r}\right)^{6}$

(b) Sketch a plot of $V(r)$.



(c) Find the derivative $\frac{dV}{dr}(r) =$

$$\frac{dV}{dr}(r) = -12\varepsilon \frac{R^{12}}{r^{13}} + 12\varepsilon \frac{R^6}{r^7} = \frac{12\varepsilon}{r}\left(\left(\frac{R}{r}\right)^6 - \left(\frac{R}{r}\right)^{12}\right),$$

(d) Where is $V(r)$ increasing? decreasing? Find its minimum location and value.

$V'(r) > 0$ if $\left(\frac{R}{r}\right)^6 > \left(\frac{R}{r}\right)^{12}$ i.e. if $\frac{R}{r} < 1$, i.e. $r > R$

$V'(r) < 0$ if $\left(\frac{R}{r}\right)^6 < \left(\frac{R}{r}\right)^{12}$ i.e. if $\frac{R}{r} > 1$, i.e. $r < R$

$\Rightarrow$ minimum at $r = R$.

(where $V(R) = \varepsilon$)