2. Limits & Asymptotes (22/9/2023)

Goals.

(1) Limits of functions
(2) Existence and nonexistence of limits: blowup
(3) Asymptotes

Last Time.

Asymptotics: Finding a new $\boxed{\text{function}}$ which gives a simpler description of the behaviour of a given function in some limit (esp $x \to \infty$, $x \to -\infty$, $x \to 0$)

Today: **Limits**, i.e. finding a number which describes the function in some limit

$\boxed{\text{Review: WS 1}}$

When a function is "well-behaved" (tech: **continuous**) as $x \to x_0$, $f(x)$ will get closer to $f(x_0)$, and $\lim_{x \to x_0} f(x) = f(x_0)$

But the values $f(x)$ can approach a value L as $x \to x_0$, even if $f(x_0)$ is undefined or if f is "bad" near x_0.

Math 100C – WORKSHEET 2
LIMITS AND ASYMPTOTES

(1) Review of asymptotics: analyze the expression $\frac{e^x+A\sin x}{e^x-x^2}$ as $x\to\infty$, $x\to 0$, $x\to-\infty$.

As $x\to\infty$ $A\sin x$ is bounded, so dominated by e^x, Also $e^x-x^2 \sim e^x$ so $\frac{e^x+A\sin x}{e^x-x^2} \sim \frac{e^x}{e^x} \sim 1$ as $x\to\infty$

As $x\to 0$, $e^x+A\sin x \sim 1+0=1$, $e^x-x^2 \sim 1$ so as $x\to 0$, $\frac{e^x+A\sin x}{e^x-x^2} \sim 1$

As $x\to-\infty$, $e^x\to 0$, so $e^x-x^2 \simeq -x^2$, $\frac{e^x+A\sin x}{e^x-x^2} \sim -\frac{e^x+A\sin x}{x^2}$

but, as $x\to-\infty$ $e^x, A\sin x$ don't dominate each other, so no simple asymptotic

1. Limits

(2) Either evaluate the limit or explain why it does not exist. Sketching a graph might be helpful.

(a) $\lim_{x\to5}(x^3-x) = 5^3-5 = 120$

Date: 13/9/2023, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

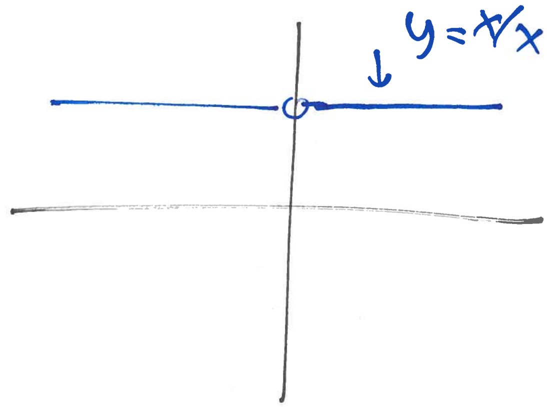
Example 1: f defined by formula, f is continuous where defined

Example 2: $\lim_{x\to 0} \frac{x}{x} = \lim_{x\to 0} 1 = 1$

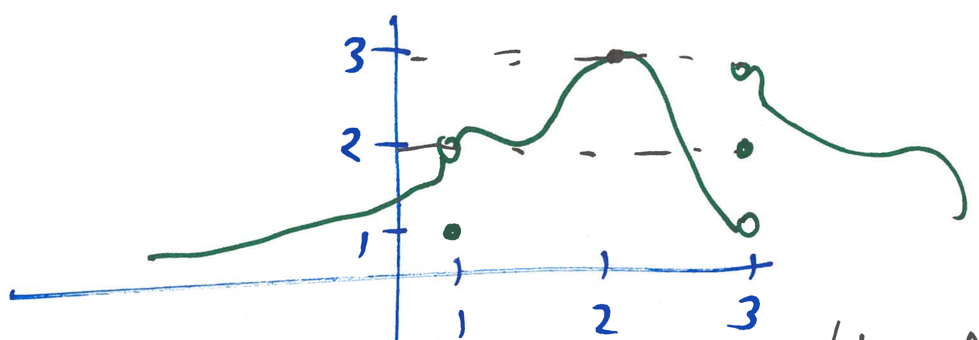
if $x \neq 0$, $\frac{x}{x} = 1$

("computing limit = plugging hole in graph")

Graphically



Example 3: f is given by



As $x \to 1$ from left & right, $f(x) \to 2$

Even though $f(1) = 1$.

as $x \to 2$

$\lim_{x\to 2} f(x) = 3$

$\lim_{x\to 3^-} f(x) = 1$

$\lim_{x\to 3^+} f(x) = 3$

$f(3) = 2$

Piecewise-defined function

(b) $\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} \sqrt{x} & 0 \le x < 1 \\ 3 & x = 1 \\ 2 - x^2 & x > 1 \end{cases}$.

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sqrt{x} = \sqrt{1} = 1$

if $x<1$
$f(x) = \sqrt{x}$

agree

so $\lim_{x\to 1} f(x) = 1$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2-x^2) = 2-1^2 = 1$

but $f(1) = 3 \neq 1$ so f is discontinuous at 1

(c) $\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} \sqrt{x} & 0 \le x < 1 \\ 1 & x = 1 \\ 4 - x^2 & x > 1 \end{cases}$.

Again $\lim_{x\to 1^-} f(x) = 1$

disagree

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (4-x^2) = 3$

$\Big\}$ so $\lim_{x\to 1} f(x)$ DNE.

Again f is discontinuous at 1

wrong: $\lim_{x\to 3} \frac{x-3}{x^2+x-12} = \frac{x-3}{(x-3)(x+4)} = \frac{1}{x+4} = \frac{1}{7}$

number ≠ function of x

(3) Let $f(x) = \frac{x-3}{x^2+x-12}$.

(a) (Final 2014) What is $\lim_{x\to 3} f(x)$?

[try: $\frac{3-3}{3^2+3-12}$ " = " $\frac{0}{0}$] ~> [$x^2+x-12 = (x-3)(x+4)$]

so $\frac{x-3}{x^2+x-12} = \frac{(x-3)}{(x-3)(x+4)} \underset{\text{if } x\neq 3}{=} \frac{1}{x+4} \xrightarrow[x\to 3]{} \frac{1}{7}$

(b) What about $\lim_{x\to -4} f(x)$?

for x near -4, $f(x) = \frac{1}{x+4}$

As x approaches -4, $x+4 \to 0$ so $\frac{1}{x+4}$ blows up

(~~takes~~ (takes very large values) so limit DNE.

In more detail, when $x > -4$, $\frac{1}{x+4} > 0$, $\lim_{x\to -4^+} f(x) = +\infty$

but when $x < -4$, $\frac{1}{x+4} < 0$, $\lim_{x\to -4^-} f(x) = -\infty$

(4) Evaluate

(a) $\lim_{x\to\infty} \frac{e^x + A\sin x}{e^x - x^2}$ $= 1$

We saw: as $x \to \infty$ $\dfrac{e^x + A\sin x}{e^x - x^2} \sim 1$

(b) $\lim_{x\to 0} \frac{e^x + A\sin x}{e^x - x^2}$ $= \dfrac{e^0 + A\sin 0}{e^0 - 0^2} = 1$

(c) $\lim_{x\to -\infty} \frac{e^x + A\sin x}{e^x - x^2}$ $= 0$

$\dfrac{e^x + A\sin x}{e^x - x^2} \sim -\dfrac{e^x + A\sin x}{x^2}$ as $x \to -\infty$ ($e^x + A\sin x$ is ~~[illegible]~~ bounded (as $x \to -\infty$) while $-x^2 \to -\infty$)

(5) Evaluate

(a) $\lim_{x\to 2} \frac{x+1}{4x^2-1}$ $= \dfrac{2+1}{4\cdot 2^2 - 1} = \dfrac{3}{15} = \dfrac{1}{5}$

(b) (Final, 2014) $\lim_{x\to -3^+} \frac{x+2}{x+3}$.

As $x \to -3^+$, $\dfrac{x+2}{x+3} \sim \dfrac{-1}{x+3} \xrightarrow[x \to -3^+]{} -\infty$

expression blows up at -3
in this limit $x+3>0$

(c) $\lim_{x\to 1} \frac{e^x(x-1)}{x^2+x-2}$

(d) $\lim_{x\to -2^-} \frac{e^x(x-1)}{x^2+x-2}$

(e) $\lim_{x\to 1} \frac{1}{(x-1)^2}$

(f) $\lim_{x\to 4} \frac{\sin x}{|x-4|}$

As $x \to 4$ $\frac{\sin x}{|x-4|} \sim \frac{\sin 4}{|x-4|} \xrightarrow[x\to 4]{} -\infty$

Since $\pi < 4 < 2\pi$, $\sin 4 < 0$

(g) $\lim_{x\to \frac{\pi}{2}^+} \tan x$, $\lim_{x\to \frac{\pi}{2}^-} \tan x$.

Examples

$$\lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} x = 0$$

$$\lim_{x\to 0} \frac{x}{x} = \lim_{x\to 0} 1 = 1$$

$$\lim_{x\to 0} \frac{x}{x^2} = \lim_{x\to 0} \frac{1}{x} \quad \text{DNE}$$

Example: WS 4

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2. Limits at infinity

(6) Evaluate

(a) $\lim_{x\to\infty} \frac{x^2+1}{x-3}$

So $x \to \infty$ $\frac{x^2+1}{x-3} \sim \frac{x^2}{x} = x \xrightarrow[x\to\infty]{} \infty$

wouldn't write

$$\lim_{x\to\infty} \frac{x^2+1}{x-3} = \lim_{x\to\infty} \frac{x^2}{x} = -$$

not equal

only asymptotic to each other

(b) (Final, 2015) $\lim_{x\to-\infty} \frac{x+1}{x^2+2x-8}$

As $x \to -\infty$, $\frac{x+1}{x^2+2x-8} \sim \frac{x}{x^2} \sim \frac{1}{x} \xrightarrow[x\to-\infty]{} 0$

(c) (Quiz, 2015) $\lim_{x\to-\infty} \frac{3x}{\sqrt{4x^2+x}-2x}$

We see: $4x^2+x \sim 4x^2$

so $\sqrt{4x^2+x} \sim -2x$

so $\frac{3x}{\sqrt{4x^2+x}-2x} \sim \frac{3x}{-2x-2x} \sim \frac{3x}{-4x} \sim -\frac{3}{4}$

and the limit is $-\frac{3}{4}$.

Let $x=-y$, $y\to\infty$

Then $\frac{-3y}{\sqrt{4y^2-y}+2y} \sim \frac{-3y}{\sqrt{4y^2}+2y} \underset{\substack{y>0 \\ \text{as } y\to\infty}}{\sim} \frac{-3y}{2y+2y} = -\frac{3}{4}$

Summary

① If as x gets closer to x_0, $f(x)$ gets closer to a value L, we say "f tends to the limit L as $x \to a$"; write

$$\lim_{x \to a} f(x) = L$$

if not, we say the limit **does not exist** (DNE)

② If f defined by formula which makes sense at a then $\lim_{x \to a} f(x) = f(a)$

③ If lim DNE, maybe one-sided limits exist

④ If lim DNE, maybe $f(x) \to \infty$ or $f(x) \to -\infty$.

("limit in extdended sense")

Say f has a <u>vertical asymptote</u> at point of blowup

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⑤ Can also compute limits as $x \to \infty$ & $x \to -\infty$

if $\lim_{\substack{x \to \infty \\ -\infty}} f(x) = L$ say f has the <u>horizontal asymptote</u> $y = L$