1A1

8. Taylor expansion (25/10/2023)

Goals.

(1) Review: ~~calculus and the shape of the graph~~ linear approximation
(2) ~~Optimization of functions~~ Taylor expansion
(3) ~~Problem solving: optimization problems~~ Manipulating expansions

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Last Time. Curve Sketching

Extract info from expression for a function, create picture featuring info

0'th derivative info: asymptotics, $f>0$, $f<0$, discontinuities, ..

1st derivative info: increase, decrease ($f'>0$, $f'<0$) $\Rightarrow$ local/global min, max, critical/singular pts

2nd derivative info: concavity, inflection pts

WS 1

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linear approx: $f(x) \approx f(a) + f'(a)\cdot(x-a)$

$\Leftrightarrow$ $f(a+h) \approx f(a) + f'(a)\cdot h$

Math 100A – WORKSHEET 8
TAYLOR EXPANSION

1. Taylor expansion

(1) (Review) Use linear approximations to estimate:

(a) $\star$ $\log\frac{4}{3}$ and $\log\frac{2}{3}$. Combine the two for an estimate of $\log 2$.

let $f(x) = \log x$, $f'(x) = \frac{1}{x}$ so $f(1) = 0$, $f'(1) = 1$

$f(x) \approx 1\cdot(x-1) = (x-1)$

so $\log\left(\frac{4}{3}\right) = \log\left(1+\frac{1}{3}\right) \approx \frac{1}{3}$

$\log\left(\frac{2}{3}\right) \approx \log\left(1-\frac{1}{3}\right) \approx 0-\frac{1}{3} \approx -\frac{1}{3}$

$\log 2 = \log\left(\frac{4/3}{2/3}\right) = \log\left(\frac{4}{3}\right) - \log\left(\frac{2}{3}\right) \approx \frac{1}{3} - \left(-\frac{1}{3}\right) = 2/3$

(b) $\star$ $\sin 0.1$ and $\cos 0.1$.

$\sin(0) = 0$, $(\sin\theta)'\big|_{\theta=0} = \cos 0 = 1$ so $\sin(0.1) \approx 0 + 1\cdot 0.1 = 0.1$

$\cos(0) = 1$, $(\cos\theta)'\big|_{\theta=0} = -\sin 0 = 0$ so $\cos(0.1) \approx 1 + 0\cdot 0.1 = 1$

Date: 25/10/2023, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(2) Let $f(x) = e^x$

(a) Find $f(0), f'(0), f^{(2)}(0), \cdots$

(b) Find a polynomial $T_0(x)$ such that $T_0(0) = f(0)$.

(c) Find a polynomial $T_1(x)$ such that $T_1(0) = f(0)$ and $T_1'(0) = f'(0)$.

(d) Find a polynomial $T_2(x)$ such that $T_2(0) = f(0)$, $T_2'(0) = f'(0)$ and $T_2^{(2)}(0) = f^{(2)}(0)$.

(e) Find a polynomial $T_3(x)$ such that $T_3^{(k)}(0) = f^{(k)}(0)$ for $0 \le k \le 3$.

$f(0)=1$, $f'(0)=1$, $f''(0)=1$, for all k, $f^{(k)}(x)=e^x$, so $f^{(k)}(0)=1$.

notation for k'th derivative

(b) Take $T_0(x)=1$

(c) add linear term to keep $T_1(0)=1$, want slope 1 so use $T_1(x)=1+x$ ("linear approximation)

(d) Try $T_2(x)=1+x+cx^2$

so $T_2(0)=1$, $T'(0)=1+2cx\big|_{x=0}=1$, $T_2''(0)=2c$

so $T_2''(0)=1$ if $2c=1$, i.e. if $c=\frac{1}{2}$, choose

$\boxed{T_2(x)=1+x+\frac{1}{2}x^2}$ quadratic correction

(e) Try $T_3(x)=1+x+\frac{1}{2}x^2+dx^3$

$T_3(0)=1$; $T_3'(0)=[1+x+3dx^2]_{x=0}=1$; $T_3^{(2)}(0)=[1+6dx]_{x=0}=1$;

$T_3^{(3)}(0)=6d$ so choose $d=\frac{1}{6}$, get $\boxed{T_3(x)=1+x+\frac{1}{2}x^2+\frac{1}{6}x^3}$

Conclusion

(If f has many derivatives at $x=a$)

Can make polynomial $T_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots + c_n(x-a)^n$

which matches f to nth order at a,

in the sense that $T^{(k)}(a) = f^{(k)}(a)$ if $0 \le k \le n$

Diff k times, plug in $x=a$ get $T^{(k)}(a) = c_k \cdot k \cdot (k-1)(k-2) \cdots 1$

so want $c_k \cdot k! = f^{(k)}(a)$, i.e.

$$\boxed{c_k = \frac{1}{k!} f^{(k)}(a)}$$

recall; $k!$ (read: "k factorial") is the product $1 \cdot 2 \cdot 3 \cdots k$:

$0! = 1$

$1! = 1$

$2! = 1 \cdot 2 = 2$

$3! = 1 \cdot 2 \cdot 3 = 6$

$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$

Call $T_n(x) = f(a) + \frac{f^{(1)}(a)}{1}(x-a) + \frac{f^{(2)}}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a$

the nth **Taylor polynomial** **of f about a**

or **nth order expansion**

Let $c_k = \frac{f^{(k)}(a)}{k!}$. The nth order Taylor expansion of $f(x)$ about $x = a$ is the polynomial

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n$$

(4) ⋆ Find the 4th order MacLaurin expansion of $\frac{1}{1-x}$ (=Taylor expansion about $x = 0$)

let $g(x) = \frac{1}{1-x}$, so $g^{(1)}(x) = (1-x)^{-2}$, $g^{(2)}(x) = 2(1-x)^{-3}$,

$g^{(3)}(x) = 3\cdot2\cdot1\cdot(1-x)^{-4}$, $g^{(4)}(x) = 4\cdot3\cdot2\cdot1\,(1-x)^{-5}$

$g(0) = 1$, $g^{(1)}(0) = 1$, $g^{(2)}(0) = 1\cdot2$, $g^{(3)}(0) = 1\cdot2\cdot3$, $g^{(4)}(0) = 1\cdot2\cdot3\cdot4$

so $T_4(x) = 1 + \frac{1}{1}\cdot x + \frac{1\cdot2}{1\cdot2}x^2 + \frac{1\cdot2\cdot3}{1\cdot2\cdot3}x^3 + \frac{1\cdot2\cdot3\cdot4}{1\cdot2\cdot3\cdot4}x^4$

$= 1 + x + x^2 + x^3 + x^4$

$k!$

<u>Fact</u> Let $T_n(x)$ be the nth order expansion

As $x \to a$, $f(x) - T_n(x) = R_n(x) \to 0$

But remainder $R_n(x)$ decays <u>faster</u> than $(x-a)^n$, typically like $(x-a)^{n+1}$!

Don't always have $a = 0$:

Expand $\log x$ about $a = 1$ set:

$$\log x \approx (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \dots$$

if x close to 1, the <u>small parameter</u> is $x-1$

<u>or</u>

let $x - 1 = h$

$$\log(1+h) \approx h - \frac{1}{2}h^2 + \frac{1}{3}h^3 - \frac{1}{4}h^4 + \dots$$

(5) ⋆⋆ Find the nth order expansion of $\cos x$, and approximate $\cos 0.1$ using a 3rd order expansion

$\cos(0) = 1$, $(\cos\theta)^{(1)} = -\sin\theta$, $(\cos\theta)^{(2)} = -\cos\theta$, $(\cos\theta)^{(3)} = \sin\theta$

$(\cos\theta)^{(4)} = \cos\theta$, so derivatives at 0 are:

$1, 0, -1, 0, 1, 0, -1, 0, 1, 0, -1, 0, \ldots$

$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \ldots$

So $\cos(0.1) \approx 1 - \frac{1}{2}(0.1)^2 = 1 - \frac{1}{200}$

(feature: quadratic approx is correct to 3^{rd} order too)

(6) (Final, 2015) $\star$ Let $T_3(x) = 24 + 6(x-3) + 12(x-3)^2 + 4(x-3)^3$ be the third-degree Taylor polynomial of some function f, expanded about $a = 3$. What is $f''(3)$?

Have: $12 = \frac{f^{(2)}(3)}{2!}$ so $f^{(2)}(3) = 24$

(7) $\star\star$ In special relativity we have the formula $E = \frac{mc^2}{\sqrt{1-v^2/c^2}}$ for the kinetic energy of a moving particle. Here m is the "rest mass" of the particle and c is the speed of light. Examine the behaviour of this formula for small velocities by expanding it to second order in the *small parameter* $x = v^2/c^2$. What is the 4th order expansion of the energy? Do you recognize any of the terms?

Have $E = mc^2(1-x)^{-\frac{1}{2}}$

$E(x=0) = mc^2$, $E'(x=0) = \frac{1}{2}mc^2$, $E''(x=0) = \frac{3}{4}mc^2$

So to 2nd order in x,

$E \simeq mc^2 + \frac{1}{2}mc^2 x + \frac{1}{2}\cdot\frac{3}{4}mc^2 \cdot x^2$

$mc^2 + \frac{1}{2}mv^2 + \frac{3}{8}mv^2 \cdot \frac{v^2}{c^2}$

$x = v^2/c^2$

"first relativistic correction"

2. New expansions from old

Near $u = 0$: $\frac{1}{1-u} = 1 + u + u^2 + u^3 + u^4 \cdots$
$\exp u = 1 + \frac{1}{1!}u + \frac{1}{2!}u^2 + \frac{1}{3!}u^3 + \frac{1}{4!}u^4 + \cdots$

(8) ⋆ (Final, 2016) Use a 3rd order Taylor approximation to estimate $\sin 0.01$. Then find the 3rd order Taylor expansion of $(x+1)\sin x$ about $x = 0$.

(earlier: $\sin(\theta) \approx \theta - \frac{1}{6}\theta^3$ to 3^{rd} order)

$\sin\left(\frac{1}{100}\right) \approx \frac{1}{100} - \frac{1}{6{,}000{,}000}$.

to 3^{rd} order, $\sin x \approx x - \frac{1}{6}x^3$

$1 + x \approx 1 + x$

so $(1+x)\cdot\sin x \approx (1+x)\left(x - \frac{1}{6}x^3\right)$

$\approx x + x^2 - \frac{1}{6}x^3 \quad \cancel{\left(-\frac{1}{6}x^4\right)}$

(9) ⋆⋆ Find the 3rd order Taylor expansion of $\sqrt{x} - \frac{1}{4}x$ about $x = 4$.

(10) ⋆⋆ Find the 8th order expansion of $f(x) = e^{x^2} - \frac{1}{1+x^3}$. What is $f^{(6)}(0)$? $\exp(u) \approx 1 + u + \frac{1}{2}u^2 + \frac{1}{6}u^3 + \frac{1}{24}u^4$

From expansion of e^u, $e^{x^2} \approx 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{24}x^8$

$\frac{1}{1-u}$, $\frac{1}{1+x^3} \approx 1 + (-x^3) + (-x^3)^2$

So $e^{x^2} - \frac{1}{1+x^3} \approx 0 + x^2 + x^3 + \frac{1}{2}x^4 - \frac{5}{6}x^6 + \frac{1}{24}x^8$
correct to 8th order.

(15) (2023 Piazza @389) Find the asymptotics as $x \to \infty$

(a) ⋆⋆ $\sqrt{x^4 + 3x^3} - x^2$

Since $\sqrt{x^4+3x^3} \sim x^2$, extract x^2:

$$\sqrt{x^4+3x^3} - x^2 = x^2\left(\sqrt{1+3/x} - 1\right)$$

Let $h = 3/x$ so $\sqrt{1+h} \approx 1 + \frac{1}{2}h - \frac{1}{8}h^2 + \ldots$ (calculate)

so $\sqrt{1+3/x} \approx 1 + \frac{1}{2}\cdot\frac{3}{x} - \frac{1}{8}\cdot\frac{9}{x^2} - \ldots$

so $x^2\left(\sqrt{1+3/x} - 1\right) = x^2\left(\frac{3}{2}\cdot\frac{1}{x} - \frac{9}{8}\frac{1}{x^2} + \ldots\right) \approx \frac{3}{2}x - \frac{9}{8} + \ldots$

$\sim 3/2\, x$

(b) ⋆⋆⋆ $\sqrt[3]{x^6 - x^4} - \sqrt{x^4 - \frac{2}{3}x^2}$