1A)

6. Applications of the Chain rule (11/10/2023)

Goals.

(1) Implicit differentiation
(2) Inverse trig functions
(3) Related rates

Last Time. **Chain rule**

Newton notation $(f(g(x)))' = f'(g(x))\cdot g'(x)$

Leibnitz notation $\frac{df}{dx} = \frac{df}{dg}\cdot\frac{dg}{dx}$

Application: If $y = \log x$, $x = e^y$ so $1 = \frac{dx}{dx} = \frac{d(e^y)}{dx} = \frac{d(e^y)}{dy}\cdot\frac{dy}{dx}$

so $\frac{d(\log x)}{dx} = \frac{dy}{dx} = \frac{1}{\frac{d(e^y)}{dx}} = \frac{1}{e^y} = \frac{1}{x}$

$\Rightarrow \frac{d(\log f)}{dx} = \frac{df}{dx}\cdot\frac{d(\log f)}{df} = \frac{1}{f}\frac{df}{dx} \Rightarrow$ $\boxed{\frac{df}{dx} = f\cdot\frac{d(\log f)}{dx}}$

Log diff. rule

WS (2)

Math 100A – WORKSHEET 6
APPLICATIONS OF THE CHAIN RULE

1. Review

(1) Differentiate
   (a) $\star$ $e^{\sqrt{\cos x}}$

(2) (Final, 2014) $\star$ Let $y = x^{\log x}$. Find $\frac{dy}{dx}$ in terms of x only.

chain rule

$$\frac{dy}{dx} = y \cdot \frac{d(\log y)}{dx} = y \frac{d(\log x \cdot \log x)}{dx} = y \cdot 2\log x \cdot \frac{1}{x}$$

log. diff. rule

$$= 2x^{\log x - 1} \cdot \log x$$

Date: 11/10/2023, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

Implicit diff

Find line tangent to $4x^2+y^2=36$, passing through $(2, 2\sqrt{5})$

Because $4x^2+y^2=36$ along the curve, we can diff both side along the curve and get:

$$\frac{d}{dx}(4x^2+y^2)=\frac{d}{dx}(36)$$

$$\Rightarrow \quad 8x+2y\cdot\frac{dy}{dx}=0 \quad \Rightarrow \quad \frac{dy}{dx}=-\frac{8x}{2y}=\frac{4x}{y}$$

chain rule

$$\frac{d(y^2)}{dx}=\frac{d(y^2)}{dy}\cdot\frac{dy}{dx}=2y\frac{dy}{dx}$$

always get linear equation for $\frac{dy}{dx}$.

(if curve meets itself this will break down)

(If $y=0$ here have tangent line, but vertical - can check that $\frac{dx}{dy}=0$ there)

(haven't solved for $y(x)$ so can't write $\frac{dy}{dx}$ as fcn of x)

At $(2, 2\sqrt{5})$ get slope $-\frac{4\cdot 2}{2\sqrt{5}}=-\frac{4}{\sqrt{5}}$

and line $\boxed{y=-\frac{4}{\sqrt{5}}(x-2)+2\sqrt{5}}$

(5) (Final 2012) Find the slope of the line tangent to the curve $y + x\cos y = \cos x$ at the point $(0, 1)$.

(6) Find y'' (in terms of x, y) along the curve $x^5 + y^5 = 10$ (ignore points where $y = 0$).

$$\frac{d}{dx}(x^5 + y^5) = \frac{d}{dx}(10)$$

so $5x^4 + 5y^4 \dfrac{dy}{dx} = 0$ so $\dfrac{dy}{dx} = -\dfrac{x^4}{y^4}$

so $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = -\frac{4x^3}{y^4} + \frac{4x^4}{y^5}\frac{dy}{dx}$$

$$= -\frac{4x^3}{y^4} + \frac{4x^4}{y^5}\cdot\left(-\frac{x^4}{y^4}\right)$$

$$= -4\frac{x^3}{y^4} - 4\frac{x^8}{y^9}$$

Sanity check: $x^5 + y^5 = 10$ means x^5, y^5 have same units, so x, y have ~~units~~, same units,

so $\frac{dy}{dx}$ should be number without units

$\left(-\frac{x^4}{y^4}\right.$ is ok$)$

so units of $\frac{d\left(\frac{dy}{dx}\right)}{dx}$ are units of $\frac{1}{x}$

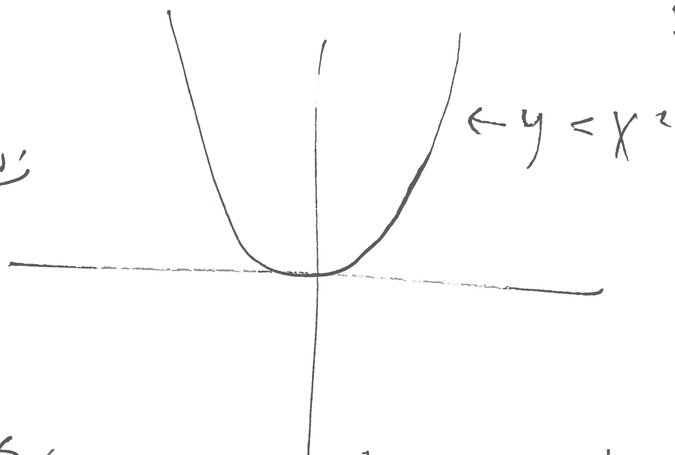
Expression is ok

Inverse trig functions

If $y=f(x)$ the **inverse function to f** is the solution x to the equation $y=f(x)$ given y.

Example: the inverse to $y=x^2$ on $[0,\infty)$ is $x=\sqrt{y}$ (domain $[0,\infty)$)

Draw:

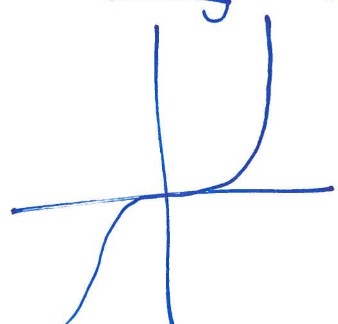


On $x\in(-\infty,\infty)$, the equation $y=x^2$ has two solutions (if $y\neq 0$), so can't have inverse fcn

So restrict domain of $y=x^2$ to $x\in[0,\infty)$ where every y value occurs once

Domain of inverse fcn is range of the function

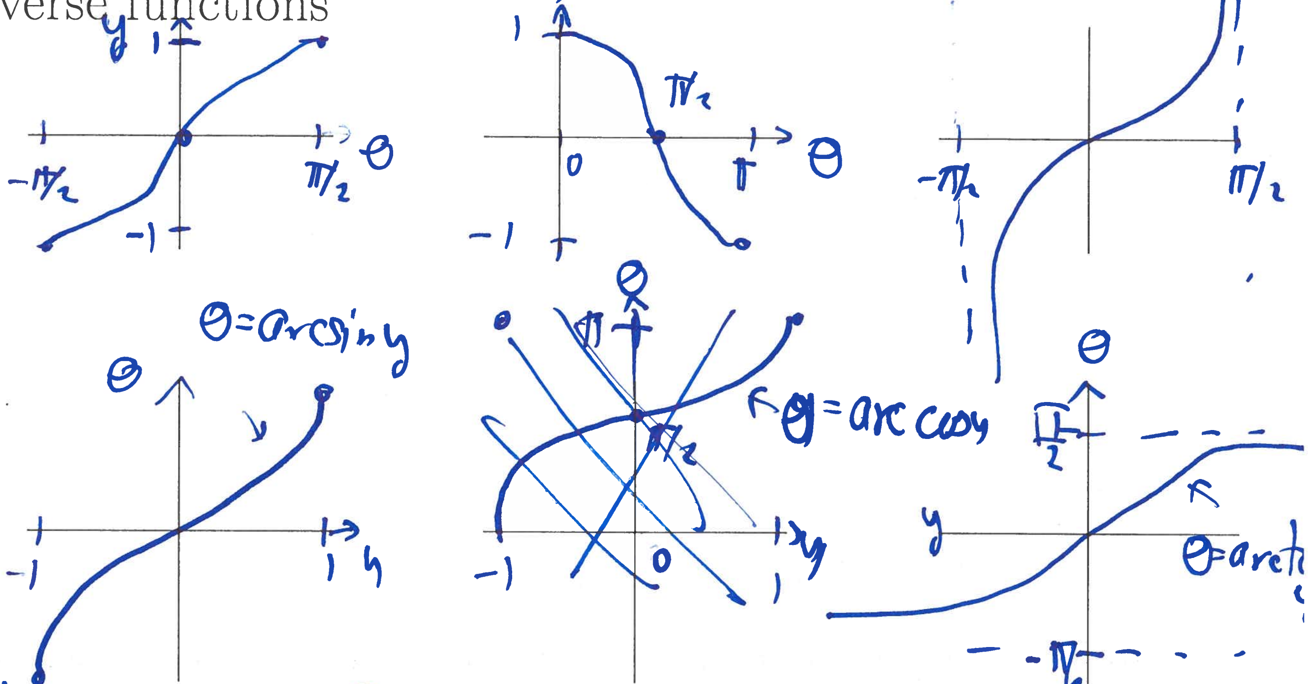
(Compare with $y=x^3$



here achieve every real number as y-value exactly once so $x=\sqrt[3]{y}$ defined on $\mathbb{R}$)

3. Inverse trig functions

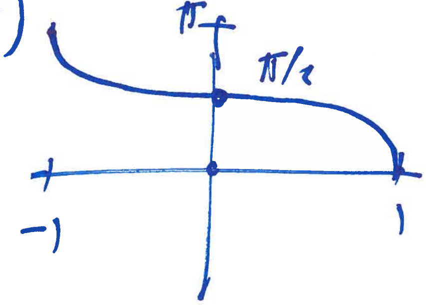
(7) Draw on the following axes graphs of $\sin\theta$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\cos\theta$ on $[0, \pi]$ and $\tan\theta$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then of their inverse functions



sinθ takes values in $[-1,1]$
So $\theta = \arcsin x$ will be defined if $x \in [-1,1]$
as the solution to $x = \sin\theta$ with $\theta \in [-\pi/2, \pi/2]$

$\theta = \arccos x$ is defined if $x \in [-1,1]$ as the solution to $x = \cos\theta$ with $\theta \in [0, \pi]$

$\theta = \arctan x$ is def for all x as solution to $\tan\theta = x$ with $\theta \in (-\pi/2, \pi/2)$



(8) Evaluation

(a) (Final 2014) Evaluate $\arcsin\left(-\frac{1}{2}\right)$ and $\arcsin\left(\sin\left(\frac{31\pi}{11}\right)\right)$.

Since $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$ so $\arcsin\left(-\frac{1}{2}\right) = -\pi/6$

(using $\sin(-\theta) = -\sin(\theta)$)

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true. $\theta = \frac{31\pi}{11}$ solves $\sin(\theta) = \sin\left(\frac{31\pi}{11}\right)$

instead: $\sin\left(\frac{31\pi}{11}\right) \underset{\uparrow}{=} \sin\left(\frac{31\pi}{11} - 2\pi\right) = \sin\left(\frac{9}{11}\pi\right) =$

$\sin(\theta + 2\pi) = \sin\theta$

$\underset{\uparrow}{=} \sin\left(\pi - \frac{9}{11}\pi\right) = \sin\left(\frac{2}{11}\pi\right)$ so $\arcsin\left(\frac{31\pi}{11}\right) = \frac{2}{11}\pi$

$\sin(\pi - \theta) = \sin\theta$

(cf $\sqrt{(-5)^2} = 5$ not -5)

Fact: If $\theta = \arcsin x$,

$$\sin\theta = x$$

diff along curve to get

$$\cos\theta \cdot \frac{d\theta}{dx} = \frac{dx}{dx} = 1$$

so $\dfrac{d\theta}{dx} = \dfrac{1}{\cos\theta} = \dfrac{1}{\sqrt{1-\sin^2\theta}} = \dfrac{1}{\sqrt{1-x^2}}$

$$\boxed{\begin{aligned} \frac{d(\arcsin x)}{dx} &= \frac{1}{\sqrt{1-x^2}} \\ \frac{d(\arccos x)}{dx} &= -\frac{1}{\sqrt{1-x^2}} \\ \frac{d(\arctan x)}{dx} &= \frac{1}{1+x^2} \end{aligned}}$$

(5) (Final 2012) Find the slope of the line tangent to the curve $y + x\cos y = \cos x$ at the point $(0, 1)$.

(for units: write $ax^5+by^5=0$)

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(6) Find y'' (in terms of x, y) along the curve $x^5 + y^5 = 10$ (ignore points where $y = 0$).

Diff along curve we get $5x^4 + 5y^4 \frac{dy}{dx} = 0$

thus $\frac{dy}{dx} = -\frac{x^4}{y^4}$ (can't be $-\frac{x^4}{y}$ because $[x]=[y]$, $\left[\frac{dy}{dx}\right]=1$)

$= -x^4y^{-4}$

So $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = -4x^3y^{-4} + 4x^4y^{-5}\frac{dy}{dx}$

$= -4x^3y^{-4} - 4x^8y^{-9}$