1. Asympototics (6/9/2023)

Today's Goals.

(1) Power laws, exponentials, and their asymptotics
(2) Asymptotics of sums
(3) Asymptotics of expressions

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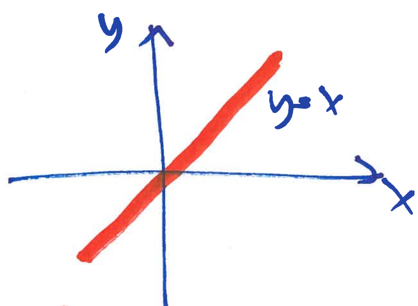
(1) Growth and decay

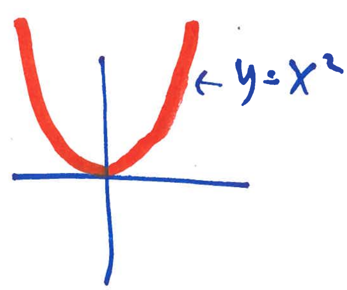
Two common patterns: power laws x^3, $x^{-\frac{1}{2}}$, $7x^7$, ..

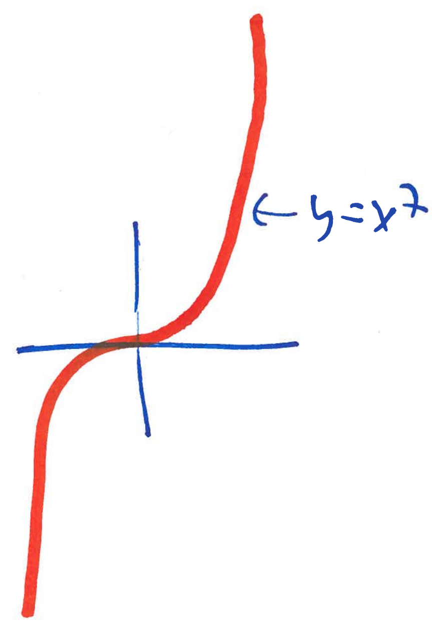
exponentials e^x, $\frac{1}{3^x}$.

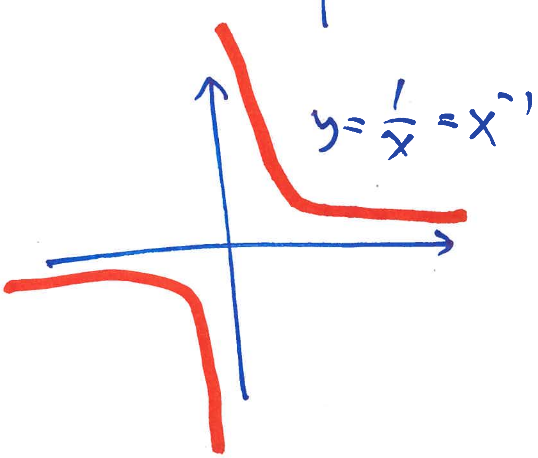
(WS 1)

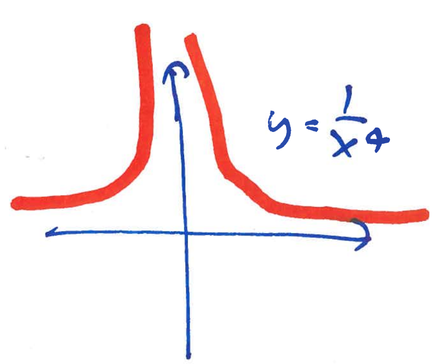
let's graph some power laws:











Math 100C – WORKSHEET 1
EXPRESSIONS AND ASYMPTOTICS

1. Asymptotics: simple expressions

(1) ⋆ Classify the following functions into *power laws / power functions* and *exponentials*: x^3, πx^{102}, e^{2x}, $c\sqrt{x}$, $-\frac{8}{x}$, 7^x, $8 \cdot 2^x$, $-\frac{1}{\sqrt{3}} \cdot \frac{1}{2^x}$, $\frac{9}{x^{7/2}}$, x^e, π^x, $\frac{A}{x^b}$.

Power Laws: x^3, πx^{102}, $[e^{2x} = (e^2)^x]$, $9 \cdot x^{-7/2}$, x^e, $A \cdot x^{-b}$

Exponentials: base, $8 \cdot 2^x$, $-\frac{1}{\sqrt{3}}\left(\frac{1}{2}\right)^x$, π^x

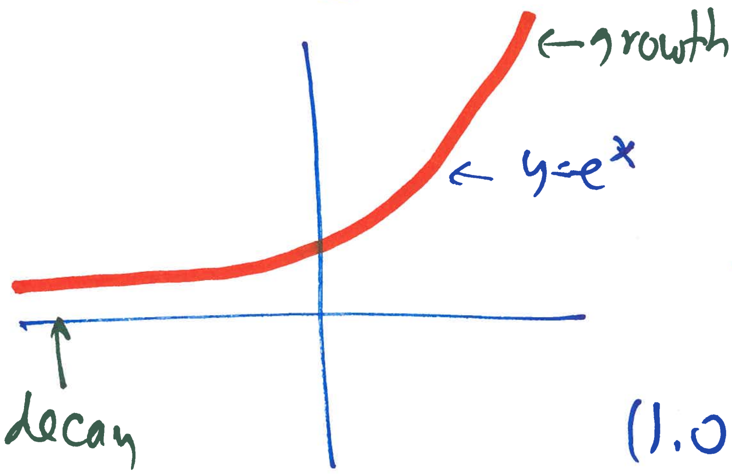
2 = exp. rate of growth

Date: 6/9/2023, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

Key idea 1: Power laws not all the same:

when x is big, x^7 is much bigger than x^2.
$1/x^4$ is much smaller than $1/x$

when x is near 0: x^7 is much smaller than x^2
$1/x^4$ is much bigger than $\frac{1}{x}$



Key idea 2: Exponentials beat power laws:

$(1.0001)^x$ will eventually dominate x^{1000}

$\Rightarrow$ decaying exponentials always decay faster than power laws

Let's compare $1000x^7$ & $\frac{1}{1000}x^2$ near 0

at $x=1$, $1000 \cdot 1^7 > \frac{1}{1000} 1^2$

② Combinations

Say $A \neq 0$. What does $A - x^3$ "look like" for large x? A: the second term will dominate,

as $x \to \infty$, $A - x^3 \sim -x^3$

"is asymptotic to"

What if x is small?

As $x \to 0$, $A - x^3 \sim A$

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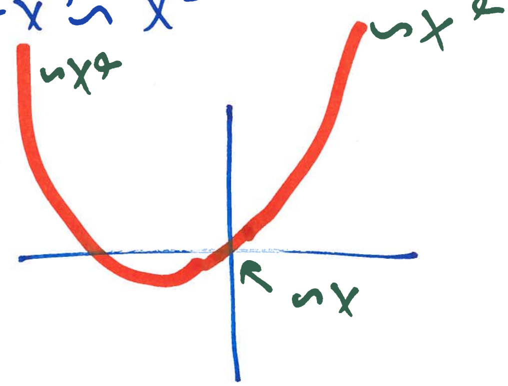
try: $x^3 + x^5 + x^7$

for x large (either $\to +\infty$ or $\to -\infty$) $x^3 + x^5 + x^7 \sim x^7$

for x small $(x \to 0)$ $x^3 + x^5 + x^7 \sim x^3$

~~plot~~ rough plot of $x + x^4$:



WS 2

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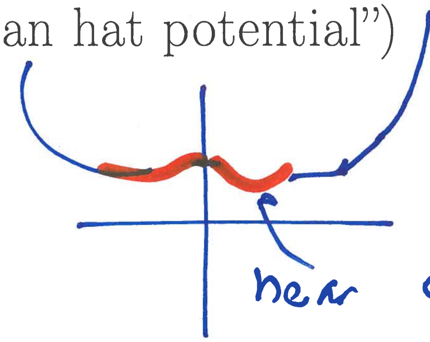
(2) ⋆ How does the each expression behave when x is large? small? what is x is large but negative? ⋆⋆ Sketch a plot

(a) $1 - x^2 + x^4$ ("Mexican hat potential")

As $x \to \infty$ $1-x^2+x^4 \sim x^4$

$x \to -\infty$ $1-x^3+x^4 \sim x^4$

$x \to 0$, $1-x^2+x^4 \sim 1$

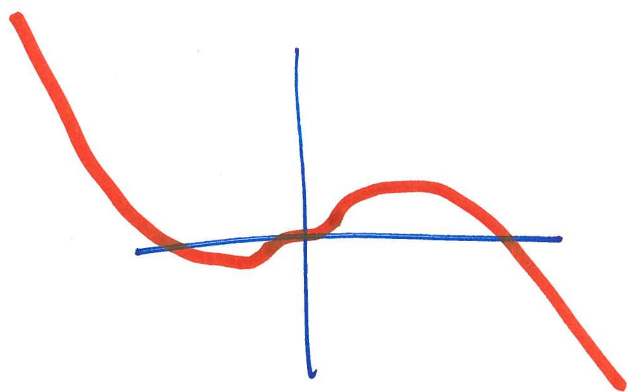


near 0, $1-x^2+x^4 < 1$
since x^2 is bigge
than x^4!

(b) $x^3 - x^5$

as $x \to 0$ $x^3 - x^5 \sim x^3$

$x \to \pm\infty$ $x^3 - x^5 \sim -x^5$



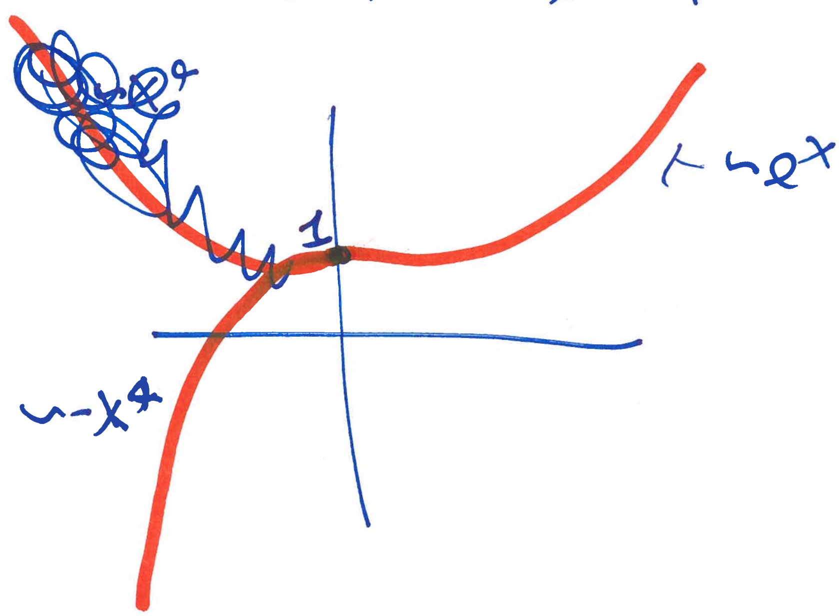
(c) $e^x - x^4$

As

$x \to \infty$, $e^x - x^4 \sim e^x$

$x \to 0$ $e^x - x^4 \sim 1$ ($e^0 = 1$, x^4 small if x near 0)

$x \to -\infty$ $e^x - x^4 \sim -x^4$ (e^x decays as $x \to -\infty$)



(e) Three strains of a contagion are spreading in a population, spreading at rates 1.05, 1.1, and 0.98 respectively. The total number of cases at time t behaves like

$$A \cdot 1.05^t + B \cdot 1.1^t + C \cdot 0.98^t \,.$$

(A, B, C are constants). Which strain dominates eventually? What would the number of infected people look like?

1.1^t grows fasterst, $A \cdot 1.05^t + B \cdot 1.1^t + C \cdot 0.98^t \sim B \cdot 1.1^t$

(also say "the number grows like 1.1^t")

(3) The (attractive) interaction between two hadrons (say protons) due to the strong nuclear force can be modeled by the *Yukawa potential* $V_{\mathrm{Y}}(r) = -g^2 \frac{e^{-\alpha m r}}{r}$ where r is the separation between the particles, and g, α, m are positive constants. The elecctrical repulsion between two protons is described by the Columb potential $V_{\mathrm{C}}(r) = kq^2 \frac{1}{r}$ where k, q are also positive constants. Which interaction will dominate for large distances? Will the net interaction be attractive or repulsive? Note that g^2 is much larger than kq^2.

As $r \to \infty$, $e^{-\alpha m r}$ decays so $-g^2 \frac{e^{-\alpha m r}}{r}$ is much smaller than $kq^2 \frac{1}{r}$

so $kq^2 \frac{1}{r} - g^2 \frac{e^{-\alpha m r}}{r} \sim kq^2 \frac{1}{r}$

~~For~~ As $r \to 0$, $e^{-\alpha m r} \sim 1$ so

$$kq^2 \frac{1}{r} - g^2 \frac{e^{-\alpha m r}}{r} \sim \frac{kq^2}{r} - \frac{g^2}{r} = (kq^2 - g^2)\frac{1}{r}$$

both of same scale

Aside: $x^2 + x^2 \sim 2x^2$ as $x \to \infty$

Last idea:

If we add $f+g$, only the larger dominates

If we multiply/divide $f \cdot g$ or f/g the behaviours multiply/divide

2. Asymptotics of complicated expressions

(4) Describe the following expressions in words

(a) $e^{|x-5|^3}$

The exponential of the cube of the absolute value of ~~xx~~ the difference of x~~x~~. and 5.

(b) $\frac{1+x}{1+2x-x^2}$

the ratio of . . -

as $x \to \infty$, $1+x \sim x$

$1+2x-x^2 \sim -x^2$ so $\frac{1+x}{1+2x-x^2} \sim \frac{x}{-x^2} \sim -\frac{1}{x}$