Math 535, Lecture 21 , 27/3/2023

Last time: G cpt Lie gp

(1) Weyl integration formula:

$$\#W \int_G f(g)\,dg = \int_T J(t)\,dt \int_G f(gtg^{-1})\,dg$$

$$J(t) = \det\left(Id - Ad_{t^{-1}} \big|\, \mathfrak{g}/\mathfrak{t} \right)$$

(2) Let $I \subset \mathfrak{t}^*$ be an additive subgp. Then $R_I = \mathbb{Z}[\{e \circ \nu\}_{\nu \in I}]$ is locally a UFD:

For any finite $A \subset I$, $R_{\langle A \rangle}$ is a UFD

Def: The singular set of $\mathfrak{t} = \mathrm{Lie}\, T$ is $\bigcup_{\beta \in \Phi} \beta^{-1}(\mathbb{Z})$, lower-dim set, complement is the regular set. Open, dense, full measure

$T_{reg} = \exp(\mathfrak{t}_{reg})$.

recall $e(z) = e^{2\pi i z}$

let $I = \{\mu \in \mathfrak{t}^* \mid \mu(\Gamma) \in \mathbb{Z}\} \supset \Lambda^*$

<u>Lemma:</u> Suppose $f \in R_I$ vanishes on $\beta^{-1}(\mathbb{Z})$ for some $\beta \in I$. Then $e \circ \beta - 1$ divides f in R_I.

<u>Cor:</u> Suppose f vanishes on t_{sing}. Then f is divisible by

$$\prod_{\beta > 0} (e \circ \beta - 1)$$

<u>Pf:</u> unique factorization, $\nearrow$ relatively prime.

<u>Def:</u> $\mathrm{sgn}: W \to \{\pm 1\}$ to be the unique hom s.t. $\mathrm{sgn}(s_\alpha) = -1$ (take $\det(w|\mathfrak{t})$).

<u>Def:</u> Call $f \in R_I$ <u>symmetric</u> if $f \circ w = f$ $\forall w \in W$, <u>alternating</u> if $f \circ w = \mathrm{sgn}(w) f$.

<u>Observe:</u> symmetric functions $R_I^W = \mathbb{Z}$-span of
$$d_\mu = \sum_{w \in W / W_\mu} e(\mu \circ w).$$

alternating f vanish on t_{sing}, since $\beta(H) \in \mathbb{Z}$ then $s_\beta(H) = H - \beta(H)\check{\alpha}$, so for all $\mu \in I$

$$\mu(s_\beta(H)) = \mu(H) - \beta(H)\mu(\alpha^\vee) \in \mu(H) + \mathbb{Z}$$

$$-f(H) = f(s_\beta H) = f(H). \quad \blacksquare$$

Example: (1) The Weyl denominator is

$$\delta(H) = \prod_{\beta>0}\left(e\left(\tfrac{1}{2}\beta(H)\right) - e\left(-\tfrac{1}{2}\beta(H)\right)\right)$$

$$= \left[e(-\rho)\cdot \prod_{\beta>0}\left(e\circ\beta - 1\right)\right](H)$$

$$= \left[e(\rho)\cdot \prod_{\beta>0}\left(1 - e\circ(-\beta)\right)\right](H)$$

(2) For $\lambda \in t^*$. $C_\lambda = \sum_{w\in W} \operatorname{sgn}(w)\cdot e(\lambda\circ w)$

Lemma: Both are alternating, δ vanishes exactly on singular set.

Pf: Clear for C_λ. For δ let $\alpha \in \Delta$. Then

$$\delta(s_\alpha H) = \prod_{\beta>0}\left(e\left(\tfrac{1}{2}\beta(s_\alpha H)\right) - e\left(-\tfrac{1}{2}\beta(s_\alpha H)\right)\right)$$

s_α permutes positive roots other than α, exchanges $\alpha \leftrightarrow -\alpha$

$\Rightarrow \delta(s_\alpha H) = -\delta(H) = \mathrm{sgn}(s_\alpha)\cdot\delta(H)$
all & W is generated by $\{s_\alpha\}_{\alpha\in\Delta}$.

Next $\delta(H)=0$ iff $\exists\beta$ st. $e(\tfrac12\beta(H)) = e(-\tfrac12\beta(H))$
iff $\exists\beta$ st. $e(\beta(H))=1$
iff $\exists\beta$ st $\beta(H)\in\mathbb{Z}$

<u>Lemma:</u> $(\bar\delta\cdot\delta)(H) = J(\exp H)$

<u>Pf:</u>

$$(\bar\delta\cdot\delta)(H) = |e(\rho(H))|\cdot\prod_{\beta>0}(1-e(\beta(H))\cdot \text{c.c.}$$

$$=\prod_{\beta\in\Phi}(1-e(\beta(H)) = \det(\mathrm{Id}-\mathrm{Ad}_{\exp(-H)}|\mathfrak{g}/\mathfrak{t})$$

$\mathfrak{g}/\mathfrak{t}\simeq\bigoplus_\beta \mathfrak{g}_\beta$, ev. of $\uparrow$ are $1-e(-\beta(H))$

∎

<u>Prop:</u> let $\lambda\in I$, set $\phi_\lambda = \dfrac{C_{\rho+\lambda}}{\delta}$.
(initially defined on $\mathfrak{t}^{\mathrm{reg}}$) Then
(1) $\phi_\lambda\in R_I$, extends to a symmetric cts fcn on $\mathfrak{t}$.
(2) If $\lambda\in\Lambda^*$, ϕ_λ is sum of characters of T.
$\Rightarrow$ fcn on T/W.

<u>Pf:</u> $e(\rho)\cdot C_{\rho+\lambda} = \sum_{w\in W}\mathrm{sgn}(w)\cdot e(\rho\circ w+\rho+\lambda\circ w)\in R_I$

since $\rho\cdot w + \rho = \underbrace{(\rho\circ w - \rho)}_{\in\,\mathbb{Z}[\Delta]} + \underbrace{2\rho}_{\in\,\mathbb{Z}[\Delta]} \subset \Lambda^*$.

vanishes on t^{reg} ($C_{\rho+\lambda}$ does), so divisible by $\prod_{\beta>0}(e\circ\beta - 1)$. But

$$\frac{e(\rho)\, C_{\rho+\lambda}}{\prod_{\beta}(e\circ\beta - 1)} = \phi_\lambda$$

so $\phi_\lambda \in R_T$, in R_{Λ^*} if $\lambda\in\Lambda^*$.

ϕ_λ symmetric as ratio of alternating functions

<u>Lemma:</u> $\mathrm{Span}_{\mathbb{Z}}\{\phi_\lambda\}_{\lambda\in I\cap C} = \mathrm{span}\{d_\lambda\}_{\lambda\in I\cap C} = R_I^W$.

<u>Pf:</u> $$\phi_\lambda = \frac{e(-\rho)\,C_{\rho+\lambda}}{\prod_{\beta>0}(1-e(-\beta))} = \left(\sum_w e(\rho\circ w - \rho + \lambda\circ w)\right)\cdot\prod_{\beta>0}\left(\sum_{m=0}^{\infty} e(-m\beta)\right)$$

in ring of infinite formal sums $\sum_\nu n_\nu e(\nu)$ with support in sets of form

$$\lambda - \sum_{\alpha\in\Delta} n_\alpha\alpha, \quad n_\alpha\in\mathbb{Z}_{\geq 0}$$

so λ highest weight in ϕ_λ, occurs with coeff 1.

So $d_\lambda - \phi_\lambda \in \operatorname{Span}\{d_\mu \mid {\mu \text{ dominant} \atop \mu < \lambda}\}$

By induction get $d_\lambda \in \operatorname{Span}\{\phi_\mu\}$.

<u>Thm:</u> (Weyl character formula) Let λ be an algebraically integral dominant weight.
Then for $H \in \mathfrak{t}$

$$\chi_\lambda(H) \overset{\text{def}}{=} \operatorname{Tr}(e(L^\lambda(H))) = \phi_\lambda(H)$$

(formally: know $\operatorname{Res}^{\operatorname{Lie} G}_{\operatorname{Lie} T} L^\lambda = \bigoplus_\mu m_\mu \, \mu$
$\uparrow$
L^λ = sum of weight space

Then $\chi_\lambda(H) = \sum_\mu m_\mu \cdot e(\mu)$.

<u>Example:</u> $\lambda = 0$, L^λ = triv. $\chi_0 = 1$ so

$$\Rightarrow e(-\rho) \cdot \underbrace{\prod_{\beta > 0} (e(\beta) - 1)}_{\delta} = C_\rho = \sum_w e(\rho \circ w) \cdot \operatorname{sgn}(w)$$

<u>Cor:</u> Weyl dimension formula

$$\dim L^\lambda = \prod_{\beta>0} \frac{\langle \beta, \lambda+\rho \rangle}{\langle \beta, \rho \rangle} = \prod_{\beta>0} \frac{(\lambda+\rho)(\beta^\vee)}{\rho(\beta^\vee)}.$$

write the character formula as

$$C_\rho \cdot \chi_\lambda = C_{\rho+\lambda}$$

want to evaluate at $H=0$, i.e. follo

$$\lim_{\substack{H\to 0 \\ H\in \mathfrak{t}^{reg}}} \frac{C_{\rho+\lambda}(H)}{C_\rho(H)}$$

Now differentiate $\#\Phi^+$, use $\prod_{\beta\in\Phi} \partial_{\beta^\vee}$.

<u>Application:</u> $H\subset G$ subgp. $T_H \subset T_G = T$.

If $\mathrm{Res}^G_H L^\lambda$ was irred then $\dim L^\lambda$ would be same for both:

$$\prod_{\beta\in\Phi^+(G)} \frac{\langle \lambda+\rho_G, \beta^\vee \rangle}{\langle \rho_G, \beta^\vee \rangle} = \prod_{\beta\in\Phi^+(H)} \frac{\langle \lambda|_H+\rho_H, \beta^\vee \rangle}{\langle \rho_H, \beta^\vee \rangle}$$

But LHS is a poly in λ of deg $\#\Phi^+(G)$
RHS " " " " " " " $\#\Phi^+(H) < \#\Phi^+(G)$

If λ is large enough $\mathrm{Res}^G_H L^\lambda$ is reducible.

Let $\lambda \in \Lambda^* \cap \mathcal{C}$, $\Phi_\lambda(t)$ the function on T/W defined by WCF, also resulting class function on G.

<u>Prop</u>: $\{\Phi_\lambda\}_{\lambda \in \Lambda^* \cap \mathcal{C}}$ are a complete o.n.b. in space of sq-int class fcns

Pf: $\langle \Phi_\mu, \Phi_\lambda \rangle_G = \frac{1}{\# W} \int_T \overline{\Phi_\mu}(t) \Phi_\lambda(t) J(t)\, dt$

$$= \frac{1}{\# W} \int_T \overline{C_{\mu+\rho}}(t) \cdot C_{\lambda+\rho}(t)\, dt$$

$J(t) = \overline{\delta} \cdot \delta(t)$

$\delta \cdot \Phi_\lambda = C_{\rho+\lambda}$

$$= \delta_{\mu,\lambda}$$

$\Rightarrow \{\Phi_\lambda\}$ are orthonormal. Complete since span dense subset of $C(T/W) = \mathrm{Span}\{d_\mu\}$.

<u>Thm</u>: If $\lambda \in \Lambda^* \cap \mathcal{C}$, L^λ is a rep'n of G.

Pf: $F = \{\lambda \in \Lambda^* \cap \mathcal{C} \mid L^\lambda$ is a rep'n of $G\}$.

We know: (1) $\{L^\lambda\}_{\lambda \in F} = \hat{G}$

(2) Peter-Weyl $\{\phi_\lambda\}_{\lambda \in F}$ is a o.n.b of square-int class functions.

But $\{\phi_\lambda\}_{\lambda \in \Lambda^* \cap \mathcal{C}}$ is also an o.n.b

So $F = \Lambda^* \cap \mathcal{C}$. ∎