Math 535, lecture 18, 17/2/2023

<u>Last time:</u> $T \simeq \mathfrak{t}/\Lambda$ torus

Lie alg. ($\simeq \mathbb{R}^n$)

lattice $= \ker(\exp_T)$

then

$$\hat{T} \simeq \Lambda^* = \mathrm{Hom}(\Lambda, \mathbb{Z})$$

"weight lattice"

So if $V_{\mathbb{C}}$ is a complex representation, have

$$V_{\mathbb{C}} \simeq \bigoplus_{\alpha \in \Lambda^*} V_\alpha$$

"weight space decomposition"

"weight spaces".

$$V_\alpha = \{ \underline{v} \in V \mid \forall H \in \mathfrak{t} : \pi(\exp H)\cdot \underline{v} = e(\alpha(H))\, \underline{v} \}$$

$e(z) = e^{2\pi i z}$

"weights of $V_{\mathbb{C}}$" $= \{ \alpha \mid V_\alpha \neq \{\underline{0}\} \}$

If V is a <u>real</u> representation, "weights of V" = "weights of $V_{\mathbb{C}}$".

On $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$ have complex conjugation exchanging $V_\alpha, V_{-\alpha}$.

Example: $V = su(2) = \left\{ \begin{pmatrix} ix & y \\ -\bar{y} & -ix \end{pmatrix} \middle| \begin{matrix} x \in \mathbb{R} \\ y \in \mathbb{C} \end{matrix} \right\}$

$T = \left\{ \begin{pmatrix} e^{ix} & 0 \\ 0 & e^{-ix} \end{pmatrix} \middle| x \in \mathbb{R}/_{2\pi\mathbb{Z}} \right\}$

$t = \left\{ \begin{pmatrix} ix & 0 \\ 0 & -ix \end{pmatrix} \middle| a \in \mathbb{R} \right\}$ $\quad \Lambda = 2\pi\mathbb{Z} \begin{pmatrix} i & \\ & -i \end{pmatrix}$

$= \mathbb{Z} \cdot \begin{pmatrix} 2\pi i & \\ & -2\pi i \end{pmatrix}$

$$\mathrm{Ad}(\exp H) \cdot X = \underbrace{\begin{pmatrix} e^{ih} & \\ & e^{-ih} \end{pmatrix}}_{\exp H} \underbrace{\begin{pmatrix} ix & y \\ -\bar{y} & -ix \end{pmatrix}}_{X \in su(2)} \begin{pmatrix} e^{-ih} & 0 \\ 0 & e^{+ih} \end{pmatrix}$$

$H = \begin{pmatrix} ih & \\ & -ih \end{pmatrix}$

$$= \begin{pmatrix} ix & e^{2ih} y \\ -e^{-2\pi ih} \bar{y} & -ix \end{pmatrix} = \begin{pmatrix} ix & e^{2\pi ih} y \\ -\overline{e^{2\pi ih} y} & -ix \end{pmatrix}$$

$$su(2) \simeq t = \mathbb{R} \cdot \begin{pmatrix} i & \\ & -i \end{pmatrix} \oplus \left\{ \begin{pmatrix} 0 & y \\ -\bar{y} & 0 \end{pmatrix} \right\}$$

$y = u + iv$ then

$$e^{2\pi ih} y = (\cos(2\pi h) + i\sin(2\pi h)) \cdot (u + iv)$$

see that $\mathrm{Ad}(\exp H)$ not diagonalble! (over $\mathbb{R}$)

But, on $su(2)_{\mathbb{C}} \cong \mathfrak{sl}_2\mathbb{C}$

$$Ad(e^{2\pi i h}) \cdot \left[\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} + i \begin{pmatrix} & i \\ i & \end{pmatrix} \right]$$

$$= \begin{pmatrix} 0 & e^{2\pi i h} \\ -e^{-2\pi i h} & 0 \end{pmatrix} + i \begin{pmatrix} & ie^{2\pi i h} \\ ie^{-2\pi i h} & \end{pmatrix}$$

$$\to \cos(2\pi h) \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} + \sin(2\pi h) \begin{pmatrix} & i \\ i & \end{pmatrix}$$

$$+ i\cos(2\pi h) \begin{pmatrix} & i \\ i & \end{pmatrix} - \sin(2\pi h) \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$

$$= (\cos(2\pi h) + i\sin(2\pi h)) \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$

$$+ (- \text{''} -) i \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$

$$= e^{2\pi i h} \cdot \left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} + i \begin{pmatrix} & i \\ i & \end{pmatrix} \right)$$

$\Rightarrow$ in $su(2)_{\mathbb{C}}$ have a joint eigenvector.

<u>Today</u>: roots in cpt Lie gps

Fix G cpt Lie gp, $T < G$ max'l torus,
$\mathfrak{g} = \mathrm{Lie}\, G$, $\mathfrak{t} = \mathrm{Lie}\, T$, $\Lambda = \ker(\exp_T)$

<u>Def:</u> The rank of G is $\mathrm{rk}_G = \dim_{\mathbb{R}} T$.
The semisimple rank of G is $\mathrm{rk}(G/Z(G))$.

<u>Def:</u> The real roots of G (wrt T) are the nonzero weights of the adjoint action of T on $\mathfrak{g}_{\mathbb{C}}$. Write $\Phi = \Phi(G:T)$ for the roots

Have $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$

weight spaces for weights α

$\mathfrak{g}_0$ = T-fixed vectors, is conjugation-inv't so is complexification of T-fixed vectors in $\mathfrak{g}$, i.e. $Z_{\mathfrak{g}}(\mathfrak{t}) = \mathfrak{t}$. So set:

$$\boxed{\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha}$$

basic structural result on $\mathfrak{g}$

We want to understand $[\cdot,\cdot]$ in this representation.

Let $H \in \mathfrak{t}$, $X_\alpha \in \mathfrak{g}_\alpha$. Then

$$\mathrm{Ad}(\exp(tH)) \cdot X_\alpha = e(\alpha(tH)) \cdot X_\alpha .$$

Diff. wrt t get $\boxed{\mathrm{ad}_H \cdot X_\alpha = 2\pi i\, \alpha(H) \cdot X_\alpha}$

__Def:__ Call $H \mapsto 2\pi i\, \alpha(H)$ the complex root associated to the real root α.

__Note:__ $\alpha \in \mathfrak{t}^*$ (vanishing on Λ)
$2\pi i \alpha \in \mathfrak{t}_{\mathbb{C}}^*$ (purely imaginary)

Real roots are natural for representation theory (incl. studying "root system"), complex roots are natural for structure theory, i.e. compute commutators

__Aside:__ Also have exponential roots $\chi_\alpha \in \hat{T}$

$$\chi_\alpha(\exp H) = e(\alpha(H)).$$

So $\mathrm{Ad}(t) \cdot X_\alpha = \chi_\alpha(t) \cdot X_\alpha$ if $t \in T$.

Lemma: For $\alpha, \beta \in \Lambda^*$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$

Pf: Let $H \in \mathfrak{t}$, $X_\alpha \in \mathfrak{g}_\alpha$, $X_\beta \in \mathfrak{g}_\beta$

$$\operatorname{ad} H \cdot [X_\alpha, X_\beta] = [H, [X_\alpha, X_\beta]] = \quad \leftarrow \text{Jacobi identity}$$

$$-[X_\alpha, [X_\beta, H]] - [X_\beta, [H, X_\alpha]]$$

$$= -[X_\alpha, -2\pi i \beta(H) X_\beta] - [X_\beta, 2\pi i \alpha(H) X_\alpha]$$

$-[H, X_\beta]$

$$= 2\pi i \beta(H) [X_\alpha, X_\beta] + 2\pi i \alpha(H) [X_\alpha, X_\beta]$$

$$= 2\pi i \cdot (\alpha+\beta)(H) [X_\alpha, X_\beta]. \quad \blacksquare$$

Thm: If $\operatorname{rk} G = 1$ the G is $SU(2)$ or $SO(3)$

Pf:

(1) If β is a root, $X_\beta \in \mathfrak{g}_\beta$, the $X_{-\beta} \overset{\text{def}}{=} \overline{X_\beta} \in \mathfrak{g}_{-\beta}$

so $[X_\beta, X_{-\beta}] \in \mathfrak{g}_{\beta+(-\beta)} = \mathfrak{g}_0 = \mathfrak{t}_{\mathbb{C}}$.

If $[X_\beta, X_{-\beta}] = 0$ then $\operatorname{span}\{X_\beta, X_{-\beta}\}$ would

be a 2d commutative subalgebra inv't by $X \to \bar{X}$,
so complexification of of commutative 2d subalg.
of $\mathfrak{g}$. But rk $G = 1$.

So write $H_\beta = [X_\beta, X_{-\beta}] \neq 0$

$$\overline{H_\beta} = \overline{[X_\beta X_{-\beta}]} = [X_{-\beta}, X_\beta] = -H_\beta$$

so $H_\beta \in i\mathfrak{t}_{\mathbb{R}}$ & $iH_\beta \in \mathfrak{t}_{\mathbb{R}}$.

(check: $X_\beta + X_{-\beta}$, $\dfrac{X_\beta - X_{-\beta}}{i}$, iH_β

give 3d real subalg. of $\mathfrak{g}_{\mathbb{R}}$)

$\cong SU(2) \cong so(3)$

(2) For any fixed $H \in \mathfrak{t}_{\mathbb{R}}$ (e.g. iH_β --),
every $\alpha \in \Phi$ determined by $\underset{\mathbb{R}}{\underset{\cap}{\alpha(H)}}$ (dim $\mathfrak{t} = 1$)

order Φ by those values