Math 535, Lecture 17, 15/2/2023

Last time: conjugacy of maximal tori

Fix ctd cpt Lie group G, $T \subset G$ max'l torus.

$W = W(G:T) = N_G(T)/Z_G(T) = N_G(T)/T$

Thm: (0) Every $g \in G$ is contained in a torus

(1) All max'l tori in G are conjugate

(2) $G/\mathrm{Ad}(G) \cong T/W$.

conjugacy classes in G W-orbits in T.

Today: roots & weights - structure theory for G.

Example: 3d Groups

Let $G = SU(2) \circlearrowright \mathbb{C}^2$. Acts on

$$S^3 = \{ \underline{z} \in \mathbb{C}^2 \mid \underbrace{|z_1|^2 + |z_2|^2}_{\|\underline{z}\|^2} = 1 \}$$

Action is transitive: given $\underline{z}$, take $\underline{w} \perp \underline{z}$ then if $g \in GL_2(\mathbb{C})$ maps $\binom{1}{0}, \binom{0}{1}$ to $\underline{z}, \underline{w}$

if $\underline{w}$ has norm 1 this is unitary map (in $U(2)$), replacing $\underline{w}$ with $\alpha\underline{w}$ ($\alpha \in S^1$) can ensure $\det g = 1$, so $g \in SU(2)$

Now
$\mathrm{Stab}_{GL_2(\mathbb{C})}\begin{pmatrix}1\\0\end{pmatrix} = \left\{\begin{pmatrix}1 & b\\ 0 & d\end{pmatrix}\right\}$ if this is unitary have $b=0$, if $\det = 1$ $d = 1$ so $\mathrm{Stab}_{SU(2)}\begin{pmatrix}1\\0\end{pmatrix} = \{id\}$

$\Rightarrow$ orbit map is a diffeo $SU(2) \to S^3$.

Cor: $SU(2)$ is simply connected

$Z_{GL_2(\mathbb{C})}(SU(2)) = \left\{\begin{pmatrix}z & 0\\ 0 & z\end{pmatrix}\right\}$ so $Z(SU(2)) = \{\pm 1\}$

$\Rightarrow$ groups covered by $SU(2)$ are $SU(2)$ and $PSU(2) = SU(2)/\mathbb{Z}_2$ = image of adjoint rep'n

$su(2) = \mathrm{Lie}\, SU(2)$ is $\left\{ X \in M_2(\mathbb{C}) \,\middle|\, \begin{matrix} X + X^\dagger = 0 \\ \mathrm{tr}\, X = 0 \end{matrix} \right\}$

has $\dim_{\mathbb{R}} su(2) = 3$: $u(2) = \left\{\begin{pmatrix} ia & b \\ -b^* & id\end{pmatrix}\right\}$

$d, a \in \mathbb{R}$, $b \in \mathbb{C}$

so $su(2) = \left\{\begin{pmatrix} ia & b \\ -b^* & -ia\end{pmatrix}\right\}$.

$\Rightarrow$ $\mathrm{Ad} : G \to GL(\mathfrak{g}) \cong GL_3(\mathbb{R})$

Since G fixes inner prdt, image is in $O(3)$.
Since G ctd, Image is in $SO(3)$

But $SO(3)$ is ctd, 3-dim, set

$SU(2)/\{\pm I\} \cong SO(3)$.

write $su(2)$ or $so(3)$ or $\mathfrak{g}$ for Lie alg

<u>Lemma:</u> Maximal tori $\Leftrightarrow$ subspaces of $\mathfrak{g}$

<u>Pf:</u> Work in $so(3) = \{X \in M_3(\mathbb{R}) \mid X + {}^tX = 0\}$

$$\left(\begin{array}{l} (I + tX + \dots)(I + tX^t + \dots) = 1 \\ \mapsto t(X + X^t) + O(t^2) = 0 \end{array} \right)$$

Let $t = \mathrm{span}\left\{ \underbrace{\begin{pmatrix} & -1 & \\ 1 & & \\ & & 0 \end{pmatrix}}_{H} \right\}$ Can compute $z_{\mathfrak{g}}(t) = t$.

<u>check:</u> Frobenius norm $\langle X, Y \rangle = \mathrm{tr}({}^tX Y)$
$= \sum_{i,j} X_{ij} Y_{ij}$

is inv't under conjugation by orthogonal matrices

$\Rightarrow$ G-inv't inner pdt on $so(3)$

So $t^\perp = \left\{ \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ -a & -b & 0 \end{pmatrix} \right\}$ check: $\text{ad}\, H$ is irred here.

$\Rightarrow$ t max'l commutative subalgebra

$\Rightarrow$ all max'l tori are 1d

In fact this shows max'l tori are max'l subgps: any subalgebra containing t is $\mathfrak{g}$ (if $t \subset \mathfrak{h} \subset \mathfrak{g}$ is a subalgebra then $\mathfrak{h}$ meets $t^\perp$ $\Rightarrow$ $\mathfrak{h}$ contains $t^\perp$ by irred $\Rightarrow$ $\mathfrak{h} = \mathfrak{g}$)

$\boxtimes$

<u>Prop:</u> Let G be a ctd 3d cpt Lie gp. Then either G is abelian or G covers $SO(3)$

<u>Pf:</u> Consider Adjoint rep'n

$$Ad : G/Z(G) \to SO(3) \subseteq GL(\mathfrak{g})$$

So either image is $SO(3)$ or contained in a torus (so at most 1d)

In first case we have a covering map

(derivative is isom of vsp)

So second case $\dim Z(G) \geq 2$
Can't have $\dim \mathfrak{g}/z_{\mathfrak{g}} = 1$, because any $X \in \mathfrak{g}$ commutes with centre

(cf. classification of finite gps of order p^3)

So G is commutative.

Weights

Let T be a torus, (π, V) a fd complex representation of T. Then

$$V = \bigoplus_{\chi \in \hat{T}} V_\chi$$

Since T is commutative, its irreps are 1d, $\hat{T} = \mathrm{Hom}_{cts}(T, S^1)$,

$$V_\chi = \{ \underline{v} \in V \mid \forall t \in T : \pi(t)\underline{v} = \chi(t)\underline{v} \}$$

Def: Call $\{ \chi \in \hat{T} \mid V_\chi \neq 0 \}$ the exponential weights of V.

Call V_χ the weight spaces

Let $\mathfrak{t} = \operatorname{Lie} T$, $\exp : \mathfrak{t} \to T$ is a grp hom with discrete kernel Λ. Call it the integral lattice

<u>Know:</u> $\chi \in \hat{T}$ has form $\chi(\exp H) = e^{2\pi i \alpha(H)}$ for some $\alpha : \mathfrak{t} \to \mathbb{R}$ st $\alpha(\Lambda) \subseteq \mathbb{Z}$

Call $\Lambda^\vee = \operatorname{Hom}(\Lambda, \mathbb{Z})$ the weight lattice of T.

Generally use weights in Λ^* rather than corresponding exponential weights in $\hat{T}$.

Now suppose that T acts on a real vsp V. Since every non-triv char of T takes complex values, V cannot realize any char.

($e^{i\theta}$ can act on $\mathbb{R}^2$ via $e^{i(-)} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ which is irred)

Let T also act on $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$. Complex conjugation in $\mathbb{C}$ extends to $\mathbb{R}$-linear complex conjugation on $V_\mathbb{C}$, also to a complex conjugation in $\mathrm{End}_\mathbb{C}(V_\mathbb{C}) = V_\mathbb{C} \otimes V_\mathbb{C}'$.

Ex: A $\mathbb{C}$-linear subspace $W \subset V_\mathbb{C}$ is of the form $W = U_\mathbb{C}$ for an $\mathbb{R}$-subspace $U \subset V$ iff $\overline{W} = W$.

T-action on V extends $\mathbb{C}$-linearly to $V_\mathbb{C}$ so have

$$V_\mathbb{C} \cong \bigoplus_{\alpha \in \Lambda^*} V_\alpha$$

Observe: If $\underline{v} \in V_\alpha$, $H \in \mathfrak{t}$, then

$$\pi(\exp H)\,\underline{v} = e^{2\pi i \alpha(H)} \cdot \underline{v}$$

Conjugating $\pi(\exp H) \cdot \overline{\underline{v}} = e^{-2\pi i \alpha(H)}\, \overline{\underline{v}}$

so $\overline{v} \in V_{-\alpha}$

Get $\mathbb{R}$-linear isom $V_\alpha \to V_{-\alpha}$

Conclusion: ① if $\alpha \neq 0$ is a weight, so is $-\alpha$

(with same multiplicity $= \dim_{\mathbb{C}} V_\alpha$)

② Isotypical real subspaces are $\mathrm{Re}(V_\alpha \oplus V_{-\alpha})$