Math 535, Lecture 15, 10/2/2023

Last time: ctd abelian Lie group has the form

$$\mathbb{R}^a \times \mathbb{T}^b \simeq \mathbb{R}^{a+b}/\Lambda \quad , \ \Lambda \text{ discrete subgp}$$

quotient map = exponential map

$\rightsquigarrow$ $\mathrm{Hom}(\mathbb{R}^n/\mathbb{Z}^n, \mathbb{R}^m/\mathbb{Z}^m) \simeq \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}^m)$

$\rightsquigarrow$ ① $\mathrm{Aut}(\mathbb{T}^n) \simeq GL_n(\mathbb{Z})$ (discrete!)

② $\mathrm{Hom}(\mathbb{T}^n, \mathbb{T}^1) \simeq \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}) = (\mathbb{Z}^n)'$

identify $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z} = S^1 \subset \mathbb{C}$
via $z \mapsto e(z) = e^{2\pi i z}$

then
$$\widehat{\mathbb{T}^n} = \{ e_{\underline{k}}(\underline{x}) = e(\underline{k}\cdot\underline{x}) \}_{\underline{k}\in(\mathbb{Z}^n)'}.$$

Starting today: structure theory of cpt groups. Key tool: maximal tori.

Preliminary 1: Ex: (HW) Tori are topologically generated by single elements

(In detail, if $\{1\} \cup \{\xi_i\}_{i=1}^n \subset \mathbb{R}$ are linearly indep / $\mathbb{Q}$ then every orbit $\{\underline{x} + j\underline{\xi}\}_{j=1}^{\infty} \subset \mathbb{T}^n$ is equidistributed.

In general, $\overline{\{j\underline{\xi}\}_{j=1}^{\infty}}$ has the form $V_{\mathbb{R}}/V_{\mathbb{R}} \cap \mathbb{Z}^n$ when $V \leq \mathbb{Q}^n$ is a subspace, $V_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Q}} V$.

Cor: $\mathbb{T}^n \times C_m$ $\leftarrow$ cyclic gp also generated by one element: $\langle \underline{\xi}, g \rangle$ $\underline{\xi}$ as above, $g \in C_m$ generator.

Preliminary 2: the exponential map

Fix cpt ctd Lie group G, Lie algebra $\mathfrak{g}$, $\mathrm{Ad}: G \to GL(\mathfrak{g})$ the adjoint rep'n.

Can equip $\mathfrak{g}$ with an Ad-inv't inner pdt.
$\cong T_eG$

Translating by G get inner pdt on every tangent space T_gG, i.e. a <u>Riemannian metric</u> on G.

<u>Observe:</u>

(1) if we define metric by left translation the metric is left - G- inv't

(2) because metric at e is inv't under conjugation, metric on G is right-inv't too

(3) map $g \to g^{-1}$ is an isometry.

$\Rightarrow$ G with this metric is a symmetric space

<u>Prop:</u> The Riemannian & Lie exponential maps agree.

<u>Pf:</u> Let $\gamma(t)$ be a Riemannian geodesic with $\gamma(0)=e$. Then $\gamma(t+t_0)$, $\gamma(t)\gamma(t_0)$
$\gamma(t_0)\gamma(t)$

all three are geodesics (metric is bi-inv't)

All three agree at $t=0$, have same derivative there. $\Rightarrow$ all three agree. So $\gamma(t)$ is a 1-param subgp (i.e. $\gamma(t) = \exp(t\cdot\dot\gamma(0))$)

<u>Cor</u>: The exponential map of a cpt ctd Lie gp is surjective

<u>Ex</u>: $\exp: \mathfrak{sl}_2\mathbb{R} \to SL_2(\mathbb{R})$ is not surjective

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<u>Maximal Tori</u>

<u>Def</u>: A **torus** in G is a subgp isomorphic to $\mathbb{T}^n$ for some n, i.e. a closed ctd abelian subgp. A **maximal torus** is a torus not properly contained by another.

<u>Observations</u>: ① a maximal torus is a maximal ctd commutative subgp (closure would be ctd, commutative)

② If $T \subsetneq T' \subset G$ are tori then $\dim T < \dim T' \leq \dim G$ (by ctd) so maximal tori exist.

<u>Lemma</u>: Every $g \in G$ is contained in a torus

Pf: $g = \exp(X)$ for some $X \in \mathfrak{g}$ (surjectivity of exp)

$\Rightarrow g \in \{\exp(tX)\}_{t\in\mathbb{R}}$ which is ctd, abelian

then $g \in \overline{\{\exp(tX)\}}_{t\in\mathbb{R}}$ which is a torus

Sketch of how to find a max'l torus:

(1) Take torus T.

(2) look at $Z_G(T)/T$

max'l commutative subgp of $G \supseteq T$

max'l commutative subgp of $Z_G(T)/T$.

T max'l torus $Z_G(T)/T$ can't have tori so is 0-dim. If $Z_G(T)$ ctd $\Rightarrow T$ max'l commutative subgp

Lemma: G cpt ctd. $T\subset G$ torus, $\mathfrak{t} = \mathrm{Lie}(T)$

(1) $Z_G(T)$ is ctd

(2) $Z_G(\mathfrak{t}) = Z_G(T)$

(3) $\mathrm{Lie}(Z_G(T)) = Z_{\mathfrak{g}}(\mathfrak{t})$

(4) $N_G(T)^0 = Z_G(T)$

Pf: Let $g \in Z_G(T)$, let $S = \overline{\langle g, T\rangle}$.

That's closed, commutative, so S^0 is a torus $\supseteq T$.

The image of g is a topological generator of S/T so also of the quotient $S/S^\circ = \pi_0(S)$

so $\quad S \cong S^\circ \times S/S^\circ$ ← cyclic gp

$\Rightarrow$ S has a topological generator h.

($h^{\mathbb{Z}} \subset S$ is a dense subgroup)

Let $\tilde{S}$ be any torus containing h. Then $h \in \tilde{S}$
so $h^{\mathbb{Z}} \subset \tilde{S}$ so $T \subset S = \overline{\langle h \rangle} \subseteq \tilde{S}$.

Thus $\quad \tilde{S} \subset Z_G(T)$ (it's commutative, so commutes with T)

also $g \in S \subseteq \tilde{S}$, so $g \in \tilde{S}$.

$\Rightarrow$ $Z_G(T)$ = union of its subtori
hence ctd.

<u>Cor of argument:</u> A max'l commutative subgp is a torus.