Math 535, lecture 13, 6/2/2023

Last time: Closed subgps

Thm: G Lie gp, $H \subset G$ closed $\Rightarrow$ H is a Lie subgp.

Application: If $f: G \to H$ is a cts gp hom, G, H Lie gps then f is smooth.

Application: covering groups

Today: Adjoint representation

Examples: f analytic on $\mathbb{R}$, then

$$f(x+t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{d^k f}{dx^k}(x)$$

$$\Updownarrow$$

$$(L_t f)(x) = \left(\left(\sum_{k=0}^{\infty} \frac{1}{k!} t^k \frac{d^k}{dx^k}\right) f\right)(x)$$

$$\Leftrightarrow \quad L_t = \exp\left(t \frac{d}{dx}\right)$$

Two interpretations of RHS:

(1) exponential map of $(\mathbb{R},+)$, value is $t \in \mathbb{R}$

(2) exponential of operator $t\frac{d}{dx}$ defined on smooth functions

Examples If $G = GL_n(\mathbb{R})$, $\mathfrak{g} \simeq M_n(\mathbb{R})$, so that

$$\exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^k \leftarrow \text{arithmetic in } M_n(\mathbb{R})$$

then $\exp(X)\exp(Y) = (1 + X + \frac{1}{2}X^2 + \frac{1}{6}X^3 + \dots)$
$(1 + Y + \frac{1}{2}Y^2 + \frac{1}{6}Y^3 + \dots)$

$$= 1 + X + Y + \frac{1}{2}(X^2 + Y^2 + 2XY) + \frac{1}{6}(X^3 + Y^3 + 3XY^2 + 3YX^2) + \dots$$

To write group law in exp co-ords need to write $\exp(X)\exp(Y) = \exp(Z)$ with $Z = Z(X, Y)$

to 1^{st} order, $Z(X, Y) \simeq X + Y$

say $Z(X, Y) = X + Y + Z_2(X, Y)$

$$\exp(X + Y + Z_2(X, Y)) \simeq 1 + X + Y + Z_2 + \frac{1}{2}(X+Y)^2$$

so $\quad Z_2 = \frac{1}{2}(X^2+Y^2+2XY) - \frac{1}{2}(X+Y)^2$

$$= \frac{1}{2}(XY - YX) = \frac{1}{2}[X,Y]$$

<u>Fact</u>: $\operatorname{Exp}(X+Y) = \operatorname{Exp}\left(\sum_{k=1}^{\infty} Z_k(X,Y)\right)$

where $Z_1(X,Y) = X+Y$, $\quad Z_2(X,Y) = \frac{1}{2}[X,Y]$

$Z_k(X,Y)$ $(k \geq 3)$ are nested commutators

(Poincaré-Birkhoff-Witt Theorem)

<u>Interpretation</u>: $[\cdot,\cdot]$ determines full Taylor expansion of $\cdot$ in exponential co-ordinates

<u>Today</u>: Show this works for all Lie groups suitably interpreted

Fix Lie gp G.

<u>Lemma</u>: The operator $R_{\operatorname{Exp}(tX)}$ on $C^\infty(G)$ has the Taylor expansion $\sum_{k=0}^{\infty} \frac{t^k}{k!} X^k$

in the sense that for $N \geq 0$, $\Omega \subset G$ cpt, for all $g \in \Omega$, $f \in C^\infty(G)$

$$(R_{\exp tX} f)(g) = f(g \cdot \exp(tX)) = \left(\sum_{k=0}^{N} \frac{t^k}{k!} X^k f\right)(g) + O_{f,\Omega}(|tX|^{k+1})$$

Pf: Since $R_\cdot$, X commute with L_g, may assume $g = 1$. Formula is then Taylor expansion with remainder of $t \mapsto f(\exp(tX))$.

(using $\frac{d}{dt}(\exp(tX)) = X \exp(tX)$ & chain rule)

$$\frac{d}{dt}(f(\exp(tX))) = (Xf)(\exp(tX)).$$

Remark: A Lie gp has a unique compatible real analytic structure; if $f \in C^\omega(G)$ then full series sums

Cor: Have

$$R_{\exp(tX)} R_{\exp(tY)} = \exp\left(t(X+Y) + \tfrac{1}{2}t^2[X,Y] + O(t^3)\right)$$

$\Rightarrow \exp(tX)\exp(tY) = \exp(t(X+Y) + \frac{1}{2}t^2[X,Y] + O(t^3))$

In general, the PBW theorem holds

<u>Cor</u>: $\exp(tX)\exp(sY)\exp(-tX)$

$= 1 + sY + ts[X,Y] + \frac{1}{2}s^2Y^2 + O(s^3, t^3, s^2t, st^2)$

<u>Def</u>: Let $g \in G$. Write $Ad_g : G \to G$ for the automorphism $Ad_g(x) = gxg^{-1}$.
Smooth gp hom, has derivative $Ad_g : \mathfrak{g} \to \mathfrak{g}$

Now $Ad_{gh} = Ad_g Ad_h$ so $Ad : G \to GL(\mathfrak{g})$

Call this the adjoint representation.

Clearly a smooth rep'n.

Write $ad : \mathfrak{g} \to End_{rsp}(\mathfrak{g})$ for its derivative.

$$ad_X \cdot Y = \frac{d}{dt}\Big|_{t=0} Ad_{\exp(tX)} \cdot Y.$$

$$= \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} Ad_{\exp(tX)} \cdot \exp(sY)$$

$$= \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} \left(\exp(tX)\exp(sY)\exp(-tX)\right)$$

$$= \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} \left(1 + sY + ts[X,Y] + \tfrac{1}{2}s^2Y^2 + O(\text{cubic})\right)$$

$$= [X, Y]$$

we have proved

Theorem: $ad_X \cdot Y = [X, Y]$

Cor: $ad_{[X,Y]} = [ad_X, ad_Y] \leftarrow$ commutator in $End_{\mathrm{vsp}}(\mathfrak{g})$

(i.e. ad is a Lie algebra rep'n)

Further connects Lie groups and algebras

Cor 2: Let $H < G$ be connected Lie group

Then $H \triangleleft G$ iff $\mathfrak{h} \subset \mathfrak{g}$ is a Lie ideal

Pf: If $H \triangleleft G$ then H is Ad-stable

$\Rightarrow \mathfrak{h}$ is Ad-stable $\Rightarrow \mathfrak{h}$ is ad-stable

Conversely, if $X \in \mathfrak{g}$, $\mathfrak{h}$ is ad_X-stable,

$\mathfrak{h}$ is $\exp(t\,\mathrm{ad}_X)$-stable:

$$\exp(t\,\mathrm{ad}_X) = \sum_{k=0}^{\infty} \frac{t^k}{k!} (\mathrm{ad}_X)^k \in \mathrm{End}(\mathfrak{g})$$

$$\parallel$$

$\mathrm{Ad}(\exp(tX))$ $\Rightarrow$ (G cts, so generated by $\exp(\mathfrak{g})$)

$\mathfrak{h}$ is Ad-stable.

For each $g \in G$, gHg^{-1} is a ctd subgp with Lie algebra $\mathrm{Ad}_g \mathfrak{h} = \mathfrak{h}$. By Lie gp-algebra corresp get $gHg^{-1} = H$.

$\blacksquare$

<u>Cor 2</u>: Let G be connected. Then
$$Z(G) = \ker(\mathrm{Ad} : G \to GL(\mathfrak{g})).$$

<u>Pf</u>: If $z \in Z(G)$, $\mathrm{Ad}_z \in \mathrm{Aut}(G)$ is trivial,
so $\mathrm{Ad}_z \in GL(\mathfrak{g})$ is trivial

Say $\mathrm{Ad}_g = \mathrm{Id}_{\mathfrak{g}}$. Then

$$\mathrm{Ad}_g(\exp(tX)) = \exp(t\,\mathrm{Ad}_g X) = \exp(tX)$$

Ad_g is a Lie gp hom

so g centralizes $\{\exp(X) : X \in \mathfrak{g}\}$ which is a generating set.

$\blacksquare$