Math 535, Lecture 11, 1/24/2023

Last time: G Lie gp, $\mathrm{Lie}\, G = \left\{ \begin{matrix} \text{left-inv't} \\ \text{vector fields} \end{matrix} \right\}$

is a Lie algebra, isom as vsp to $T_e G$.

If $f \in \mathrm{Hom}(G, H)$, $df : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra hom.

Thm: (Lie group - Lie algebra corresp.)
Have bijection $\left\{ \begin{matrix} \text{Lie subgroups} \\ \text{of } G \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \text{Lie subalgebras} \\ \text{of } \mathrm{Lie}(G) \end{matrix} \right\}$

Today: Exponential map

Thinking of $X \in \mathrm{Lie}(G)$ as a vector field on G, it has an integral curve through e. Write it as

$$e_X(t) = \exp(tX) \quad , \quad t \in \mathbb{R}$$

$e_X(t)$ solves ode $\quad \alpha'(t) = \alpha(t)_* X$

Lemma: (1) Integral curves of X live forever, (2) $\exp((t+s)X) = \exp(tX)\exp(sX)$

Pf: By Picard have $e_X(t)$ on some interval $(-\epsilon,\epsilon)$ ($e_X(0) = e$). Now if $\alpha(s)$ is any integral curve defined on (a,b), $s_0 \in (a,b)$ set

$$\tilde{\alpha}(s) = \alpha(s_0) \cdot e_X(s-s_0)$$

thus $\tilde{\alpha}(s_0) = \alpha(s_0) \cdot e_X(0) = \alpha(s_0)$

and $\frac{d}{dt} \tilde{\alpha}(s) = \alpha(s_0)_* \frac{d}{dt} e_X(s-s_0)$

left action on TG

$$= \alpha(s_0)_* \cdot e_X(s-s_0)_* X = \tilde{\alpha}(s)_* X$$

so $\tilde{\alpha}(s)$ also an integral curve through $(s_0, \alpha(s_0))$.

so $\tilde{\alpha}(s) = \alpha(s) \Rightarrow \alpha(s)$ defined on $(s_0-\epsilon, s_0+\epsilon)$

so α extends to $(a,b) \cup (s_0-\epsilon, s_0+\epsilon)$.

Apply reasoning to $\alpha(s) = e_X(s)$ get

$$e_X(s) = e_X(s_0)\, e_X(s - s_0)$$

$$e_X(t+s) \overset{!}{=} e_X(t)\, e_X(s)$$

Finally $\frac{d}{dt} e_X(at) = a\, e_X(at)_* X = e_X(at)_*(aX)$

so $e_X(at)$ is $e_{aX}(t)$

i.e. $e_X(t)$ only depends on path tX:

$$e_X(t) = e_{tX}(1)$$

$\Rightarrow$ $\forall X \in \mathfrak{g}$ have a unique Lie gp hom
$e_X : \mathbb{R} \to G$ s.t. $(d_0 e_X)(\underset{\in T_0\mathbb{R}}{1}) = X$.

Cor: can identify $\mathrm{Lie}(GL_n(\mathbb{R}))$ with $M_n(\mathbb{R})$
st. exp map is matrix exponential

$$\exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^k.$$

Observe: $\cdot : G \times G \to G$ has derivative
$d_e \cdot : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ given by $+$

Pf: $d_e \cdot$ is linear, $d_e \cdot (X, 0) = d_e(id_G)(X) = X$

$\Rightarrow$ If $f, g : \mathbb{R}^n \to G$ then
$$d(fg) = df \cdot_* g + f_* dg.$$

Thm: $\exp : \mathfrak{g} \to G$ is a local diffeomorphism with derivative id.

Pf: Solutions to ODE are diff wrt parameters.
so $\exp(X) = e_X(1)$ is diff wrt X

To compute $d(\exp)$ evaluate $\frac{d}{dt}\Big|_{t=0} \exp(tX)$.

This is X by def'n of $\exp(tX)$, by chain rule this is
$$d_e(\exp) \circ \frac{d(tX)}{dt}$$

so $d_e(\exp) \circ X = X$.

Cor: For any direct sum decomp $\mathfrak{g} = \bigoplus_{i=1}^{r} V_i$ (as Vsp), the map

$$\bigoplus_{i=1}^{r} V_i \ni (X_i)_{i=1}^{r} \mapsto \prod_{i=1}^{r} \exp(X_i)$$

is a local diffeomorphism near 0

If $\{X_i\}_{i=1}^{\dim \mathfrak{g}} \subset \mathfrak{g}$ is a basis, set:

$$\mathbb{R}^{\dim \mathfrak{g}} \ni \underline{t} \mapsto \exp\left(\sum_{i=1}^{n} t_i X_i\right)$$

$$\mapsto \prod_{i=1}^{n} \exp(t_i X_i),$$

Lemma: Homomorphisms respect exp:

if $f: G \to H$ then $f(\exp_G(tX))$ is a lie gp hom $\mathbb{R} \to H$.

$$\frac{d}{dt}\Big|_{t=0} f(\exp_G(tX)) = df \cdot \frac{d}{dt}\Big|_{t=0} (\exp_G(tX)) = df(X)$$

so $$f(\exp_G(tX)) = \exp_H(t\,df(X)) =$$

$$= \exp_H(df(tX)).$$

Closed subgroups

Thm: (Cartan 1930) Let G be a Lie gp, $H < G$ a closed topological subgp. Then H is a Lie subgp.

Pf: Let $\tilde{\mathcal{U}} \subset \mathfrak{g}$ be an open nbd of 0 on which exp is a diffeo onto $\mathcal{U} \subset G$ open $\ni e$. Let $\log: \mathcal{U} \to \mathfrak{g}$ be the inverse.

Examine:

① $\mathbb{R} \cdot \left\{ \lim_{n\to\infty} \frac{\log h_n}{|\log h_n|} \;\middle|\; \begin{matrix} \{h_n\}_n \subset H \\ h_n \to e \end{matrix} \right\}$ (all limits of such sequences)

② $\{ X \in \mathfrak{g} \mid \forall t: \exp(tX) \in H \}$

③ $\mathbb{R} \cdot \{ \log(H \cap \mathcal{U}) \}$

Clear: ② $\subseteq$ ①, ② $\subseteq$ ③

($\log e^{tX} = tX$ if t small enough)

Step 1: ① $=$ ② **Step 2:** this is a subspace $\mathfrak{h}$

Step 3: $\exp: \mathfrak{h} \to H$ is a local homeo.