Math 535, lecture 9

27/1/2023

Last time:

M smooth manifold

① sheaf of smooth functions: map $\mathcal{U} \to C^\infty(\mathcal{U})$ for open sets $\mathcal{U} \subset M$. Can restrict $f \in C^\infty(\mathcal{U})$ to open subsets $V \subset \mathcal{U}$

② sheaf of smooth vector fields: map $\mathcal{U} \to \mathrm{Der}(C^\infty(\mathcal{U}))$. Can restrict $X \in D_{C^\infty(\mathcal{U})}$ to $V \subset \mathcal{U}$.

In both cases, (1) an element $f \in C^\infty(\mathcal{U})$ / $X \in D_{C^\infty(\mathcal{U})}$ is determined by restrictions to open cover of $\mathcal{U}$.

(2) a system of elements on open cover of $\mathcal{U}$ defines an element on $\mathcal{U}$ iff restrictions to $V_i \cap V_j$ are consistent.

In local co-ordinates (i.e. on patch $\varphi : \mathcal{U} \to \mathbb{R}^n$)

(1) $C^\infty(\mathcal{U}) \cong C^\infty(\varphi(\mathcal{U}))$ via composition with φ.

(2) $D_{C^\infty(\mathcal{U})} = \left\{ \sum_{i=1}^n a_i(x) \frac{\partial}{\partial x^i} \;\middle|\; a_i \in C^\infty(\varphi(\mathcal{U})) \right\}$.

⇒ Tangent & cotangent spaces of M at p

$T_p^* M = I_p / I_p^2$ $\quad$ $I_p \subset C^\infty(M)$
$\quad\quad\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{\ker}(\delta_p)$

$T_p M =$ "derivations" $C^\infty(M) \to \mathbb{R}$:
"local derivations" $\partial(fg)(p) = \partial f \cdot g(p) + f(p) \cdot \partial g$.

spaces are in perfect pairing:
in local co-ords $T_p^* N$ represented by linear fcns dx^i
$T_p M$ " " $\frac{\partial}{\partial x^i}$.

<u>Today</u>: Derivatives

<u>Lemma</u>: Let M, N be smooth manifolds, $\varphi \in C^\infty(M, N)$, let $p \in M$. For each $v \in T_p M$, $v(f \circ \varphi)$ is a local derivation at $\varphi(p)$ on $C^\infty(N)$, map $d\varphi_p : T_p M \to T_{\varphi(p)} N$ given by $v \mapsto (f \mapsto v(f \circ \varphi))$ is linear.

<u>Def</u>: $d\varphi_p$ is called the differential or derivative of φ at p

Ex: In local co-ordinates this is the usual Jacobian matrix

Pf: composition with φ is an algebra hom
$C^\infty(N) \to C^\infty(M)$
so $(v, f) \mapsto v(f \circ \varphi)$ is bilinear. Also

$$v((fg)\circ\varphi)) = v((f\circ\varphi)\cdot(g\circ\varphi)) = v(f\circ\varphi)\cdot(g\circ\varphi)(p) + (f\circ\varphi)(p)\cdot v(g\circ\varphi)$$
$$= v(f\circ\varphi)\cdot g(\varphi(p)) + f(\varphi(p))\cdot v(g\circ\varphi).$$

Prop: (1) $d\varphi : TM \to TN$ is smooth

(2) Chain rule holds (=construction is functorial)

$$d(\varphi\circ\psi)_p = d\varphi_{\psi(p)} \circ d\psi_p.$$

Thm: (Inverse & Implicit function theorems)
(1) Suppose $d\varphi_p$ is injective. Then $\varphi|_U$ is injective for some nbd U of p.
(2) " " " surjective. Then φ is an open map in a nbd of p

(3) If $d\varphi_p$ is a isom then $\exists$ nbd U of p, V of $\varphi(p)$ s.t. $\varphi|_U : U \to V$ is a diffeomorphism.

(4) Suppose $d\varphi_p$ is surjective. The level set $P = \varphi^{-1}(\varphi(p))$ is (near p) a submanifold

<u>Def:</u> A smooth map $\varphi : M \to N$ is an

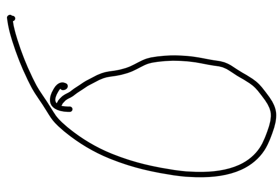
(1) immersion if $d\varphi_p$ is injective for all p

$\Updownarrow$

(2) local embedding if for all $p \in M$ $\exists$ nbd U s.t. $\varphi|_U$ is a diffeo onto its image

(3) an embedding if it's an immersion which is a homeomorphism onto its image

<u>Example:</u>



<u>Def:</u> A parametrized submanifold of N is a pair (M, f) where M is a manifold, $f : M \to N$ is an injective immersion

Two parametrizations are equivalent if conjugate by a diffeo of the source manifolds

A submanifold is an equivalence class

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if $f_1 : M_1 \to N$ is an injective immersion

if $\psi : M_2 \to M_1$ is a diffeo.

then $f_1 \circ \psi : M_2 \to N$ is an injective immersion

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usually write $M \subset N$ for a submanifold.

Then for $p \in M$, $T_p M \subset T_p N$.

<u>Def:</u> Let $\gamma : [a,b] \to N$ be a smooth curve

its derivative is $\dot{\gamma}(t) = d\gamma\left(\frac{d}{dt}\right)_t \in T_{\gamma(t)} N$

Say γ is an integral curve of $X \in \mathcal{D}_M$ if

$$\dot{\gamma}(t) = X\restriction_{\gamma(t)}.$$

<u>Thm:</u> (Picard) For any X, any $p \in M$ there is an integral curve of X with $\gamma(0) = p$ defined in some nbd of 0, unique when defined.

Def: A distribution on N is a smooth choice of a subspace $V_p \subset T_pN$ of dim k for each k.

$\Leftrightarrow$ cover N with nbds, on each choose k vector fields $\{X_i\}_{i=1}^k \subset \mathcal{D}_U$ s.t. $\{X_i(p)\}_{i=1}^k$ are indep in T_pN for each $p \in U$

Call a submanifold (M^k, f) tangent to a distribution V if $df(T_pM) = V_{f(p)}$ for all $p \in M$

Observation: If $X, Y \in \mathcal{D}(N)$ are sections of V (i.e. $X(p), Y(p) \in V_p$ for all p) can think of X, Y as vector fields on M. Then $[X, Y]$ is a vector field on M, hence tangent to V.

Thm (Frobenius) (1) Through each $p \in N$ there is a unique submanifold tangent to V.

(2) The distribution is integrable: if X, Y sections of V so is $[X, Y]$