Math 535, lecture 5 . 18/1/2023

Last time: f.d. rep'ns of compact groups:

① schur's lemma ② schur orthogonality

$\Rightarrow$ Def $\mathcal{C}(\sigma) = \{ \Phi^{\sigma}_{v,v'}(g) \mid \begin{smallmatrix} v \in V_\sigma \\ v' \in V_\sigma' \end{smallmatrix} \}$

Then $\bigoplus_{\substack{\sigma \text{ f.d.} \\ \text{irrep}}} \mathcal{C}(\sigma) \subset L^2(G)$ is an orthogonal sum

(with this pov & equipping each V_σ with the inv't ip, the map $\Phi_{\cdot,\cdot} : V \otimes V' \to L^2(G)$ is a unitary intertwining operator of $G \times G$-reps)

Today: Cor: Every f.d. rep'n of G is a subrep'n of $L^2(G)$

Show: ① $L^2(G) = \widehat{\bigoplus_{\substack{\sigma \text{ f.d.} \\ \text{irrep}}}} \mathcal{C}(\sigma)$ (i.e. the sum above is dense)

② Every irrep of G is f.d.

③ Every rep'n of G "is" a direct sum of irreps

Example: $G = \mathbb{R}/\mathbb{Z}$ rep'ns are $e_k(x) = e^{2\pi i k x}$
$e_k(x) \in U(1)$ (acts on $\mathbb{C}$)

$\bigoplus_k \mathbb{C} e_k$ = "trigonometric polynomials".

Fourier series: these are dense in $L^2(S^1)$
$C(S^1)$

Look up "abstract harmonic analysis"
"non commutative Fourier analysis".

∞-dim rep'ns of cpt gps

Let (π, V) be a cts rep'n of G (suppose V is a quasi-complete locally convex TVS)

Lemma: TFAE for $v \in V$:

(1) $\dim_{\mathbb{C}} \operatorname{Span}_{\mathbb{C}} \{\pi(g) v \mid g \in G\} < \infty$

(2) $\exists$ a f.d. G-inv't subspace $W \subset V$ with $v \in W$.

[Call such v G-finite]

Also, the set V_G of G-finite vectors is a G-inv't algebraic subspace of V.

Pf: $\text{Span}_{\mathbb{C}}\{\pi(g)v\}$ is a G-inv't subspace so (1) ⇒ (2), and it is contained in every G-inv't subspace containing v, so (2) ⇒ (1).

If W_1, W_2 are inv't f.d. subspaces, so is W_1+W_2, so if v_1, v_2 are G-finite so is $\alpha v_1 + v_2$. ∎

Prop: $\bigoplus_\sigma \mathcal{C}(\sigma) = L^2(G)_G$

Pf: $\mathcal{C}(\sigma)$ is a d_σ^2-dim inv't subspace, so

$$\bigoplus_\sigma \mathcal{C}(\sigma) \subseteq C(G)_G \subseteq L^2(G)_G.$$

Conversely let $W \subset L^2(G)_G$ be a f.d. subspace inv't by $(R_g f)(x) = f(xg)$. Want: $W \subset \bigoplus_\sigma \mathcal{C}(\sigma)$

Maschke: $W = \bigoplus$ irreps so wlog W is irred

let $\{f_i\}_{i=1}^d \subset W$ be an o.n.b. for each $f \in W$ each $g \in G$ $R_g f \in W$ so have $a_i(g)$ s.t.
$$R_g f = \sum_i a_i(g) f_i$$

Observe: $a_i(g) = \langle f_i, R_g f \rangle = \Phi^W_{f,f_i}(g)$

so $R_g f = \sum_i \Phi^W_{f,f_i}(g) f_i$ in $L^2(G)$

$\Rightarrow$ for a.e $x \in G$ $f(xg) = \sum_i \Phi^W_{f,f_i}(g) f_i(x)$

want to set $x=e$, get

$$f(g) = \sum_i f_i(e)\, \Phi^W_{f,f_i}(g)$$

can't quite.

Instead interpret identity in $L^2(G \times G)$:

① both sides are in $L^2(G \times G)$

② for each g, both sides are equal for a.e x

$\Rightarrow$ equal in $L^2(G \times G)$

③ so for a.e. x, $f(xg) = \sum_{i=1}^{d_W} f_i(x)\, \Phi^W_{f,f_i}(g)$

hold for a.e. g.

so for a.e x, $(g \mapsto f(xg)) \in \mathcal{C}(W)$ but $\mathcal{C}(W)$ also inv't under left translation $\blacksquare$

Def: For $f \in C(G)$, $\underline{v} \in V$ set

$$\pi(f)\underline{v} = \int_G f(g) \cdot (\pi(g)\underline{v})\, dg.$$

Lemma: $\pi(f): V \to V$ is a cts linear map,
$f \mapsto \pi(f)$ is a cts algebra hom $C(G) \to End(V)$

Pf: May assume $|f(g)| \le 1$ for all g.
If $g \in G$ can find convex nbd $\underline{0} \in \mathcal{U}_g \subset V$
nbd $g \in W_g \subset G$
st if $x \in W_g$, $\underline{v} \in \mathcal{U}_g$ then $\pi(x)\underline{v} \in \mathcal{U}$
where $\mathcal{U} \subset V$ $\underbrace{\text{fixed}}_{\text{convex}}$ nbd of $\underline{0}$

cover G with $\{W_{g_i}\}_{i=1}^{r}$, let $\tilde{\mathcal{U}} = \bigcap_{i=1}^{r} \mathcal{U}_{g_i}$.

St $x \in G$, $\underline{v} \in \tilde{\mathcal{U}}$ then $\pi(x)\underline{v} \in \mathcal{U}$

By convexity of $\mathcal{U}$, $\int_G f(x)\,\pi(x)\underline{v}\,dx \in \mathcal{U}$.

$\vdots$

Cor: Let $\{f_n\} \subset C(G)$ st $f_n \to \delta_e$
then $\pi(f_n)\underline{v} \to \underline{v}$ | $f_n \ge 0$, supported in decreasing nbds of e
$\int f_n = 1$

Example: Let $V \subset L^2(G)$ be a closed G-inv't subspace. Then $C(G) \cap V$ are dense in V.

Pf: if $\varphi \in V$, $f \in C(G)$ then $\pi(f)\varphi$ is cts:

$$(\pi(f)\varphi)(x) = \int f(g)\,\varphi(g^{-1}x)\,dg$$
$$= \int f(x^{-1}g)\;(g^{-1})\,dg$$

$$\Rightarrow \left|(\pi(f)\varphi)(x) - (\pi(f)\varphi)(y)\right| \leq \int \left|f(x^{-1}g) - f(y^{-1}g)\right|$$
$$|\varphi(g)^{-1}|\,dg$$

f unif cts, so for x close to y $|f(x^{-1}g) - f(y^{-1}g)| < \varepsilon$

then

$$|\quad - \quad| \underset{\text{C-S}}{\leq} \varepsilon \cdot \sqrt{\|\varphi\|_2}$$