Lior Silberman's Math 412: Problem Set 7 (due 10/3/2023)

M1. (Minimal polynomials)

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}$.

(a) Find the minimal polynomial of A and show that the minimal polynomial of B is $x^2(x-1)^2$.

(b) Find a 3×3 matrix whose minimal polynomial is x^2.

M2. For each of A, B find its eigenvalues and the correpsonding generalized eigenspaces.

Triangular matrices

1. Let U be an upper-triangular square matrix with non-zero diagonal entries.
   (a) Give a "backward-substitution" algorithm for solving $U\underline{x} = \underline{b}$ efficiently.
   (b) Explicitely use your algorithm to solve $\begin{pmatrix} 1 & 4 & 5 \\ & 2 & 6 \\ & & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$.
   (c) For a general upper-triangular U give a formula for U^{-1}, proving in particular that U is invertible and that U^{-1} is again upper-triangular.
   RMK It's possible to show that if $\mathcal{A} \subset M_n(F)$ is a subspace containing the identity matrix and closed under matrix multiplication, then the inverse of any matrix in $\mathcal{A}$ belongs to $\mathcal{A}$. This applies, in particular, to the set of upper-triangular matrices.

The minimal polynomial

2. Let $D \in M_n(F) = \operatorname{diag}(a_1, \ldots, a_n)$ be diagonal.
   (a) For any polynomial $p \in F[x]$ show that $p(D) = \operatorname{diag}(p(a_1), \ldots, p(a_n))$.
   (b) Show that the minimal polynomial of D is $m_D(x) = \prod_{j=1}^{r}(x - a_{i_j})$ where $\left\{a_{i_j}\right\}_{j=1}^{r}$ is an enumeration of the distinct values among the a_i.
   (c) Show that (over any field) the matrix B from problem P1 is not similar to a diagonal matrix.
   (d) Now suppose that U is an upper-triangular matrix with diagonal D. Show that for any $p \in F[x]$, $p(U)$ has diagonal $p(D)$. In particular, $m_D | m_U$.

3. Let $T \in \operatorname{End}(V)$ be diagonable. Show that every generalized eigenspace is simply an eigenspace.

4. Let $S \in \operatorname{End}(U)$, $T \in \operatorname{End}(V)$. Let $S \oplus T \in \operatorname{End}(U \oplus V)$ be the "block-diagonal map".
   (a) For $f \in F[x]$ show that $f(S \oplus T) = f(S) \oplus f(T)$.
   (b) Show that $m_{S\oplus T} = \operatorname{lcm}(m_S, m_T)$ ("least common multiple": the polynomial of smallest degree which is a multiple of both).
   (c) Conclude that $\operatorname{Spec}_F(S \oplus T) = \operatorname{Spec}_F(S) \cup \operatorname{Spec}_F(T)$.
   RMK See also problem B below.

Supplementary problems

A. Let $R \in \mathrm{End}(U \oplus V)$ be "block-upper-triangular", in that $R(U) \subset U$.

(a) Define a "quotient linear map" $\bar{R} \in \mathrm{End}(U \oplus V/U)$.

(b) Let S be the restriction of R to U. Show that both m_S, $m_{\bar{R}}$ divide m_R.

(c) Let $f = \mathrm{lcm}[m_S, m_{\bar{R}}]$ and set $T = f(R)$. Show that $T(U) = \{\underline{0}\}$ and that $T(V) \subset U$.

(d) Show that $T^2 = 0$ and conclude that $f \mid m_R \mid f^2$.

(e) Show that $\mathrm{Spec}_F(R) = \mathrm{Spec}_F(S) \cup \mathrm{Spec}_F(\bar{R})$.

B. (Cholesky decomposition)

(a) Let A be a positive-definite square matrix. Show that $A = LL^\dagger$ for a unique lower-triangular matrix L with positive entries on the diagonal.

DEF For $\varepsilon \in \pm 1$ define $D^\varepsilon \in M_n(\mathbb{R})$ by $D^\varepsilon_{ij} = \begin{cases} \varepsilon & j = i + \varepsilon \\ -\varepsilon & j = i \\ 0 & j \neq i, i+\varepsilon \end{cases}$ and let $A = -D^- D^+$ be the (positive) discrete Laplace operator.

(b) To $f \in C^\infty(0,1)$ associate the vector $\underline{f} \in \mathbb{R}^n$ where $\underline{f}(i) = f(\frac{i}{n})$. Show that $\frac{1}{n} D^+ \underline{f}$ and $\frac{1}{n} D^- \underline{f}$ are both close to $\underline{f'}$ (so that both are discrete differentiation operators). Show that $\frac{1}{n^2} D^- D^+$ is an approximation to the second derivative.

(c) Find a lower-triangular matrix L such that $LL^\dagger = A$.

C. (Rational canonical form part I) Let $T \in \mathrm{End}(V)$. For monic irreducible $p \in F[x]$ define $V_p = \left\{ \underline{v} \in V \mid \exists k : p(T)^k \underline{v} = \underline{0} \right\}$.

(a) Show that V_p is a T-invariant subspace of V and that $m_{T \restriction V_p} = p^k$ for some $k \geq 0$, with $k \geq 1$ iff $V_p \neq \{\underline{0}\}$. Conclude that $p^k | m_T$.

(b) Show that if $\{p_i\}_{i=1}^r \subset F[x]$ are distinct monic irreducibles then the sum $\bigoplus_{i=1}^r V_{p_i}$ is direct.

(c) Let $\{p_i\}_{i=1}^r \subset F[x]$ be the prime factors of $m_T(x)$. Show that $V = \bigoplus_{i=1}^r V_{p_i}$.

(d) Suppose that $m_T(x) = \prod_{i=1}^r p_i^{k_i}(x)$ is the prime factorization of the minimal polynomial. Show that $V_{p_i} = \mathrm{Ker}\, p_i^{k_i}(T)$.