Math 100C – WORKSHEET 8
DIFFERENTIAL EQUATIONS

1. Manipulating Taylor expansions

Let $c_k = \frac{f^{(k)}(a)}{k!}$. The nth order Taylor expansion of $f(x)$ about $x = a$ is the polynomial

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n$$

In addition we have the following expansions about $x = 0$:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots; \qquad \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

(1) (Final, 2016) Use a 3rd order Taylor approximation to estimate $\sin 0.01$. Then find the 3rd order Taylor expansion of $(x+1)\sin x$ about $x = 0$.

(2) Find the 3rd order Taylor expansion of $\sqrt{x} - \frac{1}{4}x$ about $x = 4$.

(3) Expand $\frac{e^{x^2}}{1+x}$ to second order about $x = 1$.

(4) Find the 8th order expansion of $f(x) = e^{x^2} - \frac{1}{1+x^3}$. What is $f^{(6)}(0)$?

(5) Show that $\log \frac{1+x}{1-x} \approx 2(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots)$. Use this to get a good approximation to $\log 3$ via a careful choice of x.

Date: 3/11/2022, Worksheet by Lior Silberman.

2. Differential equations

(6) For each equation: Is $y = 3$ a solution? Is $y = 2$ a solution? What are *all* the solutions?

$$y^2 = 4 \qquad ; \qquad y^2 = 3y$$

(7) For each equation: Is $y(x) = x^2$ a solution? Is $y(x) = e^x$ a solution?

$$\frac{dy}{dx} = y \qquad ; \qquad \left(\frac{dy}{dx}\right)^2 = 4y$$

(8) Which of the following (if any) is a solution of $\frac{dz}{dt} + t^2 - 1 = z$ (challenge: find more solutions):

A. $z(t) = t^2$; B. $z(t) = t^2 + 2t + 1$

(9) The balance of a bank account satisfies the differential equation $\frac{dy}{dt} = 1.04y$ (this represents interest of 4% compounded continuously). Sketch the solutions to the differential equation. What is the solution for which $y(0) = \$100$?

(10) Suppose $\frac{dy}{dx} = ay$, $\frac{dz}{dx} = bz$. Can you find a differential equation satisfied by $w = \frac{y}{z}$? Hint: calculate $\frac{dw}{dx}$.

3. Solutions by massaging and ansatze

(11) For which value of the constant ω is $y(t) = \sin(\omega t)$ a solution of the oscillation equation $\frac{d^2y}{dt^2} + 4y = 0$?

(12) (The quantum harmonic oscillator) For which value of the constants A, B (with $B > 0$) does the function $f(x) = Axe^{-Bx^2}$ satisfy $-f'' + x^2 f = 3f$? What if we also insist that $f(1) = 1$?

(13) Consider the equation $\frac{dy}{dt} = a(y - b)$.

(a) Define a new function $u(t) = y(t) - b$. What is the differential equation satisfied by uv?

(b) What is the general solution for $u(t)$?

(c) What is the general solution for $y(t)$?

(d) Suppose $a < 0$. What is the asymptotic behaviour of the solution as $t \to \infty$?

(e) Suppose we are given the *initial value* $y(0)$. What is C? What is the formula for $y(t)$ using this?