Math 100C – SOLUTIONS TO WORKSHEET 6
CURVE SKETCHING; TAYLOR EXPANSION

1. Curve sketching

Let $f(x) = \frac{x^3+2}{x^2+1}$; and that $f''(x) = -2\frac{x^3-6x^2-3x+2}{(x^2+1)^3}$

(1) Zeroeth derivative questions
  (a) Where is f defined?
  Solution: f is defined on the entire axis since $x^2+1>0$ for all x.
  (b) List the vertical asymptotes of f, if any?
  Solution: No; f is defined by formula hence continuous everywhere and does not blow up.
  (c) What are the asymptotic behaviours of f at $\pm\infty$?
  Solution: When x is large (whether negative or positive) we have $x^3+2 \sim x^3$ and x^2+1 so $f(x) \sim \frac{x^3}{x^2} = x$ on both ends.
  (d) Where does f meet the axes?
  Solution: $f(0)=2$; $f(x)=0$ iff $x^3=-2$ that is at $x=-\sqrt[3]{2}$.

(2) It is a fact that $f'(x) = \frac{x(x-1)(x^2+x+4)}{(x^2+1)^2}$
  (a) Where is f differentiable?
  Solution: f' is defined on the entire axis since $x^2+1>0$ for all x.
  (b) Where does $f'(x)=0$? Where it is positive? Negative?
  Solution: Clearly $f'(0)=f'(1)=0$. Now $x^2+x+4=(x+\frac{1}{2})^2+\frac{15}{4}$ is positive everywhere so the only zeroes of the derivative are $0,1$. The sign of the derivative is then the sign of $x(x-1)$ so the derivative is positive when $x<0$ or $x>1$ and negative when $0<x<1$.
  (c) Where are the local extrema of f? What are the values at those points?
  Solution: $x=0$ is a local maximum, since f is increasing on its left and decreasing on its right. $x=1$ is a local minimum for the same reasons. $f(0)=2$, $f(1)=\frac{3}{2}$.

(3) It is a fact that $f''(x) = -2\frac{x^3-6x^2-3x+2}{(x^2+1)^3}$.
  (a) Where is f'' positive/negative? Where does it vanish? Say as much as you can.
  Solution: The sign of f'' is the sign of $h(x) = -\left(x^3-6x^2-3x+2\right)$. Now $h(x) \sim -x^3$ at infinity, so h is positive for $x \ll 0$ and negative for $x \gg 0$. Next, $h(0)=-2<0$ and $h(1)=6>0$. Since $h(-1)=2>0$ we conclude that f'' is initially positive, crosses the axis somewhere on $(-1,0)$ to become negative, crossess the axis again on $(0,1)$, and the crosses the axis a final time to become negative somewhere on $(1,\infty)$. Since h is cubic polynomial it has at most three roots, so those are the only sign changes of h hence of f''.
  (b) Where is f concave up/down? Where are its inflection points?
  Solution: By part (a) we conclude that f is initially concave down, has an inflection point somehwere on $(-1,0)$ after which it is concave up, has a second inflection point on $(0,1)$ after which it is concave down, and then has a third inflection point after which it is concave up.

(4) Draw a sketch of the graph of f, incorporating all the features you have identified in questions 1-3.

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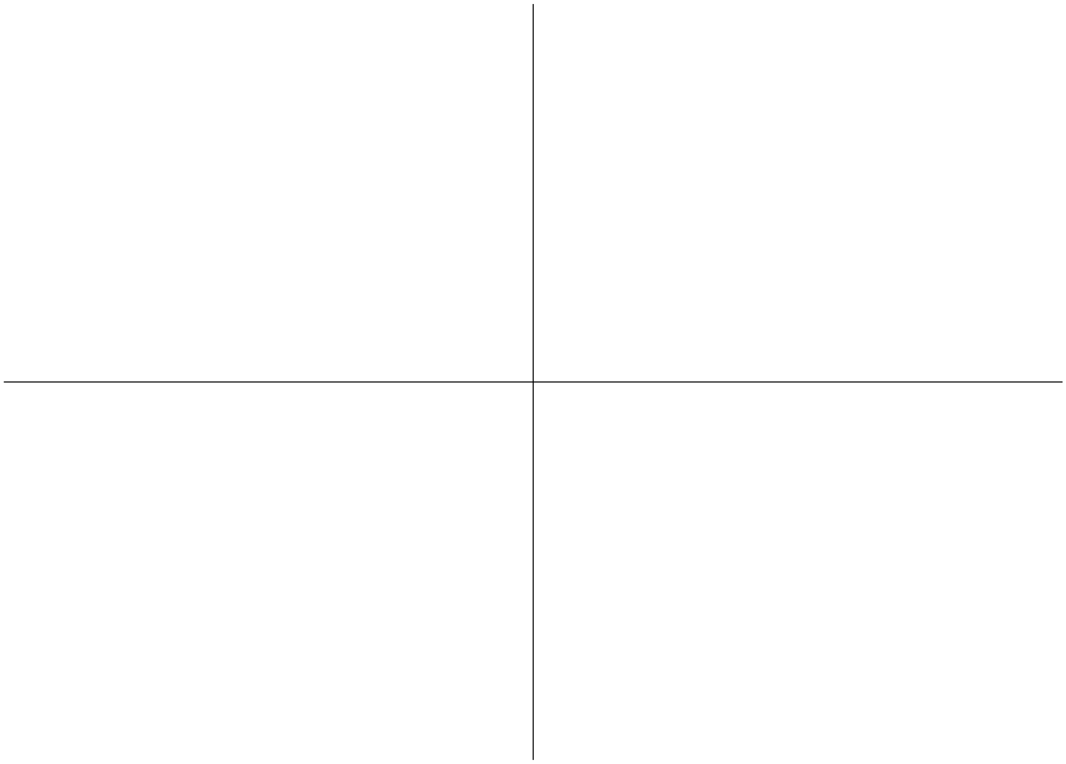

- Extra credit: Find the constant b so that $f(x) \approx x+b$ as $x \to \infty$ (in the sense that $f(x)-x-b \to 0$). We call this line a *slant asymptote* for f.

  Solution: $\frac{x^3+2}{x^2+1} - x = \frac{2-x}{x^2+1} \sim -\frac{1}{x} \to 0$ so $f(x) \approx x$ is actually correct.

  Solution: We have $\frac{x^3+2}{x^2+1} = x\frac{1+\frac{2}{x^2}}{1+\frac{1}{x^2}} = x\left(1+\frac{2}{x^2}\right)\left(1-\frac{1}{x^2}+\frac{1}{x^4}+\cdots\right) = x\left(1+\frac{1}{x^2}-\frac{1}{x^4}+\frac{1}{x^6}-\frac{1}{x^8}+\cdots\right)$ from which we can read off $f(x) \approx x + \frac{1}{x}$ as $|x| \to \infty$.

2. Taylor expansion

(5) (Review) Use linear approximations to estimate:

  (a) $\log\frac{4}{3}$ and $\log\frac{2}{3}$. Combine the two for an estimate of $\log 2$.

  Solution: Let $f(x) = \log x$ so that $f'(x) = \frac{1}{x}$. Then $f(1) = 0$ and $f'(1) = 1$ so $f(1+\frac{1}{3}) \approx \frac{1}{3}$ and $f(1-\frac{1}{3}) \approx -\frac{1}{3}$. Then $\log 2 = \log\frac{4}{3}\big/\frac{2}{3} = \log\frac{4}{3} - \log\frac{2}{3} \approx \frac{2}{3}$.

  (b) $\sin 0.1$ and $\cos 0.1$.

  Solution: Let $f(x) = \sin x$ so that $g(x) = f'(x) = \cos x$ and $g'(x) = -\sin x$. Then $f(1) = 0$ and $g(0) = f'(0) = \cos 0 = 1$ while $g'(0) = -\sin 0 = 0$. So $f(0.1) \approx 0 + 1\cdot 0.1 \approx 0.1$ and $g(0.1) \approx 1 - 0\cdot 0.01 = 1$.

(6) Let $f(x) = e^x$

  (a) Find $f(0), f'(0), f^{(2)}(0), \cdots$

  (b) Find a polynomial $T_0(x)$ such that $T_0(0) = f(0)$.

  (c) Find a polynomial $T_1(x)$ such that $T_1(0) = f(0)$ and $T_1'(0) = f'(0)$.

  (d) Find a polynomial $T_2(x)$ such that $T_2(0) = f(0)$, $T_2'(0) = f'(0)$ and $T_2^{(2)}(0) = f^{(2)}(0)$.

  (e) Find a polynomial $T_3(x)$ such that $T_3^{(k)}(0) = f^{(k)}(0)$ for $0 \le k \le 3$.

  Solution: $f(x) = f'(x) = f^{(2)}(x) = \cdots = e^x$ so $f(0) = f'(0) = f''(0) = \cdots = 1$. Now $T_0(x) = 1$ works, as does $T_1(x) = 1+x$. If $T_2(x) = 1+x+cx^2$ then $T_2''(x) = 2c = 1$ means $c = \frac{1}{2}$ and $T_2(x) = 1+x+\frac{1}{2}x^2$. Finally, $T_3(x) = 1+x+\frac{1}{2}x^2+dx^3$ works if $6d = 1$ so if $d = \frac{1}{6}$.

(7) Do the same with $f(x) = \log x$ about $x = 1$.

  Solution: $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$, $f'''(x) = \frac{2}{x^3}$ so $f(1) = 0$, $f'(1) = 1$, $f''(1) = -1$, $f'''(1) = 2$. Try $T_3(x) = a + bx + cx^2 + dx^3$ (can truncate later). Need $a = 0$ to make $T_3(x) = 0$. Diff we get

$T_3'(x) = b + 2cx + 3dx^2$, setting $x = 0$ gives $b = 1$. Diff again gives $T_3''(x) = 2c + 6dx$ so $2c = -1$ and $c = -\frac{1}{2}$. Diff again give $T_3'''(x) = 6d = 2$ so $d = \frac{1}{3}$ and $T_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$. Truncate this to get T_0, T_1, T_2.

Let $c_k = \frac{f^{(k)}(a)}{k!}$. The nth order Taylor expansion of $f(x)$ about $x = a$ is the polynomial

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n$$

(8) Find the 4th order MacLaurin expansion of $\frac{1}{1-x}$ (=Taylor expansion about $x = 0$)

Solution: $f'(x) = \frac{1}{(1-x)^2}$, $f''(x) = \frac{2}{(1-x)^3}$, $f^{(3)}(x) = \frac{6}{(1-x)^4}$, $f^{(4)}(x) = \frac{24}{(1-x)^5}$ $f^{(k)}(0) = k!$ and the Taylor expansion is $1 + x + x^2 + x^3 + x^4$.

(9) Find the nth order expansion of $\cos x$, and approximate $\cos 0.1$ using a 3rd order expansion

Solution: $(\cos x)' = -\sin x$, $(\cos x)^{(2)} = -\cos x$, $(\cos x)^{(3)} = \sin x$, $(\cos x)^{(4)}(x) = \cos x$ and the pattern repeats. Plugging in zero we see that the derivatives at 0 (starting with the zeroeth) are $1, 0, -1, 0, 1, 0, -1, 0, \ldots$ so the Taylor expansion is

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$

In particular, $\cos 0.1 \approx 1 - \frac{1}{2}(0.1)^2 = 0.995$.

(10) (Final, 2015) Let $T_3(x) = 24 + 6(x-3) + 12(x-3)^2 + 4(x-3)^3$ be the third-degree Taylor polynomial of some function f, expanded about $a = 3$. What is $f''(3)$?

Solution: We have $c_2 = \frac{f^{(2)}}{2!} = 12$ so $f^{(2)} = 24$.

(11) In labour economics, the *CES production function* is the functional form $Q(K, E) = \left[\alpha K^\delta + (1-\alpha)E^\delta\right]^{1/\delta}$. Here K is capital, E is employment, and $\delta < 1$ measures the degree of substitution between labour and capital. Find the linear and quadratic expansions of Q in the variable E about the point $(K_0, E_0) = \left(\frac{1}{2}, \frac{1}{2}\right)$ if $\alpha = \frac{1}{2}$.

Solution: $\frac{\partial Q}{\partial E} = \frac{1-\alpha}{\delta}\left[\alpha K^\delta + (1-\alpha)E^\delta\right]^{1/\delta - 1} \delta E^{\delta-1} = (1-\alpha)\left[\alpha K^\delta + (1-\alpha)E^\delta\right]^{1/\delta-1} E^{\delta-1}$. Thus

$$\begin{aligned}
\frac{\partial^2 Q}{\partial E^2} &= (1-\alpha)(1-\delta)\left[\alpha K^\delta + (1-\alpha)E^\delta\right]^{1/\delta-2} E^{2(\delta-1)} + (1-\alpha)(1-\delta)\left[\alpha K^\delta + (1-\alpha)E^\delta\right]^{1/\delta-1} E^{\delta-2} \\
&= (1-\alpha)(1-\delta)\left[\alpha K^\delta + (1-\alpha)E^\delta\right]^{1/\delta-2} E^{\delta-2}\left[\alpha K^\delta + (2-\alpha)E^\delta\right].
\end{aligned}$$

Plugging in $\alpha = K = E = \frac{1}{2}$ gives $Q(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$; $Q'(\frac{1}{2}, \frac{1}{2})$ =2, $Q''(\frac{1}{2}, \frac{1}{2}) = 8(1-\delta)$ so for E close to $\frac{1}{2}$ we have

$$Q(\frac{1}{2}, E) \approx \frac{1}{2} + 2(E - \frac{1}{2}) + 4(1-\delta)(E - \frac{1}{2})^2$$

correct to second order.

3. New expansions from old

(12) (Final, 2016) Use a 3rd order Taylor approximation to estimate $\sin 0.01$. Then find the 3rd order Taylor expansion of $(x+1)\sin x$ about $x = 0$.

Solution: Let $f(x) = \sin x$. Then $f'(x) = \cos x$, $f^{(2)}(x) = -\sin x$ and $f^{(3)}(x) = -\cos x$. Thus $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f^{(3)}(0) = -1$ and the third-order expansion of $\sin x$ is $0 + \frac{1}{1!}x + \frac{0}{2!}x^2 + \frac{(-1)}{3!}x^3 = x - \frac{1}{6}x^3$. In particular $\sin 0.1 \approx 0.1 - \frac{1}{6000}$. We then also have, correct to third order, that

$$(x+1)\sin x \approx (x+1)\left(x - \frac{1}{6}x^3\right) = x + x^2 - \frac{1}{6}x^3 - \frac{1}{6}x^4 \approx x + x^2 - \frac{1}{6}x^3.$$

(13) Find the 3rd order Taylor expansion of $\sqrt{x} - \frac{1}{4}x$ about $x = 4$.

Solution: Let $f(x) = \sqrt{x}$. Then $f'(x) = \frac{1}{2\sqrt{x}}$, $f^{(2)}(x) = -\frac{1}{4x^{3/2}}$ and $f^{(3)}(x) = \frac{3}{8}x^{-5/2}$. Thus $f(4) = 2$, $f'(4) = \frac{1}{4}$, $f^{(2)}(4) = -\frac{1}{32}$, $f^{(3)}(4) = \frac{3}{256}$ and the third-order expansions are

$$\sqrt{x} \approx 2 + \frac{1}{4}(x-4) - \frac{1}{32 \cdot 2!}(x-4)^3 + \frac{3}{256 \cdot 3!}(x-4)^3$$
$$\frac{1}{4}x \approx 1 + \frac{1}{4}(x-4)$$

so that

$$\sqrt{x} - \frac{1}{4}x \approx 1 - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3 .$$

(14) Find the 8th order expansion of $f(x) = e^{x^2} - \frac{1}{1+x^3}$. What is $f^{(6)}(0)$?

Solution: To fourth order we have $e^u \approx 1 + u + \frac{u^2}{2} + \frac{u^3}{6} + \frac{u^4}{24} + \frac{u^5}{120}$ so $e^{x^2} \approx 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \frac{x^8}{24}$ to 8th order. We also know that $\frac{1}{1-u} \approx 1 + u + u^2 + u^3$ so $\frac{1}{1+x^3} \approx 1 - x^3 + x^6$ correct to 8th order. We conclude that

$$e^{x^2} + \cos(2x) \approx \left(1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \frac{x^8}{24}\right) - \left(1 - x^3 + x^6\right)$$
$$\approx x^2 - x^3 + \frac{1}{2}x^4 - \frac{5}{6}x^6 + \frac{1}{24}x^8 .$$

In particular, $\frac{f^{(6)}(0)}{6!} = -\frac{5}{6}$ so $f^{(6)}(0) = -720 \cdot \frac{5}{6} = -600$.

(15) Show that $\log \frac{1+x}{1-x} \approx 2(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots)$. Use this to get a good approximation to $\log 3$ via a careful choice of x.

Solution: Let $f(x) = \log(1+x)$. Then $f'(x) = \frac{1}{1+x}$, $f^{(2)}(x) = -\frac{1}{(1+x)^2}$, $f^{(3)}(x) = \frac{1 \cdot 2}{(1+x)^3}$, $f^{(4)}(x) = -\frac{1 \cdot 2 \cdot 3}{(1+x)^4}$ and so on, so $f^{(k)}(x) = (-1)^{k-1} \cdot \frac{(k-1)!}{(1+x)^k}$. We thus have that $f(0) = 0$ and for $k \geq 1$ that $f^{(k)}(0) = (-1)^{k-1}(k-1)!$. Then $\frac{f^{(k)}(0)}{k!} = \frac{(-1)^{k-1}}{k}$ so

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots .$$

Plugging $-x$ we get:

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \cdots$$

so

$$\log \frac{1+x}{1-x} = \log(1+x) - \log(1-x) = 2x + 2\frac{x^3}{3} + 2\frac{x^5}{5} + \cdots .$$

In particular

$$\log 3 = \log \frac{1+\frac{1}{2}}{1-\frac{1}{2}} = 2\left(\frac{1}{2} + \frac{1}{24} + \frac{1}{160} + \cdots\right) = 1 + \frac{1}{12} + \frac{1}{80} + \cdots \approx 1.096$$