Math 100C – SOLUTIONS TO WORKSHEET 5
THE CHAIN RULE ETC

1. The Chain Rule

(1) We know $\frac{d}{dy}\sin y = \cos y$.
  (a) Expand $\sin(y+h)$ to linear order in h. Write down the linear approximation to $\sin y$ about $y = a$.
  Solution: $\sin(y+h) \approx \sin y + h\cos y$ and $\sin y \approx \sin a + (y-a)\cos a$.
  (b) Now let $F(x) = \sin(3x)$. Expand $F(x+h)$ to linear order in h. What is the derivative of $\sin 3x$?
  Solution: $F(x+h) = \sin(3(x+h) = \sin(3x+3h)$ so we use $y = 3x$ in the previous example to get

$$\begin{aligned} F(x+h) &= \sin\left(3(x+h)\right) \\ &= \sin\left(3x+3h\right) \\ &\approx \sin(3x) + (3h)\cos(3x) \\ &= \sin(3x) + (3\cos(3x))h \end{aligned}$$

  so the derivative is $\boxed{3\cos(3x)}$.

(2) Write each function as a composition and differentiate
  (a) e^{3x}
  Solution: This is $f(g(x))$ where $g(x) = 3x$ and $f(y) = e^y$. The derivative is thus

$$e^{3x} \cdot \frac{\mathrm{d}(3x)}{\mathrm{d}x} = 3e^{3x}\,.$$

  (b) $\sqrt{2x+1}$
  Solution: This is $f(g(x))$ where $g(x) = 2x+1$ and $f(y) = \sqrt{y}$. Thus

$$\frac{\mathrm{d}f(g(x))}{\mathrm{d}x} = f'(g(x))g'(x) = \frac{1}{2\sqrt{g}} \cdot 2 = \frac{1}{\sqrt{2x+1}}\,.$$

  (c) (Final, 2015) $\sin(x^2)$
  Solution: This is $f(g(x))$ where $g(x) = x^2$ and $f(y) = y^2$. The derivative is then

$$\cos(x^2) \cdot 2x = 2x\cos(x^2)\,.$$

  (d) $(7x + \cos x)^n$.
  Solution: This is $f(g(x))$ where $g(x) = 7x + \cos x$ and $f(y) = y^n$. The derivative is thus

$$n\left(7x+\cos x\right)^{n-1} \cdot \left(7 - \sin x\right)\,.$$

(3) (Final, 2012) Let $f(x) = g(2\sin x)$ where $g'(\sqrt{2}) = \sqrt{2}$. Find $f'\left(\frac{\pi}{4}\right)$.
  Solution: By the chain rule, $f'(x) = g'(2\sin x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}(2\sin x) = 2g'(2\sin x)\cos x$. In particular,

$$\begin{aligned} f'\left(\frac{\pi}{4}\right) &= 2g'\left(2\sin\frac{\pi}{4}\right)\cos\frac{\pi}{4} = 2g'\left(2\frac{\sqrt{2}}{2}\right) \cdot \frac{\sqrt{2}}{2} \\ &= 2\sqrt{2} \cdot \frac{\sqrt{2}}{2} = 2. \end{aligned}$$

(4) Differentiate

Date: 13/10/2022, Worksheet by Lior Silberman.

(a) $7x + \cos(x^n)$

Solution: We apply linearity and then the chain rule:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(7x + \cos(x^n)\right) &= \frac{\mathrm{d}(7x)}{\mathrm{d}x} + \frac{\mathrm{d}\cos(x^n)}{\mathrm{d}x} \\ &= 7 + \frac{\mathrm{d}\cos(x^n)}{\mathrm{d}(x^n)} \cdot \frac{\mathrm{d}(x^n)}{\mathrm{d}x} \\ &= 7 - \sin(x^n) \cdot nx^{n-1}\,.\end{aligned}$$

(b) $e^{\sqrt{\cos x}}$

Solution: We repeatedly apply the chain rule:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x} e^{\sqrt{\cos x}} &= e^{\sqrt{\cos x}} \frac{\mathrm{d}}{\mathrm{d}x}\sqrt{\cos x} \\ &= e^{\sqrt{\cos x}} \frac{1}{2\sqrt{\cos x}} \frac{\mathrm{d}}{\mathrm{d}x} \cos x \\ &= -e^{\sqrt{\cos x}} \frac{\sin x}{2\sqrt{\cos x}}\,.\end{aligned}$$

(c) (Final 2012) $e^{(\sin x)^2}$

Solution: By the chain rule:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{(\sin x)^2}\right) &= e^{(\sin x)^2} \frac{\mathrm{d}}{\mathrm{d}x}\left((\sin x)^2\right) \\ &= e^{(\sin x)^2} 2\sin x \frac{\mathrm{d}}{\mathrm{d}x} \sin x \\ &= e^{(\sin x)^2} 2 \sin x \cos x \\ &= e^{(\sin x)^2} \sin(2x).\end{aligned}$$

(5) Suppose f, g are differentiable functions with $f(g(x)) = x^3$. Suppose that $f'(g(4)) = 5$. Find $g'(4)$.

Solution: Applying the chain rule we have $f'(g(x)) \cdot g'(x) = 3x^2$. Plugging in $x = 4$ we get $5g'(4) = 3 \cdot 4^2$ and hence $g'(4) = \frac{48}{5}$.

2. Logarithmic differentiation

(6) $\log\left(e^{10}\right) =$ $\qquad$ $\log(2^{100}) =$

Solution: $\log e^{10} = 10$ while $\log(2^{100}) = 100 \log 2$.

(7) Differentiate

(a) $\frac{\mathrm{d}(\log(ax))}{\mathrm{d}x} =$ $\qquad$ $\frac{\mathrm{d}}{\mathrm{d}t}\log\left(t^2 + 3t\right) =$

Solution: By the chain rule, the derivatives are: $\frac{1}{ax} \cdot a = \frac{1}{x}$ and $\frac{1}{t^2+3t} \cdot (2t+3) = \frac{2t+t}{t^2+3t}$. We can also use the logarithm laws first: $\log(ax) = \log a + \log x$ so $\frac{\mathrm{d}}{\mathrm{d}x}(\log ax) = \frac{\mathrm{d}}{\mathrm{d}x}(\log a) + \frac{\mathrm{d}}{\mathrm{d}x}(\log x) = \frac{1}{x}$ since $\log a$ is constant if a is. Similarly, $\log(t^2+3t) = \log t + \log(t+3)$ so its derivative is $\frac{1}{t} + \frac{1}{t+3}$.

(b) $\frac{\mathrm{d}}{\mathrm{d}x} x^2 \log(1 + x^2) =$ $\qquad$ $\frac{\mathrm{d}}{\mathrm{d}r}\frac{1}{\log(2+\sin r)} =$

Solution: Applying the product rule and then the chain rule we get: $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\log(1+x^2)\right) = 2x\log(1+x^2) + x^2\frac{1}{1+x^2} \cdot 2x = 2x\log(1+x^2) + \frac{2x^3}{1+x^2}$. Using the quotient rule and the chain rule we get

$$\frac{\mathrm{d}}{\mathrm{d}r}\frac{1}{\log(2+\sin r)} = -\frac{1}{\log^2(2+\sin r)} \cdot \frac{1}{2+\sin r} \cdot \cos r = -\frac{\cos r}{(2+\sin r)\log^2(2+\sin r)}\,.$$

(8) (Logarithmic differentiation) differentiate
$y = \left(x^2+1\right) \cdot \sin x \cdot \frac{1}{\sqrt{x^3+3}} \cdot e^{\cos x}$.

Solution: We have

$$\log y = \log\left(x^2+1\right) + \log(\sin x) + \log\left(\frac{1}{\sqrt{x^3+3}}\right) + \log\left(e^{\cos x}\right)$$
$$= \log\left(x^2+1\right) + \log\left(\sin x\right) - \frac{1}{2}\log\left(x^3+3\right) + \cos x\,.$$

Differentiating with respect to x gives:

$$\frac{y'}{y} = \frac{2x}{x^2+1} + \frac{\cos x}{\sin x} - \frac{1}{2}\frac{3x^2}{x^3+3} - \sin x$$

and solving for y' finally gives

$$y' = \left(\frac{2x}{x^2+1} + \frac{\cos x}{\sin x} - \frac{3x}{2(x^3+3)} - \sin x\right)\cdot\left(x^2+1\right)\cdot\sin x\cdot\frac{1}{\sqrt{x^3+3}}\cdot e^{\cos x}\,.$$

(9) Differentiate using $\boxed{f' = f\times(\log f)'}$

(a) x^n

Solution: If $y = x^n$ then $\log y = n\log x$. Differentiating with respect to x gives $\frac{1}{y}y' = \frac{n}{x}$ so $y' = y\frac{n}{x} = nx^{n-1}$.

Solution: By the rule, $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^n\right) = x^n\frac{\mathrm{d}}{\mathrm{d}x}\left(\log(x^n)\right) = x^n\left(\frac{n}{x}\right) = nx^{n-1}$.

(b) x^x

Solution: If $y = x^x$ then $\log y = x\log x$. Differentiating with respect to x gives $\frac{1}{y}y' = \log x + x\cdot\frac{1}{x} = \log x + 1$ so $y' = y\left(\log x + 1\right) = x^x\left(\log x + 1\right)$.

Solution: By the rule, $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^x\right) = x^x\frac{\mathrm{d}}{\mathrm{d}x}\left(\log(x^x)\right) = x^x\left(\log x + 1\right)$.

Solution: We have $x^x = \left(e^{\log x}\right)^x = e^{x\log x}$. Applying the chain rule we now get $\left(x^x\right)' = e^{x\log x}\left(\log x + 1\right) = x^x\left(\log x + 1\right)$.

(c) $(\log x)^{\cos x}$

Solution: By the logarithmic differentiation rule we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\log x\right)^{\cos x} = \left(\log x\right)^{\cos x}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos x\log(\log x)\right)$$
$$= -\sin x\log\log x\left(\log x\right)^{\cos x} + \left(\log x\right)^{\cos x}\cos x\frac{1}{\log x}\frac{1}{x}$$
$$= -\sin x\log\log x\left(\log x\right)^{\cos x} + \cos x\left(\log x\right)^{\cos x - 1}\frac{1}{x}\,.$$

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(d) (Final, 2014) Let $y = x^{\log x}$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x only.

Solution: By the logarithmic differentiation rule we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y\frac{\mathrm{d}\log y}{\mathrm{d}x} = x^{\log x}\frac{\mathrm{d}}{\mathrm{d}x}\left(\log x\cdot\log x\right)$$
$$= x^{\log x}\left(2\log x\cdot\frac{1}{x}\right) = 2\log x\cdot x^{\log x - 1}\,.$$

3. Implicit Differentiation

(10) Find the line tangent to the curve $y^2 = 4x^3 + 2x$ at the point $(2,6)$.

Solution: Differentiating with respect to x we find $2y\frac{\mathrm{d}y}{\mathrm{d}x} = 12x^2 + 2$, so that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6x^2+1}{y}$. In particular at the point $(2,6)$ the slope is $\frac{25}{6}$ and the line is

$$y = \frac{25}{6}(x-2) + 6\,.$$

(11) (Final, 2015) Let $xy^2 + x^2y = 2$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ at the point $(1,1)$.

Solution: Differentiating with respect to x we find $y^2 + 2xy\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy + x^2\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ along the curve. Setting $x = y = 1$ we find that, at the indicated point,

$$3 + 3\frac{\mathrm{d}y}{\mathrm{d}x}\Big|_{(1,1)} = 0$$

so

$$\frac{\mathrm{d}y}{\mathrm{d}x}\Big|_{(1,1)} = -1\,.$$

(12) (Final 2012) Find the slope of the line tangent to the curve $y + x\cos y = \cos x$ at the point $(0, 1)$.
Solution: Differentiating with respect to x we find $y' + \cos y - x\sin y \cdot y' = -\sin x$, so that $y' = -\frac{\sin x+\cos y}{1-x\sin y} = \frac{\sin x+\cos y}{x\sin y-1}$. Setting $x = 0$, $y = 1$ we get that at that point $y' = \frac{\cos 1}{-1} = -\cos 1$.

(13) Find y'' (in terms of x, y) along the curve $x^5 + y^5 = 10$ (ignore points where $y = 0$).
Solution: Differentiating with respect to x we find $5x^4 + 5y^4y' = 0$, so that $y' = -\frac{x^4}{y^4}$. Differentiating again we find

$$y'' = -\frac{4x^3}{y^4} + \frac{4x^4y'}{y^5} = -\frac{4x^3}{y^4} - \frac{4x^8}{y^9}\,.$$

4. Inverse trig functions

(14) Evaluation
(a) (Final 2014) Evaluate $\arcsin\left(-\frac{1}{2}\right)$; Find $\arcsin\left(\sin\left(\frac{31\pi}{11}\right)\right)$.
Solution: $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ so $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$. Also $\sin\left(\frac{31\pi}{11}\right) = \sin\left(\frac{31\pi}{11} - 2\pi\right) = \sin\left(\frac{9\pi}{11}\right) = \sin\left(\pi - \frac{9\pi}{11}\right) = \sin\left(\frac{2\pi}{11}\right)$ and $\frac{2\pi}{11} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so $\arcsin\left(\sin\left(\frac{31\pi}{11}\right)\right) = \frac{2\pi}{11}$.
(b) (Final 2015) Simplify $\sin(\arctan 4)$
Solution: Consider the right-angled triangle with sides 4, 1 and hypotenuse $\sqrt{1+4^2} = \sqrt{17}$. Let θ be the angle opposite the side of length 4. Then $\tan\theta = 4$ and $\sin\theta = \frac{4}{\sqrt{17}}$ so $\sin(\arctan 4) = \sin\theta = \frac{4}{\sqrt{17}}$.
(c) Find $\tan(\arccos(0.4))$
Solution: Consider the right-angled triangle with sides 0.4, $\sqrt{1-0.4^2}$ and hypotenuse 1. Let θ be the angle between the side of length 0.4 and the hypotenuse. Then $\cos\theta = \frac{0.4}{1} = 0.4$ and $\tan\theta = \frac{\sqrt{1-0.4^2}}{0.4} = \frac{\sqrt{0.84}}{0.4} = \sqrt{\frac{0.84}{0.16}} = \sqrt{5.25}$.

(15) Differentiation
(a) Find $\frac{\mathrm{d}}{\mathrm{d}x}(\arctan x)$
Solution: Let $\theta = \arctan x$. Then $x = \tan\theta$ so by the chain rule $1 = \frac{dx}{dx} = \frac{d\tan\theta}{dx} = \frac{d\tan\theta}{d\theta}\cdot\frac{d\theta}{dx} = (1+\tan^2\theta)\frac{d\theta}{dx}$ so

$$\frac{d(\arctan x)}{dx} = \frac{d\theta}{dx} = \frac{1}{1+\tan^2\theta} = \frac{1}{1+x^2}\,.$$

(b) Find $\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin(2x))$
Solution: Since $\frac{\mathrm{d}}{\mathrm{d}x}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$, the chain rule gives

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin(2x)) = \frac{2}{\sqrt{1-4x^2}}\,.$$

Alternatively, let $\theta = \arcsin 2x$, so that $\sin\theta = 2x$. Differentiating both sides we get

$$\cos\theta\cdot\frac{\mathrm{d}\theta}{\mathrm{d}x} = 2$$

so that

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{2}{\cos\theta} = \frac{2}{\sqrt{1-\sin^2\theta}} = \frac{2}{\sqrt{1-4x^2}}\,.$$

(c) Find the line tangent to $y = \sqrt{1 + (\arctan(x))^2}$ at the point where $x = 1$.

Solution: Since $\frac{\mathrm{d}}{\mathrm{d}x}\arctan(x) = \frac{1}{1+x^2}$, the chain rule gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{1+(\arctan(x))^2} = \frac{1}{2\sqrt{1+(\arctan(x))^2}} \cdot 2\arctan(x) \cdot \frac{1}{1+x^2}$$
$$= \frac{\arctan x}{(1+x^2)\sqrt{1+(\arctan(x))^2}} \, .$$

Now $\arctan 1 = \frac{\pi}{4}$ so the line is

$$y = \frac{\pi}{8\sqrt{1+\frac{\pi^2}{16}}}(x-1) + \sqrt{1+\frac{\pi^2}{16}} \, .$$

(d) Find y' if $y = \arcsin\left(e^{5x}\right)$. What is the domain of the functions y, y'?
Solution: From the chain rule we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin\left(e^{5x}\right) = \frac{1}{\sqrt{1-e^{10x}}}5e^{5x} = \frac{5e^{5x}}{\sqrt{1-e^{10x}}} \, .$$

The function y itself is defined when $-1 \le e^{5x} \le 1$, that is when $5x \le 0$, that is when $x \le 0$. The derivative is defined when $-1 < e^{10x} < 1$, that is when $x < 0$. The point is that since $\sin\theta$ has horizontal tangents at $\pm\frac{\pi}{2}$, $\arcsin x$ has vertical tangents at ± 1.
Solution: We can write the identity as $\sin y = e^{5x}$ and differentiate both sides to get $y' \cos y = 5e^{5x}$ so that

$$y' = \frac{5e^{5x}}{\cos y} = \frac{5e^{5x}}{\sqrt{1-\sin^2 y}} = \frac{5e^{5x}}{\sqrt{1-e^{10x}}} \, .$$