5. The Chain rule etc (13/10/2022)

Goals.

(1) The Chain Rule
(2) Logarithmic differentiation
(3) Implicit differentiation
(4) Inverse trig functions

Last Time.

Combining linear approx:

$$f(x+h) \approx f(x) + f'(x)\cdot h$$

$$g(x+h) \approx g(x) + g'(x)\cdot h$$

can add / multiply $\Rightarrow$ $(f+g)'(x) = f'(x) + g'(x)$

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

(can also divide) $\Rightarrow \left(\frac{f}{g}\right)'(x) = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{(g(x))^2}$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

WS1

Math 100C – WORKSHEET 5
THE CHAIN RULE ETC

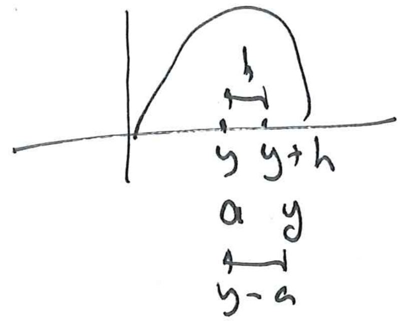
1. The Chain Rule

(1) We know $\frac{d}{dy}\sin y = \cos y$.

(a) Expand $\sin(y+h)$ to linear order in h. Write down the linear approximation to $\sin y$ about $y = a$.

$\sin(y+h) \approx \sin y + (\cos y)h$

$\sin y \approx \sin a + (\cos a)\cdot(y-a)$



(b) Now let $F(x) = \sin(3x)$. Expand $F(x+h)$ to linear order in h. What is the derivative of $\sin 3x$?

$F(x+y) = \sin(3(x+h)) = \sin(3x+3h)$

if $y=3x$ have $\sin(y+3h) \approx \sin y + (\cos y)\cdot 3h$

so $\sin(3(x+h)) \approx \sin(3x) + (\cos(3x)3)h$

so $\boxed{\frac{d(\sin 3x)}{dx} = \cos(3x)\cdot 3}$

Date: 13/10/2022, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

In general

Say $g(x+h) \approx g(x) + g'(x)h$

say $f(y+k) = f(y) + f'(y)k$

$$f(g(x+h)) \approx f(\underbrace{g(x)}_{y} + \underbrace{g'(x)h}_{k}) \approx f(g(x)) + f'(g(x))(g'(x)h)$$

$$\approx f(g(x)) + (f'(g(x))\cdot g'(x))h$$

So $\dfrac{d(f(g(x)))}{dx} = \left.\dfrac{df}{dy}\right|_{y=g(x)} \cdot \dfrac{dg}{dx}$ ("chain rule")

$\Rightarrow \dfrac{dz}{dx} = \dfrac{dz}{dy} \cdot \dfrac{dy}{dx}$ if $y = g(x)$, $z = f(y)$

Another formulation ↑

Applying chain rule

(a) e^{3x} composition of $y = 3x$

$z = e^y$

$\frac{dy}{dx} = 3$, $\frac{dz}{dy} = e^y$ so $\frac{d(e^{3x})}{dx} = e^{3x} \cdot 3$

or:

$$\frac{d(e^{3x})}{dx} = \frac{d(e^{3x})}{d(3x)} \cdot \frac{d(3x)}{dx} = e^{3x} \cdot 3$$

or: $(e^{3x})' = e^{3x} \cdot 3$

(b) $\sqrt{2x+1}$: composition of $y = 2x+1$

$z = \sqrt{y} = y^{1/2}$

$$\Rightarrow \frac{d\sqrt{2x+1}}{dx} = \frac{1}{2\sqrt{y}} \cdot 2 = \frac{1}{\sqrt{2x+1}}$$

don't want function of y!

(c) $\sin(x^2)$: composition of $g(x) = x^2$

$f(y) = \sin y$

So $(\sin(x^2))' = \cos(x^2) \cdot 2x$

(4) Differentiate

(a) $7x + \cos(x^n)$

$y = \cos x$

$z = \sqrt{y}$

$w = e^z$

$\frac{dw}{dx} = \frac{dw}{dz} \cdot \frac{dz}{dy} \cdot \frac{dy}{dx}$

$= e^z \cdot \frac{1}{2\sqrt{y}}(-\sin x)$

(b) $e^{\sqrt{\cos x}}$

chain rule chain rule

$\left(e^{\sqrt{\cos x}}\right)' = e^{\sqrt{\cos x}} \cdot \left(\sqrt{\cos x}\right)' = e^{\sqrt{\cos x}} \frac{1}{2\sqrt{\cos x}} (\cos x)' =$

chain rule

$= -e^{\sqrt{\cos x}} \cdot \frac{1}{2\sqrt{\cos x}} \cdot \sin x$

(c) (Final 2012) $e^{(\sin x)^2}$

(5) Suppose f, g are differentiable functions with $f(g(x)) = x^3$. Suppose that $f'(g(4)) = 5$. Find $g'(4)$.

Say $y = \log x$ want $\frac{dy}{dx}$.

Then $e^y = x$ so $1 = \frac{dx}{dx} = \frac{d(e^y)}{dx} = \frac{d(e^y)}{dy} \cdot \frac{dy}{dx} = e^y \frac{dy}{dx}$

solve for $\frac{dy}{dx}$: $\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$

so: $\boxed{\frac{d(\log x)}{dx} = \frac{1}{x}}$

<u>Aside:</u> want $(7.3 \cdot 10^8) \cdot (3.7 \cdot 10^{-2})$

Take log. $\log_{10}(\quad)$

$= 8 + \log_{10} 7.3 + \log_{10} 3.7 - 2$

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2. Logarithmic differentiation

(6) $\log\left(e^{10}\right) = 10$ $\qquad$ $\log(2^{100}) = 100 \log 2$

(7) Differentiate

(a) $\frac{\mathrm{d}(\log(ax))}{\mathrm{d}x} =$

$\overset{\text{by chain rule}}{=} \frac{1}{ax} \cdot a = \frac{1}{x}$

$\frac{\mathrm{d}}{\mathrm{d}t} \log\left(t^2 + 3t\right) \overset{\text{by chain rule}}{=}$

$= \frac{1}{t^2+3t} \cdot (2t+3)$

(b) $\frac{\mathrm{d}}{\mathrm{d}x} x^2 \log(1 + x^2) =$

$= 2x \cdot \log(1+x^2) + x^2 \left(\log(1+x^2)\right)'$

product rule

chain rule $\rightarrow = 2x \log(1+x^2) + x^2 \cdot \frac{1}{1+x^2} \cdot 2x$

$\frac{\mathrm{d}}{\mathrm{d}r} \frac{1}{\log(2+\sin r)} =$

Logarithmic Differentiation

Idea: $\log(fg) = \log f + \log g$

Example: $y = (x^2+1)\cdot\sin x\cdot\frac{1}{\sqrt{x^3+3}}\cdot e^{\cos x}$

want $\frac{dy}{dx}$, can use pdt rule.

But: $\log y = \log(1+x^2) + \log(\sin x) - \frac{1}{2}\log(x^3+3) + \cos x$

diff wrt x:

$$\underbrace{\frac{1}{y}\cdot\frac{dy}{dx}}_{\frac{d(\log y)}{dx}} = \frac{2x}{1+x^2} + \frac{\cos x}{\sin x} - \frac{3x^2}{2(x^3+3)} - \sin x$$

so $$\frac{dy}{dx} = (x^2+1)\sin x\cdot\frac{1}{\sqrt{x^3+3}}e^{\cos x}\left(\frac{2x}{1+x^2} + \frac{\cos x}{\sin x} - \frac{3x^2}{2(x^3+3)} - \sin x\right)$$

In general:

$$\frac{d(\log y)}{dx} = \frac{dy}{dx}\frac{1}{y} \quad \text{so} \quad \boxed{\frac{dy}{dx} = y\cdot\frac{d(\log y)}{dx}}$$

"log diff rule"

(b) x^x $\log(x^x) = x\log x$

so $\frac{d(\log x^x)}{dx} = \frac{d(x\log x)}{dx} = \log x + x\cdot\frac{1}{x} = \log x + 1$

so $\frac{d(x^x)}{dx} = x^x(\log x + 1)$

$$\boxed{\begin{array}{l} \text{If } y = x^n \qquad\qquad (1) \\ \log y = n\log x \\ \text{so } \frac{d(\log y)}{dx} = \frac{n}{x} \\ \text{so } \frac{dy}{dx} = \frac{n}{x}\cdot x^n = nx^{n-1} \end{array}}$$

(c) $(\log x)^{\cos x}$

(d) (Final, 2014) Let $y = x^{\log x}$. Find $\frac{dy}{dx}$ in terms of x only.

(i)! $\frac{d}{dx}(\log x)^7 = 7\cdot(\log x)^6 \frac{d(\log x)}{dx} = 7\cdot(\log x)^6\cdot\frac{1}{x}$

chain rule

Implicit differentiation

Suppose we have curve $x^2+4y^2=1$

want slope at point on curve.

Idea: can differentiate along curve then solve for $\frac{dy}{dx}$

Here

$$\frac{d}{dx}\left(x^2+4y^2\right) = \frac{d(1)}{dx} = 0$$

$$\| $$

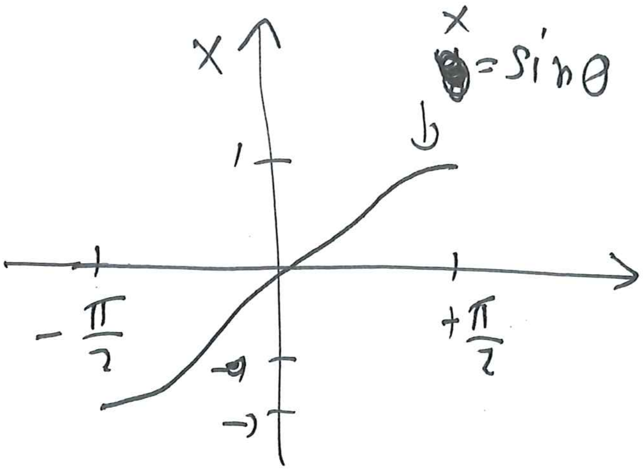
$$2x + 8y\frac{dy}{dx} = 0$$

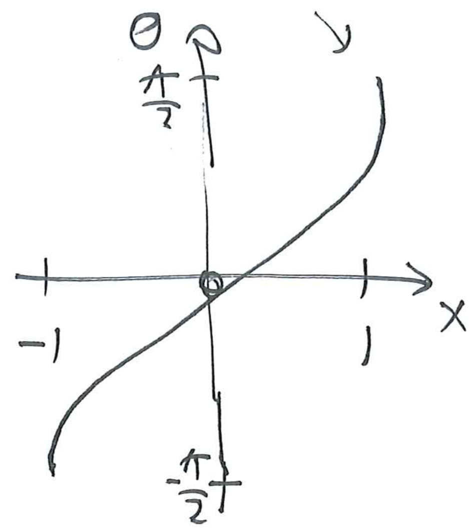
So $$\frac{dy}{dx} = -\frac{2x}{8y} = -\frac{x}{4y}$$

(if we have a point (x,y) on curve can plug in x,y values, get slope at the point)

Inverse trig functions

$\theta = \arcsin(x)$

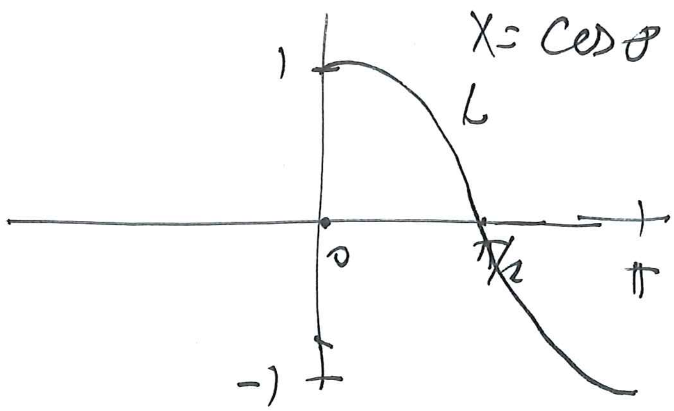


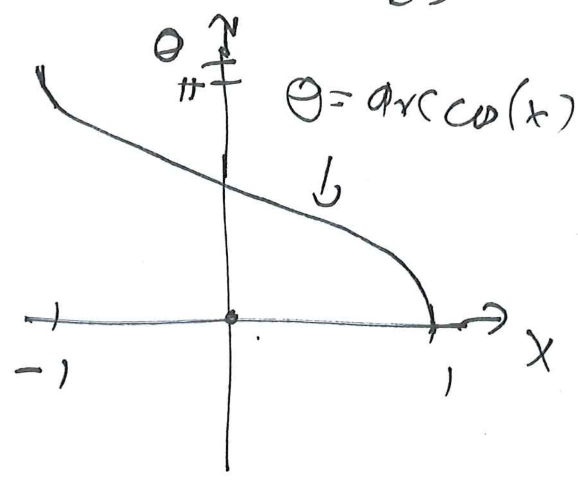


$\sin\theta$ is periodic:

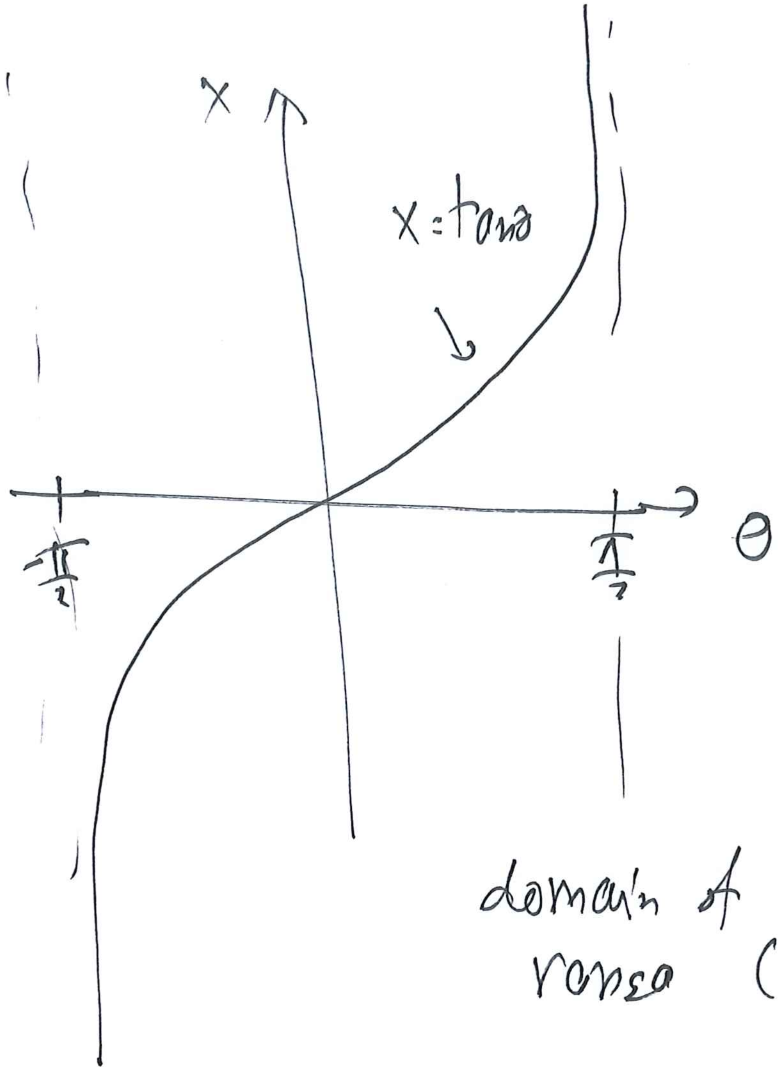
<u>many</u> solutions to $\sin\theta = x$

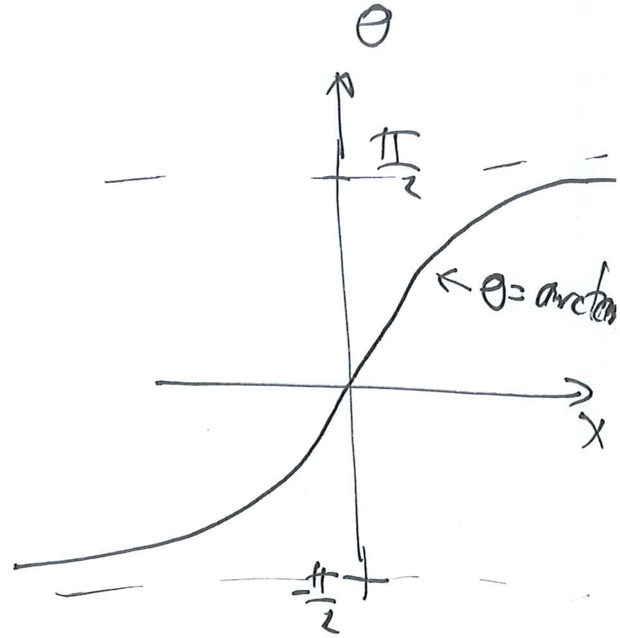
arcsin(x) is the solution $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$





arc cos, arc sin have domain $[-1, 1]$





domain of arctan is $(-\infty, \infty)$
range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Need to know: $$\frac{d(\arcsin x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d(\arccos x)}{dx} = -\frac{1}{\sqrt{1-x^2}} = -\frac{d(\arcsin x)}{dx}$$

$$\frac{d(\arctan x)}{dx} = \frac{1}{1+x^2}$$