Lior Silberman's Math 223: Problem Set 10 (due 28/3/2022)

Practice problems

Section 5.1: all problems are suitable
Section 5.2: all problems are suitable

Calculation

M1. Find the characteristic polynomial of the following matrices.

(a) $\begin{pmatrix} 5 & 7 \\ -3 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} \pi & e \\ \sqrt{7} & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -a_0 & \cdots & \cdots & -a_{n-2} & -a_{n-1} \end{pmatrix}$.

M2. For each of the following matrices find its spectrum and a basis for each eigenspace.

(a) $\begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}$ (b) $\frac{1}{3}\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.

Projections

Fix a vector space V.

1. Let $T, T' \in \mathrm{End}(V)$ be similar. Show that $p_T(x) = p_{T'}(x)$. (Hint: show that $x\mathrm{Id} - T$, $x\mathrm{Id} - T'$ are similar)

2. Let $T \in \mathrm{End}(V)$.
   (a) Let $p \in \mathbb{R}[x]$, and let $\underline{v} \in V$ be an eigenvector of T with eigenvalue λ. Show that $\underline{v}$ is an eigenvector of $p(T)$ with eigenvalue $p(\lambda)$.
   (b) Suppose $p(T) = 0$. Show that $p(\lambda) = 0$ for all eigenvalues λ of V.
   (c) Show that the only eigenvalue of a nilpotent map is 0.

3. Let $P \in \mathrm{End}(V)$ satisfy $P^2 = P$. Such maps are called *projections*.
   (a) Apply problem 2(b) to show that $\mathrm{Spec}(P) \subset \{0, 1\}$.
   REVIEW In the extra credit part of PS5 we showed that the the eigenspaces V_0 and V_1 of a projection span V (we call P the projection *onto* V_1 *along* V_0) and conversely that for any decomposition $V = V_0 \oplus V_1$ there is a unique projection for which these are the eigenspaces.
   (b) Let $V_0 = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right\}$ $V_1 = \mathrm{Span}\left\{\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}$ so that $\mathbb{R}^3 = V_0 \oplus V_1$ [no need to check this separately]. Let P be the projection onto V_1 along V_0. Find the matrix of P with respect to the *standard* basis of $\mathbb{R}^3$.
   Hint: By diagonalization $P = S \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix} S^{-1}$ where S is the matrix of eigenvectors.

The Quantum Harmonic Oscillator, I

PRAC In physics a "parity operator" is a map $R \in \mathrm{End}(V)$ such that $R^2 = I$ (we use the shorthand $I = \mathrm{Id}_V$).

RMK This was problem 4, but it is for practice, not for submission.

(a) Show that $\pm I$ are (uninteresting) parity operators.

— For parts (b)-(d) fix a parity operator R.

(b) Show that the eigenvalues of R are in $\{\pm 1\}$; let $V_\pm$ be the corresponding eigenspaces.

(c) Show that $\frac{I+R}{2}$, $\frac{I-R}{2}$ are the projections onto V_+, V_- along the other subspace, respectively. *Hint:* compute $(I+R)^2$ using that $R^2 = I$.

(d) Conclude that $V = V_+ \oplus V_-$ and hence that every parity operator is diagonalizable.

(e) Let X be a set and let $\tau\colon X \to X$ be an *involution*: a map such that $\tau^2 = \mathrm{id}_X$ (identity permutation). Let $R_\tau \in \mathrm{End}(\mathbb{R}^X)$ be the linear map $f \mapsto f \circ \tau$. Show that P_τ is a parity operator.

(f) Let $X = \mathbb{R}$, $\tau(x) = -x$. Explain how (b)-(e) relate to the concepts of *odd* and *even* functions.

5. Let $V = \left\{ p(x)e^{-x^2/2} \mid p \in \mathbb{R}[x] \right\}$ and for $n \geq 1$ let $V_n = \left\{ p(x)e^{-x^2/2} \mid p \in \mathbb{R}[x]^{<n} \right\} \subset V$. Let $H \in C^\infty(\mathbb{R})$ be the operator ("quantum Hamiltonian") $H = -D^2 + M_{x^2}$. Concretely we have $Hf = -\frac{d^2 f}{dx^2} + x^2 f$.

PRAC Show that $V_n \subset V$ are subspaces of $C^\infty(\mathbb{R})$, the space of infinitely differentiable functions.

(a) Show that $HV \subset V$ and $HV_n \subset V_n$.

(b) Let $H_n = H \restriction_{V_n} \in \mathrm{End}(V_n)$ be the restriction of H to V_n. Show that H_n has an upper-triangular basis with respect to an appropriate basis of V_n and determine its eigenvalues.

(c) Show that H_n is diagonable.

(d) Show that $HR = RH$ for the parity operator of 4(f).

(*e) Show that every eigenfunction of H_n is either even or odd. Which is which?

(f) Show that $V = \left\{ p(x)e^{-x^2/2} \mid p \in \mathbb{R}[x] \right\}$ has a basis of eigenfunctions of H, and that each eigenfunction is either even or odd.

Extra credit: the generalized eigenvalue decomposition and the Cayley–Hamilton Theorem

Fix a vector space V and a linear map $T \in \mathrm{End}_F(V)$.

A. DEF For a number λ define the *generalized λ-eigenspace* to be the set of vectors $\underline{v} \in V$ killed by some power of $T - \lambda$ (possibly depending on $\underline{v}$):

$$\tilde{V}_\lambda = \left\{ \underline{v} \in V \mid \exists k\colon (T-\lambda)^k \underline{v} = \underline{0} \right\}.$$

(a) Show that $\tilde{V}_\lambda$ is a subspace containing V_λ.
(b) Show that $\tilde{V}_\lambda \neq \{\underline{0}\}$ iff $V_\lambda \neq \{\underline{0}\}$ ("every generalized eigenvalue is a regular eigenvalue").
(c) Show that V_λ and $\tilde{V}_\lambda$ are T-invariant: if $\underline{v} \in \tilde{V}_\lambda$ then $T\underline{v} \in \tilde{V}_\lambda$ as well, and similarly for V_λ.
(d) Let $\mu \neq \lambda$. Show that $T \restriction_{\tilde{V}_\lambda} - \mu \in \mathrm{End}(\tilde{V}_\lambda)$ is injective ("no other eigenvalues in $\tilde{V}_\lambda$ except λ"). Using a factorization into linear terms conclude that for any polynomial p if $p(\lambda) \neq 0$ then $p\left(T \restriction_{\tilde{V}_\lambda}\right) \in \mathrm{End}(\tilde{V}_\lambda)$ is injective there.
(**e) Show that $\left\{\tilde{V}_\lambda\right\}_{\lambda \in \mathrm{Spec}(T)}$ are linearly independent.

COR The sum $\tilde{V} = \bigoplus_{\lambda \in \mathrm{Spec}(T)} \tilde{V}_\lambda$ is direct.

B. Continuing the previous problem, suppose now that V is finite-dimensional.

(a) Show that $p_{T\restriction_{\tilde{V}_\lambda}}(x) = (x-\lambda)^{\dim \tilde{V}_\lambda}$ and that $\left(T \restriction_{\tilde{V}_\lambda} - \lambda\right)^{\dim \tilde{V}_\lambda} = 0_{\tilde{V}_\lambda}$.
(b) Let $m(x) = \prod_{\lambda \in \mathrm{Spec}(T)} (x-\lambda)^{\dim \tilde{V}_\lambda}$. Show that $m(x) = p_{T\restriction_{\tilde{V}}}(x)$ and that $m(T \restriction_{\tilde{V}}) = 0$.
(c) Suppose that $\tilde{V} \neq V$. Show that setting $\bar{T}(\underline{v} + \tilde{V}) = T\underline{v} + \tilde{V}$ gives a well-defined linear map $\bar{T}$ on the quotient vector space $W = V/\tilde{V}$.
(d) Let μ be a root of $p_{\bar{T}}(x)$, and let $W_\mu \subset W$ be the corresponding eigenspace. Show that $\prod_{\lambda \in \mathrm{Spec}(T)\setminus\{\mu\}} (\bar{T} - \lambda)^{\dim \tilde{V}_\lambda}$ is an invertible map there. Conclude that if $\underline{v} + \tilde{V} \in W_\mu$ with $\underline{v} \notin \tilde{V}$ then $\underline{u} = \prod_{\lambda \in \mathrm{Spec}(T)\setminus\{\mu\}} (T - \lambda)^{\dim \tilde{V}_\lambda} \underline{v} \notin \tilde{V}$ but $\underline{u} + \tilde{V} \in W_\mu$.
(e) Suppose μ is not an eigenvalue of T. Show that $(T-\mu)\underline{u} = \underline{0}$, a contradiction to $\underline{u} \notin \tilde{V}$.
Hint: In this case the polynomial in the definition of $\underline{u}$ is exactly $m(T)$).
(f) Suppose μ is an eigenvalue of T. Show that $(T-\mu)^{1+\dim \tilde{V}_\mu} \underline{u} = \underline{0}$ showing that $\underline{u} \in \tilde{V}_\mu \subset \tilde{V}$, a contradiction.

C. It follows that $V = \tilde{V}$ so that $T \restriction_{\tilde{V}} = T$. Problem B(b) now gives two corollaries:

(a) The algebraic multiplicity of $\lambda \in \mathrm{Spec}(T)$ is equal to $\dim \tilde{V}_\lambda$ (and since $V_\lambda \subset \tilde{V}_\lambda$ we get a new proof that the algebraic multipicity is at least the geometric multiplicity).
(b) (Cayley–Hamilton Theorem) $p_T(T) = 0$.